The gamma–Weibull distribution revisited

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ABSTRACT

The five parameter gamma–Weibull distribution has been introduced by Leipnik and Pearce (2004). Nadarajah and Kotz (2007) have simplified it into four parameter form, using hypergeometric functions in some special cases. We show that the probability distribution function, all moments of positive order and the characteristic function of gamma–Weibull distribution of a random variable can be explicitly expressed in terms of the incomplete confluent Fox–Wright Psi–function, which is recently introduced by Srivastava and Pogány (2007). In the same time, we generalize certain results by Nadarajah and Kotz that follow as special cases of our findings.

Key words: gamma distribution, Weibull distribution, confluent Fox–Wright $_1\Psi_0$, incomplete confluent Fox–Wright $_1\psi_0$.

1 INTRODUCTION AND RESULTS REQUIRED

Multiplying and then renormalizing the product of densities of gamma and Weibull distributions (Leipnik and Pearce 2004) defined the so–called *gamma–Weibull* distribution. The related probability density function (**pdf** in the sequel) was simplified by Nadarajah and Kotz (2007, Eq. (1.1)) in the form

$$f_{gW}(x) = Kx^{\alpha-1}e^{-\mu x - ax^r}\chi_{(0,\infty)}(x) \qquad \left(\theta := (\alpha, \mu, a, r) > 0\right), \tag{1}$$

where $\chi_A(x)$ denotes the indicator of $\{x \in A\}$, while $K = K(\alpha, \mu, a, r)$ is the normalizing constant. In what follows we write $\xi \sim gW(\theta)$ when the r.v. ξ has the gamma–Weibull distribution with parameter θ.

Leipnik and Pearce obtained the characteristic function and the moments of gamma–Weibull distribution. Nadarajah and Kotz simplified these results in a modestly simpler form using finite sums of generalized hypergeometric function $_{p+1}F_q$ (Nadarajah and Kotz 2007, Eqs. (2.1–2)), when $r = p/q$; $p, q \geq 1$ are co–prime integers.

We will show here that the cumulative distribution function (**CDF**), the moments of all orders having positive real parts and the characteristic function (**chf**) can be represented explicitly in terms of the so–called *confluent Fox–Wright function* $_1\Psi_0$ and its *incomplete variant form* $_1\psi_0$ for all $r > 0$, thereby simplifying and generalizing mainly the results (Nadarajah and Kotz 2007). The incomplete $_p\psi_q$ is studied very recently by Srivastava and Pogány (2007). In what follows, the symbol $_p\Psi_q$ stands for the Fox–Wright generalization of the hypergeometric function $_pF_q$, with p numerator and q denominator parameters, defined by

$$
\begin{aligned}
&_p\Psi_q\left[\begin{array}{c} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) \\ (\beta_1, B_1), \cdots, (\beta_q, B_q) \end{array} \middle| x\right] \\
&= {}_p\Psi_q\left[\begin{array}{c} (\alpha_p, A_p) \\ (\beta_q, B_q) \end{array} \middle| x\right] := \sum_{m=0}^{\infty} \frac{\prod_{\ell=1}^{p} \Gamma(\alpha_\ell + A_\ell m)}{\prod_{\ell=1}^{q} \Gamma(\beta_\ell + B_\ell m)} \frac{x^m}{m!} \\
&\left(A_\ell, B_j \in \mathbb{R}_+, \ \ell = \overline{1, p}, \ j = \overline{1, q}; \quad 1 + \sum_{\ell=1}^{q} B_\ell - \sum_{j=1}^{p} A_j > 0\right),
\end{aligned}
\tag{2}
$$

for suitably bounded values of $|x|$ and in terms of Euler's Gamma function

$$
\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-z} dz \qquad \left(\Re\{z\} > 0\right),
$$

where an empty product in (2) is to be interpreted (as usual) to be 1 (Mathai and Saxena 1978), (Srivastava et al. 1982). In terms of the analytically continued Gamma–function $\Gamma(z)$, $z \in \mathbb{C} \setminus \mathbb{Z}_-^0 := \{0, -1, -2, \cdots\}$), the *incomplete Gamma–function* $\gamma(z, a)$ (Gradshteyn and Ryzhik 2000, **8.350** 1.) is given by truncating the integration domain of Gamma–function to $[0, \omega]$, i.e.

$$
\gamma(z, \omega) := \int_0^{\omega} x^{z-1} e^{-x} dx \qquad \left(\Re\{z\} > 0, \ \omega \geq 0\right).
$$

Following (Srivastava and Pogány 2007, Eq. (6)), we define the so–called incomplete Fox–Wright function $_p\psi_q$ in the form

$$
_p\psi_q\left[\begin{array}{c} (\alpha_p, A_p) \\ (\beta_q, B_q) \end{array} \middle|_\gamma (x, \omega)\right] := \sum_{m=0}^{\infty} \frac{\prod_{\ell=1}^{p} \gamma(\alpha_\ell + A_\ell m, \omega)}{\prod_{\ell=1}^{q} \gamma(\beta_\ell + B_\ell m, \omega)} \frac{x^m}{m!}
\tag{3}
$$

$$
\mathcal{P} := \left(A_\ell, B_j \in \mathbb{R}_+, \ \ell = \overline{1, p}, \ j = \overline{1, q}; \quad 1 + \sum_{\ell=1}^{q} B_\ell - \sum_{j=1}^{p} A_j > 0; \quad \omega \geq 0\right).
$$

It is interesting to remark, that the definition (3) immediately yields the limit relationship

$$
\lim_{\omega \to \infty} {}_p\psi_q\left[\begin{array}{c} (\alpha_p, A_p) \\ (\beta_q, B_q) \end{array} \middle|_\gamma (x, \omega)\right] \equiv {}_p\Psi_q\left[\begin{array}{c} (\alpha_p, A_p) \\ (\beta_q, B_q) \end{array} \middle| x\right].
\tag{4}
$$

2   THE GAMMA–WEIBULL $gW(\theta)$ DISTRIBUTION

Cutting off to $(0, z)$ the integration domain $\mathbb{R}_+$ of the integral $\mathcal{I}(\alpha, \mu, a, r)$ introduced by Nadarajah and Kotz (2007, Eq. (2.1)), we get the resulting integral

$$
\mathcal{I}_{gW}(\theta | \omega) := \int_0^{\omega} x^{\alpha-1} \exp\left\{-\mu x - a x^r\right\} dx \qquad \left(\theta > 0\right),
\tag{5}
$$

such that it is closely connected to $gW(\theta)$ distribution. We express now $\mathcal{I}_{gW}(\theta|\omega)$ in terms of the incomplete confluent Fox–Wright $_1\psi_0$ for positive real $r \in \mathbb{R}_+$ extending the related result by Nadarajah and Kotz (2007, Eq. (2.2)), which was obtained for rational positive $r \in \mathbb{Q}_+$.

LEMMA 1.

$$\mathcal{I}_{gW}(\theta|\omega) = \begin{cases} \mu^{-\alpha} {}_1\psi_0\left[\begin{array}{c|c} (\alpha, r) \\ \hline \quad \end{array} \Big|_\gamma \left(-a\mu^{-r}, \omega \right) \right] & 0 < r < 1 \\[2ex] \dfrac{\gamma\left(\alpha, (\mu+a)\omega\right)}{(\mu+a)^\alpha} & r = 1 \\[2ex] \dfrac{1}{ra^{\alpha/r}} {}_1\psi_0\left[\begin{array}{c|c} (\alpha/r, 1/r) \\ \hline \quad \end{array} \Big|_\gamma \left(-\mu a^{-1/r}, \omega^r \right) \right] & r > 1 \end{cases} . \quad (6)$$

PROOF. Expanding $\exp\{-ax^r\}$ into Maclaurin series, interchanging the order of the integration and summation and finally using definition of $\gamma(z; \omega)$, then we easily confirm that

$$\mathcal{I}_{gW}(\theta|\omega) = \frac{1}{\mu^\alpha} \sum_{n=0}^\infty \gamma(\alpha + rn, \omega) \frac{(-a\mu^{-r})^n}{n!} \qquad (r \in (0,1)). \quad (7)$$

which is the assertion of the Lemma. Since the constraint $\mathcal{P}$ in (3) being now ensured with $r < 1$, so the series (7) converges. The case $r = 1$ results in the incomplete gamma function.

Finally, when $r > 1$, replacing $x \mapsto x^r$ in the integral (5), we easily deduce

$$\mathcal{I}_{gW}(\alpha, \mu, a, r|\omega) = r^{-1} \mathcal{I}_{gW}\left(\alpha/r, a, \mu, 1/r|\omega^r\right).$$

This confirms the assertion of the Lemma.   □

With the help of (4), we clearly deduce the limiting case

$$\mathcal{I}_{gW}(\theta) := \lim_{\omega \to \infty} \mathcal{I}_{gW}(\theta|\omega), \quad (8)$$

and *a fortiori* the limiting results of Lemma 1. We, therefore, arrive at

COROLLARY 1.1.

$$\mathcal{I}_{gW}(\theta) = \begin{cases} \mu^{-\alpha} {}_1\Psi_0\left[\begin{array}{c|c} (\alpha, r) \\ \hline \quad \end{array} \Big| -\dfrac{a}{\mu^r} \right] & 0 < r < 1 \\[2ex] \dfrac{\Gamma(\alpha)}{(\mu+a)^\alpha} & r = 1 \\[2ex] \dfrac{1}{ra^{\alpha/r}} {}_1\Psi_0\left[\begin{array}{c|c} (\alpha/r, 1/r) \\ \hline \quad \end{array} \Big| -\dfrac{\mu}{a^{1/r}} \right] & r > 1 \end{cases} . \quad (9)$$

REMARK 1. The case $r = 1$ in (Nadarajah and Kotz 2007, Eq. (2.2)) is erroneous – there should be $\Gamma(\alpha)$ in the numerator.

THEOREM 1. *Let the random variable* $\xi \sim gW(\theta), \theta > 0$ *be with* **pdf** (1). *Then, the* **pdf** $f_{gW}(x)$ *and the* **PDF** $F_{gW}(x)$ *of r.v.* ξ *are of the form:*

$$f_{gW}(x) = \frac{x^{\alpha-1}\exp\{-\mu x - ax^r\}}{\mathcal{I}_{gW}(\theta)}\chi_{(0,\infty)}(x),\tag{10}$$

$$F_{gW}(x) = \chi_{(0,\infty)}(x)\begin{cases}\dfrac{{}_1\psi_0\left[\begin{array}{c}(\alpha,r)\\ \underline{\hspace{1cm}}\end{array}\bigg|_\gamma\left(-a\mu^{-r},x\right)\right]}{{}_1\Psi_0\left[\begin{array}{c}(\alpha,r)\\ \underline{\hspace{1cm}}\end{array}\bigg|-\dfrac{a}{\mu^r}\right]} & 0 < r < 1 \\[2em] \dfrac{\gamma\left(\alpha,(\mu+a)x\right)}{\Gamma(\alpha)} & r = 1 \\[2em] \dfrac{{}_1\psi_0\left[\begin{array}{c}(\alpha/r,1/r)\\ \underline{\hspace{1cm}}\end{array}\bigg|_\gamma\left(-\mu a^{-1/r},x^r\right)\right]}{{}_1\Psi_0\left[\begin{array}{c}(\alpha/r,1/r)\\ \underline{\hspace{1cm}}\end{array}\bigg|-\dfrac{\mu}{a^{1/r}}\right]} & r > 1\end{cases}\tag{11}$$

respectively.

PROOF. To prove (10), we need the exact value of the normalizing constant K, such that appears in (1). But it is straightforward that $K^{-1} = \mathcal{I}_{gW}(\theta)$. As $\mathsf{P}\{\xi \le 0\} = 0$ hence, we deduce that

$$F_{gW}(x) = \frac{1}{\mathcal{I}_{gW}(\theta)}\int_0^x t^{\alpha-1}\exp\{-\mu t - at^r\}\mathrm{d}t = \frac{\mathcal{I}_{gW}(\theta\,|\,x)}{\mathcal{I}_{gW}(\theta)}.$$

Expanding the last relation into three formulæ with respect to the related domains of r, by the Lemma 1 and Corollary 1.1 we conclude (11). □

3   MOMENTS AND CHARACTERISTIC FUNCTION

We now derive the moments $\mathsf{E}\xi^\beta$ and the characteristic function $\phi_{gW}(t)$ of the Gamma–Weibull distribution. All these results are expressed in terms of the confluent Fox–Wright function ${}_1\Psi_0$ *via* the integral $\mathcal{I}_{gW}(\theta)$.

THEOREM 2. *The moment $\mathsf{E}\xi^\beta$ of order β of the r.v. $\xi \sim gW(\theta)$ is given by*

$$
\mathsf{E}\xi^\beta =
\begin{cases}
\dfrac{{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha+\beta,r)\\ \hline \end{array}\; -\dfrac{a}{\mu^r}\right]}{\mu^\beta\,{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha,r)\\ \hline \end{array}\; -\dfrac{a}{\mu^r}\right]} & 0 < r < 1 \\[6mm]
\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,(\mu+a)^\beta} & r = 1 \\[6mm]
\dfrac{{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha/r+\beta/r,\,1/r)\\ \hline \end{array}\; -\dfrac{\mu}{a^{1/r}}\right]}{a^{\beta/r}\,{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha/r,\,1/r)\\ \hline \end{array}\; -\dfrac{\mu}{a^{1/r}}\right]} & r > 1
\end{cases},
\tag{12}
$$

whenever $\alpha + \Re\{\beta\} > 0$.

Moreover, the associated **chf** $\phi_{gW}(t)$ *is given by*

$$
\phi_{gW}(t) =
\begin{cases}
\left(\dfrac{\mu}{\mu-it}\right)^\alpha \cdot \dfrac{{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha,r)\\ \hline \end{array}\; -\dfrac{a}{(\mu-it)^r}\right]}{{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha,r)\\ \hline \end{array}\; -\dfrac{a}{\mu^r}\right]} & 0 < r < 1 \\[6mm]
\left(\dfrac{\mu+a}{\mu+a-it}\right)^\alpha & r = 1 \\[6mm]
\dfrac{{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha/r,\,1/r)\\ \hline \end{array}\; \dfrac{it-\mu}{a^{1/r}}\right]}{{}_1\Psi_0\!\left[\begin{array}{c|c}(\alpha/r,\,1/r)\\ \hline \end{array}\; -\dfrac{\mu}{a^{1/r}}\right]} & r > 1
\end{cases}.
\tag{13}
$$

PROOF. Let us take a certain parameter β, which satisfies $\alpha + \Re\{\beta\} > 0$. Then, we have

$$
\mathsf{E}\xi^\beta = K \int_0^\infty x^\beta \, f_{gW}(x)\,\mathrm{d}x = K\,\mathcal{I}_{gW}(\alpha+\beta,\mu,a,r).
$$

Finally, for the **chf**, we have

$$
\phi_{gW}(t) = \mathsf{E}(e^{it\xi}) = K \int_0^\infty e^{itx} f_{gW}(x)\mathrm{d}x = K\,\mathcal{I}_{gW}(\alpha,\mu-it,a,r).
$$

Now, replacing $\mu - it \mapsto \mu$ in $\mathcal{I}_{gW}(\theta)$, we immediately obtain (13). $\qquad\square$

4   CONCLUDING REMARKS

A. The confluent Fox–Wright functions $_1\Psi_0, \, _1\psi_0$ are defined by series containing gamma function and incomplete gamma function assuming $r \in (0, 1)$, we have

$$
_1\psi_0\left[\begin{array}{c}(\alpha, r) \\ \underline{\quad\quad} \end{array}\bigg|_{\gamma} \; (x, \omega)\right] = \sum_{n=0}^{\infty} \frac{\gamma(\alpha + rn, \omega)}{n!} x^n = \omega^{\alpha} \sum_{n=0}^{\infty} {}_1F_1\left[\begin{array}{c}\alpha + rn \\ \alpha + rn + 1\end{array}\bigg| \omega\right]\frac{(\omega^r x)^n}{(\alpha + rn)\, n!}, \quad (14)
$$

because of (Srivastava et al. 1982, Eq. A25, p. 244)

$$
\gamma(a, \omega) = a^{-1}\, \omega^a \, {}_1F_1\left[\begin{array}{c}a \\ a+1\end{array}\bigg| \omega\right] \qquad (\Re\{a\} > 0).
$$

Finally, applying now the limiting process $\omega \to \infty$ to (14) and using (3), we deduce

$$
_1\Psi_0\left[\begin{array}{c}(\alpha, r) \\ \underline{\quad\quad}\end{array}\bigg| \, x\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + rn)}{n!} x^n .
$$

The gamma–, incomplete gamma– and the confluent hypergeometric $_1F_1$ are implemented in *Mathematica* as the subroutines `Gamma[z]`, `Gamma[a,0,z]` and `Hypergeometric1F1[a,b,z]`, respectively.

B. In practice, the parameter vector $\theta = (\alpha, \mu, a, r)$ is unknown, and it has to be estimated from a random sample $\mathbf{X} = (X_1, \cdots, X_n)$, such that it consists of independently sampled identical random repliacæ of the r.v. $\xi \sim gW(\theta)$. The traditional parameter estimation procedure is the Maximum Likelihood (ML) method because the ML estimator possesses numerous desired properties, such as asymptotic normality and consistency. Unfortunately, this procedure can be used in a special case only for $r = 1$, when $gW(\theta)$ becomes the ordinary gamma–distribution with parameters α and $\beta := \mu + a$. It is known that the related ML–estimators are given by

$$
\ln\widehat{\alpha} - \psi(\widehat{\alpha}) = \ln \frac{\overline{X}}{\sqrt[n]{\prod_{j=1}^{n} X_j}}, \qquad \widehat{\beta} = \frac{\widehat{\alpha}}{\overline{X}},
$$

where $\psi = (\ln\Gamma)' = \Gamma'/\Gamma$ stands for the familiar digamma function, and $\overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$ denotes the sample mean.

In the case when $r \neq 1$, in general the ML estimator does not exist. Indeed, let $0 < r < 1$. Then, the log–likelihood function becomes

$$
\ell_{\theta}(\mathbf{X}) = \ln \prod_{j=1}^{n} f_{gW}(X_j) = n\left(\alpha \ln\mu - \ln {}_1\Psi_0(\alpha, r) - (\alpha - 1)\overline{\ln X} - \mu\overline{X} - a\overline{X^r}\right).
$$

where

$$
_1\Psi_0(\alpha, r) = {}_1\Psi_0\left[\begin{array}{c}(\alpha, r) \\ \underline{\quad\quad}\end{array}\bigg| -\frac{a}{\mu^r}\right].
$$

Equating to zero the partial derivatives of $\ell_\theta(\mathbf{X})$ with respect to μ and a, we conclude

$$_1\Psi_0(\alpha, r)\,\mu^r\left(\alpha - \mu\overline{X}\right) = {}_1\Psi_0(\alpha + r, r)\,a\,r$$

$$_1\Psi_0(\alpha + r, r) = \mu^r\,\overline{X^r}\,{}_1\Psi_0(\alpha, r)$$

such that it results in

$$\alpha - \mu\overline{X} = a\,r\,\overline{X^r}\,, \tag{15}$$

when $_1\Psi_0(\alpha, r) \cdot {}_1\Psi_0(\alpha + r, r) \neq 0$. Now, it is not difficult to find values of θ for any fixed sample $\mathbf{X}$ outside of the manifold (15).

Similar procedure leads to counterexamples when $r > 1$.

Since the ML method is not efficient when $r \neq 1$, following the recent method by Marković et al. (2009) developed for the three–parameter Weibull distribution, Least Squares parameter estimation method could be derived for the gamma–Weibull distribution. However, this problem deserves a separate article.

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RESUMO

A distribuição gamma–Weibull a cinco parâmetros foi introduzida for Leipnik e Pearce (2004). Nadarajah e Kotz (2007) a simplificaram para uma forma a quatro parâmetros usando funções hipergeométricas em alguns casos especiais. Nós mostramos que a função de distribuição cumulativa, todos os momentos de ordem positiva e a função característica da distribuição gamma–Weibull de uma variável randômica pode ser explicitamente expressa em termos da função–Psi de Fox–Wright confluente incompleta, recentemente introduzida por Srivastava and Pogány (2007). Ao mesmo tempo, generalizamos certos resultados de Nadarajah e Kotz que decorrem como casos especiais de nossos achados.

Palavras-chave: distribuição gamma, distribuição de Weibull, função $_1\Psi_0$ de Fox–Wright confluente, função $_1\psi_0$ de Fox–Wright confluente incompleta.

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