Anais da Academia Brasileira de Ciências (2010) 82(4): 813-822
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

# Injectivity of the Dirichlet-to-Neumann Functional and the Schwarzian Derivative

**FERNANDO A.F.C. SILVA**[1], **PEDRO A.G. VENEGAS**[2] **and RAMÓN O.M. AHUMADA**[1]

[1]Universidade Federal de Pernambuco, CCEN – Departamento de Matemática, Av. Prof. Luiz Freire s/n
Cidade Universitária, 50740-540 Recife, PE, Brasil
[2]Universidade Federal da Paraíba, CCEN – Departamento de Matemática, Cidade Universitária
58059-970 João Pessoa, PB, Brasil

*Manuscript received on October 1, 2008; accepted for publication on August 11, 2010*

**ABSTRACT**

In this article, we show the relation between the Schwartz kernels of the Dirichlet-to-Neumann operators associated to the metrics $g_0$ and $h = F^*(e^{2\varphi}g_0)$ on the circular annulus $A_R$, and the Schwarzian Derivative of the argument function $f$ of the restriction of the diffeomorphism $F$ to the boundary of $A_R$.

**Key words:** annulus, Dirichlet-to-Neumann Functional, Schwarzian Derivative.

## 1 INTRODUCTION

Let $\mathcal{M}(\overline{\Omega})$ denote the space of all Riemannian metrics on a compact manifold $\overline{\Omega}$, with $C^\infty$ boundary $\partial\Omega$, and denote by $\mathcal{O}_p(\partial\Omega)$ the space of continuous linear operators acting on $C^\infty(\partial\Omega)$.

The Dirichlet-to-Neumann functional $\Lambda$ is a mapping from $\mathcal{M}(\overline{\Omega})$ into $\mathcal{O}_p(\partial\Omega)$ such that, for each $g \in \mathcal{M}(\overline{\Omega})$, $\Lambda_g$ takes Dirichlet boundary values to Neumann boundary values. More precisely, if $u \in C^\infty(\overline{\Omega})$ is the unique solution of the Dirichlet problem $\Delta_g u = 0$ in $\Omega$, $u|_{\partial\Omega} = \varphi \in C^\infty(\partial\Omega)$, then $\Lambda_g(\varphi) = du(\nu_g) \in C^\infty(\partial\Omega)$, where $\Delta_g$ (resp. $\nu_g$) is the Laplace-Beltrami operator (resp. unit interior normal vector field) associated to the metric $g$. The study of this functional goes back to the seminal paper of (Calderón 1980).

It is known (Lee and Uhlmann 1989) that $\Lambda_g$ is in fact an elliptic self-adjoint pseudo-differential operator of order one, whose principal symbol is $|\xi|_{h_0}$, $\xi \in T^*\partial\Omega$, and $h_0 := g\,|_{\partial\Omega}$.

Let $\mathcal{D}(\overline{\Omega})$ be the group of diffeomorphism of $\overline{\Omega}$. The semi-direct product $\mathcal{D}(\overline{\Omega}) \ltimes \mathcal{C}^\infty(\overline{\Omega})$ (Polyakov 1987) of the groups $\mathcal{D}(\overline{\Omega})$ and $\mathcal{C}^\infty(\overline{\Omega})$ defined by

$$(F, \alpha) \bullet (H, \beta) = (F \circ H, \beta \circ F^{-1} + \alpha),$$

Correspondence to: Ramón Orestes Mendoza Ahumada
E-mail: ramon@dmat.ufpe.br

provides a natural right action on $\mathcal{M}(\overline{\Omega})$, given by

$$g \bullet (F, \varphi) = F^* e^{2\varphi} g,$$

where $F^*$ denotes the pull-back of $F$.

The main obstruction to injectivity, in the two-dimensional case, is the semidirect product of the groups of diffeomorphisms that restricts to the identity on the boundary, and the Abelian group of real-valued functions that equals zero on it. In fact, as formula (2.1) shows, the Dirichlet-to-Neumann Functional is constant on the orbits determined by $\mathcal{D}_0(\overline{\Omega}) \ltimes \mathcal{C}_0^\infty(\overline{\Omega})$; this is a normal subgroup of $\mathcal{D}(\overline{\Omega}) \ltimes \mathcal{C}^\infty(\overline{\Omega})$.

With respect to the determination of the metric $g$ from the Dirichlet-to-Neumann Operator, we recommend the papers (Lee and Uhlmann 1989), (Lassas and Uhlmann 2001) and (Lassas et al. 2003). In these papers, they solve, in a more general setting, the problem of recovering the manifold and the metric.

In the case of a fixed annulus, all metrics can be written as $h = F^* e^{2\varphi} g_0$, for $g_0$ coming from the pull-back of the euclidean metric in the annulus of radius 1 and $R^2$, $R > 1$. We prove, in this special case, that the equality of the Dirichlet-to-Neumann Operators associated to both metrics $h$ and $g_0$ gives us a relation involving the Schwarzian derivative of $f$ ($f$ the lifting to $\mathbb{R}$ of the restriction to the boundary of the diffeomorphism $F$).

Furthermore, we also show that the conformal factor restricted to the boundary of the annulus is determined by $f$.

More precisely, we shall prove in Section 2 that, if $\Omega$ is the annulus

$$A_R = \left\{ z \in \mathbb{C} : \frac{1}{R} <\mid z \mid< R \right\},$$

$g_0 \in \mathcal{M}(\overline{A_R})$ is conformal to the euclidean metric, $h = F^*(e^{2\varphi} g_0)$, where $F \in \mathcal{D}(\overline{A_R})$ and $\varphi \in \mathcal{C}^\infty(\overline{A_R})$; the equality of the Schwartz kernels of $\Lambda_{g_0}$ and of $\Lambda_h$ implies that the argument function $f$, of the restriction of $F$ to $\partial A_R$, satisfies the differential equation

$$S(f) = \lambda(R)\big([f']^2 - 1\big) \quad \text{and} \quad e^{-\varphi \circ F} = f',$$

and $S(f)$ denotes the Schwarzian Derivative of $f$. It follows that, if $\lambda(R) \geq 0$, then $f' = 1$, $f(\theta) = \theta + c$ and $\varphi$ equals zero on the boundary.

## 2 GEOMETRIC FORMULATION

Here on we will denote by $\mathcal{N}_g$ the Schwartz kernel of $\Lambda_g$. We start with two lemmas.

LEMMA 2.1. *Given a two-dimensional compact manifold $\overline{\Omega}$ with $\mathcal{C}^\infty$ boundary and $F \in \mathcal{D}(\overline{\Omega})$, $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ and $g \in \mathcal{M}(\overline{\Omega})$, we have*

$$\Lambda_{F^* e^{2\varphi} g} = F^* \circ e^{-\varphi} \circ \Lambda_g \circ F^{-1^*}. \tag{2.1}$$

PROOF. See (Gómez and Mendoza 2006). □

LEMMA 2.2. *Let $\overline{\Omega}$ be a two-dimensional compact manifold with $\mathcal{C}^\infty$ boundary, $h = F^*(e^{2\phi}g)$ where $F \in \mathcal{D}(\overline{\Omega})$, $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$, $g \in \mathcal{M}(\overline{\Omega})$ and $E$ the unitary vector field to $\partial\Omega$, with respect to the metric $g|_{\partial\Omega}$. Then,*

$$\begin{cases} \mathcal{N}_{F^*g}(x, y) &= \mathcal{N}_g(F(x), F(y))F'(y), \\ \mathcal{N}_{e^{2\varphi}g}(x, y) &= e^{-\varphi(x)}\mathcal{N}_g(x, y). \end{cases}$$

*where $F'$ denotes the real, valued function on $\partial\Omega$ such that $F_*E = F'\, E \circ F$.*

PROOF. Let $x \in \partial\Omega$,

$$\begin{aligned} \Lambda_{F^*g}(\psi)(x) &= \int_{\partial\Omega} \mathcal{N}_{F^*g}(x, y)\psi(y)v_h(y) \\ &= F^* \circ \Lambda_g \circ \left(F^{-1}\right)^* \circ \psi(x) \\ &= F^* \circ \Lambda_g\big(\psi \circ F^{-1}\big)(x) \\ &= \Lambda_g\big(\psi \circ F^{-1})(F(x)\big) \\ &= \int_{\partial\Omega} \mathcal{N}_g(F(x), z)\big(\psi \circ F^{-1}\big)(z)v_h(z) \end{aligned}$$

changing variables $F(y) = z$ we get:

$$\Lambda_{F^*g}(\psi)(x) = \int_{\partial\Omega} \mathcal{N}_g(F(x), F(y))\psi(y)F^*(v_h)(y)$$

$$\Lambda_{F^*g}(\psi)(x) = \int_{\partial\Omega} \mathcal{N}_g(F(x), F(y))\psi(y)F'(y)(v_h)(y)$$

where $F'$ denotes the unique real, valued function defined on $\partial\Omega$ such that

$$F_*E = F'\, E \circ F$$

and $E$ is the tangent unitary vector field on $\partial\Omega$ such that $h(E, E) = 1$ and $v_h(E) = +1$. The above equation means at every point $p \in \partial\Omega$ the following: $F_*E(p)$ and $E \circ F(p)$ belong to the same one-dimensional tangent space $T_{F(p)}(\partial\Omega)$; consequently, the first one is a real multiple of the second. In fact, this multiple is unique and it is equal to $F'(p)$.

For the second equality,

$$\begin{aligned} \Lambda_{e^{2\varphi}g}(\psi)(x) &= \int_{\partial\Omega} \mathcal{N}_{e^{2\varphi}g}(x, y)\psi(y)v_h(y) \\ &= e^{-\varphi(x)}\Lambda_g(\psi)(x) \\ &= e^{-\varphi(x)} \int_{\partial\Omega} \mathcal{N}_g(x, y)\psi(y)v_h(y) \end{aligned}$$

finishing the proof. □

The next Lemma establish, a relation between $\mathcal{N}_g$ and the Green function $G(z, z')$ of the Laplacian $\Delta_g$ with Dirichlet condition on $\partial\Omega$ (Guillarmou and Sá Barreto 2009).

LEMMA 2.3. *The Schwartz kernel $\mathcal{N}_g(y, y')$ of $\Lambda_g$ is given for $y, y' \in \partial\Omega$, $y \neq y'$, by*

$$\mathcal{N}_g(y, y') = \partial_n \partial_{n'} G(z, z')\big|_{z=y, z'=y'}$$

*where $\partial_n$, $\partial_{n'}$ are, respectively, the inward pointing vector fields to the boundary in variable $z$ and $z'$.*

PROOF. Let $x$ be the distance function to the boundary in $\overline{\Omega}$; it is smooth in a neighborhood of $\partial\Omega$ and the normal vector field to the boundary is the gradient $\partial_n = \nabla^g x$ of $x$. The flow $e^{t\partial_n}$ of $\nabla^g x$ induces a diffeomorphism $\phi : [0, \epsilon)_t \times \partial\Omega \to \phi([0, \epsilon)_t \times \partial\Omega)$ defined by $\phi(t, y) := e^{t\partial_n}(y)$, and we have $x(\phi(t, y)) = t$. This induces natural coordinates $z = (x, y)$ near the boundary, these are normal geodesic coordinates. The function $u$ is the unique solution of the Dirichlet problem $\Delta_g u = 0$ in $\Omega$, and $u|_{\partial\Omega} = \varphi \in \mathcal{C}^\infty(\partial\Omega)$ can be obtained by taking

$$u(z) := \chi(z) - \int_{\overline{\Omega}} G(z, z')(\Delta_g \chi)(z')dz'$$

where $\chi$ is any smooth function on $\overline{\Omega}$ such that $\chi = \varphi + O(x^2)$. Now, using Green's formula and $\Delta_g(z)G(z, z') = \delta(z - z') = \Delta_g(z')G(z, z')$, where $\delta(z - z')$ is the Dirac mass on the diagonal, we obtain for $z \in \Omega$

$$u(z) = \int_{\partial\Omega} \left(\partial_{n'} G(z, z')\chi(z')\right)\big|_{z'=y'} dy' - \int_{\partial\Omega} \left(G(z, z')(\partial_n \chi)(z')\right)\big|_{z'=y'} dy'$$

$$u(z) = \int_{\partial\Omega} \left(\partial_{n'} G(z, z')\right)\big|_{z'=y'} \varphi(y')dy'.$$

We have Taylor expansion $u(x, y) = \varphi(y) + x\Lambda_g \varphi(y) + O(x^2)$ near the boundary. Let $y \in \partial\Omega$ and take $\varphi \in C^\infty(\Omega)$ supported near $y$. Thus, pairing with $\phi \in C^\infty(\partial\Omega)$ gives

$$\int_{\partial\Omega} u(x, y)\phi(y)dy = \int_{\partial\Omega} \varphi(y)\phi(y)dy - x\int_{\partial\Omega} \phi(y)\Lambda_g \varphi(y)dy + O(x^2). \tag{2.2}$$

Now taking $\phi$ with support disjoint to the support of $\varphi$, thus $\phi\varphi = 0$, and differentiating (2.2) in $x$, we see, in view of the fact that Green's function $G(z, z')$ is smooth outside the diagonal, that

$$\int_{\partial\Omega} \phi(y)\Lambda_g \varphi(y)dy = \int_{\partial\Omega}\int_{\partial\Omega} \left(\partial_n \partial_{n'} G(z, z')\right)\big|_{z=y, z'=y'} \varphi(y')\phi(y)dy\, dy',$$

which proves the claim. $\square$

Let $(\partial\Omega, g)$ be a Riemannian manifold, and let us denote by $d_g(x, y)$ the geodesic distance between $x, y \in \partial\Omega$, and we denote $[d_g(x, y)]^2 = d_g^2(x, y)$. If

$$\lim_{y \to x} d_g^2(F(x), F(y))\mathcal{N}_g(F(x), F(y)) \neq 0$$

does not depend on $x$, we have the following result:

COROLLARY 2.4. *If* $\Lambda_{F*e^{2\varphi}g} = \Lambda_g$ *then* $e^{-\varphi\circ F(x)} = F'(x)$ *for* $x \in \partial\Omega$.

PROOF. Using the equalities of the Dirichlet-to- Neumann operators and Lemma 2.2 we have

$$\frac{d_g^2(x,y)}{d_g^2(F(x),F(y))}e^{-\phi\circ F(x)}d_g^2(F(x),F(y))\mathcal{N}_g(F(x),F(y))F'(y) = d_g^2(x,y)\mathcal{N}_g(x,y) \tag{2.3}$$

On the other hand, since

$$\lim_{y\to x}\frac{d_g(x,y)}{d_g(F(x),F(y))} = \frac{1}{F'(x)},$$

then, taking the limit when $y \to x$ in (2.3), the demonstration follows. □

REMARK 2.5. From Lemma 2.2 and Corollary 2.4 we have the following equation,

$$\mathcal{N}_g(F(x),F(y))F'(x)F'(y) = \mathcal{N}_g(x,y). \tag{2.4}$$

The set of solutions of equation (2.4) is a group with multiplication law given by composition of functions, that is, if $F$ and $G$ are solutions of the equation (2.4), then, $G\circ F$ is solution of (2.4). In fact,

$$\begin{aligned}
&\mathcal{N}_g((G\circ F)(x),(G\circ F)(y))(G\circ F)'(x)(G\circ F)'(y)\\
&= \mathcal{N}_g(G(F(x)),(G(F(y)))G'(F(x))G'(F(y))F'(x)F'(y)\\
&= \mathcal{N}_g(F(x),F(y))F'(x)F'(y) = \mathcal{N}_g(x,y).
\end{aligned}$$

In what follows, we use an explicit formula for the Green's Function of $\Delta_{g_0}$ on the annulus $A_R$ (Bârza and Guisa 1998). There, $g_0$ is given in polar coordinates by:

$$g_0 = \frac{1}{2}\left(1+\frac{1}{\rho^2}\right)\left(d\rho^2+\rho^2 d\theta^2\right), \tag{2.5}$$

and it is conformal to the euclidean metric, with conformal factor $f(\rho,\theta) = \frac{1}{2}\left(1+\frac{1}{\rho^2}\right)$.

Then, the normal derivative of $u \in \mathcal{C}^\infty(\overline{A_R})$, with respect to $g_0$ on $|z| = R$, is:

$$\left.\frac{\partial u}{\partial\nu_{g_0}}\right|_{\rho=R} = \frac{2R^2}{1+R^2}\left.\frac{\partial u}{\partial\rho}\right|_{\rho=R}$$

Analogously, the normal derivative of $u$, with respect to $g_0$ on $|z| = \frac{1}{R}$, is:

$$\left.\frac{\partial u}{\partial\nu_{g_0}}\right|_{\rho=\frac{1}{R}} = \frac{-2}{1+R^2}\left.\frac{\partial u}{\partial\rho}\right|_{\rho=\frac{1}{R}}$$

The Green's function of $\overline{A}_R$ is given by

$$\begin{aligned}
G(z,\zeta) \;=\;& \ln(rR) + \sum_{n=1}^{\infty}\frac{1}{n}\frac{r^n+(-r)^{-n}}{R^n}\frac{\rho^n+(-\rho)^{-n}}{R^n+(-R)^{-n}}\cos n(\theta-\alpha)\\
&- \ln|\rho e^{i\theta}-re^{i\alpha}| - \ln\left|\frac{1}{\rho}e^{i(\theta+\pi)}-re^{i\alpha}\right|,
\end{aligned}$$

where $z = \rho e^{i\theta}$, $\frac{1}{R} \leq \rho \leq R$, $\zeta = re^{i\alpha}$, $\frac{1}{R} < r < R$, $0 < \theta < 2\pi$, $0 < \alpha < 2\pi$.

LEMA 2.6. *The Schwartz kernel of $\Lambda_{g_0}$, $g_0 \in \mathcal{M}(\overline{A_R})$ being of the form* (2.5)*, is*

$$\begin{aligned} \mathcal{N}_{g_0}(Re^{i\theta}, Re^{i\alpha}) &= \frac{4R^2}{\left(1+R^2\right)^2} \sum_{n=1}^{\infty} n \frac{R^n - (-R)^{-n}}{R^n} \frac{R^n - (-R)^{-n}}{R^n + (-R)^{-n}} \cos n(\theta - \alpha) \\ &+ \frac{2R^2}{\left(1+R^2\right)^2} \frac{1}{1 - \cos(\theta-\alpha)} - \frac{4R^4}{\left(1+R^2\right)^2} \frac{R^{-4}\cos(\theta-\alpha) + \cos(\theta-\alpha) + 2R^{-2}}{\left(R^{-2} + 2\cos(\theta-\alpha) + R^2\right)^2} \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathcal{N}_{g_0}\left(R^{-1}e^{i\theta}, R^{-1}e^{i\alpha}\right) &= \frac{4R^2}{\left(1+R^2\right)^2} \sum_{n=1}^{\infty} n \frac{\left(\frac{1}{R}\right)^n - \left(-\frac{1}{R}\right)^{-n}}{R^n} \frac{\left(\frac{1}{R}\right)^n - \left(-\frac{1}{R}\right)^{-n}}{R^n + (-R)^{-n}} \cos n(\theta - \alpha) \\ &+ \frac{2R^2}{\left(1+R^2\right)^2} \frac{1}{1 - \cos(\theta-\alpha)} - \frac{4}{\left(1+R^2\right)^2} \cdot \frac{R^{4}\cos(\theta-\alpha) + \cos(\theta-\alpha) + 2R^{2}}{\left(R^{2} + 2\cos(\theta-\alpha) + R^{-2}\right)^2} \end{aligned} \tag{2.7}$$

The equality above is in the distributions sense.

PROPOSITION 2.7. *Let $p, q \in \partial A_R$, then,*

$$\lim_{p \to q} d^2_{\overline{g}_{eucl}}(p,q)\mathcal{N}_{g_0}(p,q) = \frac{-4R^4}{\left(1+R^2\right)^2} \qquad on \quad |z| = R \tag{2.8}$$

*and*

$$\lim_{p \to q} d^2_{\overline{g}_{eucl}}(p,q)\mathcal{N}_{g_0}(p,q) = \frac{-4}{\left(1+R^2\right)^2} \qquad on \quad |z| = \frac{1}{R}, \tag{2.9}$$

*where $d_{\overline{g}_{eucl}}$ denotes the geodesic distance between $p$ and $q$ with respect to the Euclidean metric in $\partial A_R$.*

PROOF. In order to prove equation (2.8), we write

$$a_n = \frac{\left(R^n - (-R)^{-n}\right)^2}{R^n\left(R^n + (-R)^{-n}\right)}.$$

Then, the sequence $b_n = n(a_n - 1)$ has the following property: $|b_n| < \frac{4n}{R^n} < C(k,R)n^{-k}$ for all $n \in \mathbb{N}$ where $C(k,R)$ is a constant that depends only on $k$ and $R$. In fact, $\frac{n^{k+1}}{R^n} < \frac{(k+1)!}{(\ln R)^{k+1}}$. Hence, the series $\sum_{n=1}^{\infty} b_n \cos n(\theta - \alpha)$ represents a $\mathcal{C}^\infty$ function. On the other hand, using the Fourier series of the function $f(x) = \ln(|\sin(\frac{x}{2})|)$, with $0 < x < \pi$, we have that

$$\ln\left(\left|\sin\left(\frac{x}{2}\right)\right|\right) = -\left\{\ln 2 + \sum_{n=1}^{\infty} \frac{\cos nx}{n}\right\},$$

that is,

$$\frac{1}{2}\ln\left(\frac{1-\cos x}{2}\right) = -\left\{\ln 2 + \sum_{n=1}^{\infty} \frac{\cos nx}{n}\right\},$$

which implies:

$$\sum_{n=1}^{\infty} n \cos nx = -\frac{1}{1-\cos x},$$

the equality being in the distributions sense.

Then, multiplying (2.6) by $d^2_{g_{eucl}}(p, q)$ and taking the limit as $q \to p$, we get the following:

$$\lim_{\theta\to\alpha} \frac{-2R^2}{\left(1+R^2\right)^2} \cdot \frac{R^2(\theta-\alpha)^2}{1-\cos(\theta-\alpha)} = \frac{-4R^4}{\left(1+R^2\right)^2}.$$

Analogously, we get (2.9). □

REMARK 2.8. It follows from the proof of the Proposition (2.6) that the Schwartz kernel of $\Lambda_{g_0}$ can be written as:

$$\mathcal{N}_{g_0}\big(Re^{i\theta}, Re^{i\alpha}\big) = H\big(Re^{i\theta}, Re^{i\alpha}\big) - \frac{2R^2}{\left(1+R^2\right)^2} \cdot \frac{1}{1-\cos(\theta-\alpha)} \qquad \text{on} \quad |z| = R$$

$$\mathcal{N}_{g_0}\big(R^{-1}e^{i\theta}, R^{-1}e^{i\alpha}\big) = H\big(R^{-1}e^{i\theta}, R^{-1}e^{i\alpha}\big) - \frac{2R^2}{\left(1+R^2\right)^2} \cdot \frac{1}{1-\cos(\theta-\alpha)} \qquad \text{on} \quad |z| = \frac{1}{R},$$

where $H$ is a $\mathcal{C}^\infty$ function given by

$$H\big(Re^{i\theta}, Re^{i\alpha}\big) = \frac{4R^2}{\left(1+R^2\right)^2}\left\{\sum_{n=1}^{\infty} b_n \cos n(\theta-\alpha) - \frac{R^{-2}\cos(\theta-\alpha) + R^2\cos(\theta-\alpha)+2}{\left(R^{-2}+2\cos(\theta-\alpha)+R^2\right)^2}\right\}. \quad (2.10)$$

TEOREMA 2.9. *Let $g_0$ be a metric as in* (2.5)*, $h = F^*(e^{2\varphi}g_0)$ where $F \in \mathcal{D}(\overline{A_R})$, $\varphi \in \mathcal{C}^\infty(\overline{A_R})$ and $F(Re^{i\theta}) = Re^{if(\theta)}$. If $\Lambda_h = \Lambda_{g_0}$, then,*

$$\begin{cases} S(f) = \lambda\left(\left[f'\right]^2 - 1\right) \\ e^{-\varphi\circ F} = f' \end{cases} \qquad (2.11)$$

*where $S(f)$ denotes the Schwarzian Derivative of $f$ (see* (2.18) *and the line right after it).*

PROOF. Using the equality of the Dirichlet-to-Neumann operators, it follows from Lemma 2.2 that

$$\mathcal{N}_{F^*e^{2\varphi}g_0}(x, y) = e^{-\varphi\circ F(x)}\mathcal{N}_{g_0}(F(x), F(y))F'(y) = \mathcal{N}_{g_0}(x, y).$$

Writing $x = Re^{i\theta}$, $y = Re^{i\alpha}$ and using (2.8), we have that

$$e^{-\varphi\circ F\left(Re^{i\theta}\right)}\left\{H\big(Re^{if(\theta)}, Re^{if(\alpha)}\big) - \frac{2R^2}{\left(1+R^2\right)^2}\frac{1}{1-\cos(f(\theta)-f(\alpha))}\right\} f'(\alpha)$$

$$= H\big(Re^{i\theta}, Re^{i\alpha}\big) - \frac{2R^2}{\left(1+R^2\right)^2}\frac{1}{1-\cos(\theta-\alpha)}.$$

On the other hand, we have from Corollary 2.4 and (2.8) that $e^{-\varphi\circ F} = f'$ on the boundary. Hence,

$$\left\{H\big(Re^{if(\theta)}, Re^{if(\alpha)}\big) - \frac{2R^2}{\left(1+R^2\right)^2}\frac{1}{1-\cos(f(\theta)-f(\alpha))}\right\} f'(\alpha)f'(\theta)$$

$$= H\big(Re^{i\theta}, Re^{i\alpha}\big) - \frac{2R^2}{\left(1+R^2\right)^2}\frac{1}{1-\cos(\theta-\alpha)}.$$

We obtain, then,

$$H\big(Re^{if(\theta)}, Re^{if(\alpha)}\big)f'(\alpha)f'(\theta) - H\big(Re^{i\theta}, Re^{i\alpha}\big) = \frac{2R^2}{\left(1+R^2\right)^2}\left\{\frac{f'(\alpha)f'(\theta)}{1-\cos(f(\theta)-f(\alpha))} - \frac{1}{1-\cos(\theta-\alpha)}\right\}. \tag{2.12}$$

Since the left hand side of the equation (2.12) is the $C^\infty$ component of the Schwartz kernel, then if we take $\alpha \to \theta$, we get

$$H(R,R)\left\{\left[f'(\theta)\right]^2 - 1\right\}.$$

In what concerns the right hand side of the equation (2.12), we use Taylor expansion of order 4 of the expression in brackets, for $\alpha$ near $\theta$; we get, with $\delta = \alpha - \theta$,

$$\frac{f'(\theta)\left\{f'(\theta) + f''(\theta)\delta + f'''(\theta)\frac{\delta^2}{2!}\right\}}{\left[f'(\theta)\right]^2\frac{\delta^2}{2!} + 3f'(\theta)f''(\theta)\frac{\delta^3}{3!} + \left\{-\left[f'(\theta)\right]^4 + 3\left[f''(\theta)\right]^2 + 4f'(\theta)f'''(\theta)\right\}\frac{\delta^4}{4!}} - \frac{1}{\frac{\delta^2}{2!} - \frac{\delta^4}{4!}},$$

which can be written as follows,

$$\frac{\left\{-\left[f'(\theta)\right]^2 - 3\left[f''(\theta)\right]^2 + 2f'(\theta)f'''(\theta) + \left[f'(\theta)\right]^4\right\}\frac{\delta^4}{4!} + \mathcal{O}\big(\delta^5\big)}{\left[f'(\theta)\right]^2\frac{\delta^4}{4} + \mathcal{O}\big(\delta^5\big)}. \tag{2.13}$$

Since the limit exists, when $\delta \to 0$, we obtain from (2.12) and (2.13) that

$$3!\frac{\left(1+R^2\right)^2}{2R^2}H(R,R)\left\{\left[f'(\theta)\right]^2 - 1\right\} = \left[f'(\theta)\right]^2 - 1 - 3\left[\frac{f''(\theta)}{f'(\theta)}\right]^2 + 2\frac{f'''(\theta)}{f'(\theta)},$$

which implies:

$$\left\{\left[f'(\theta)\right]^2 - 1\right\}\left\{3!\frac{\left(1+R^2\right)^2}{2R^2}H(R,R) - 1\right\} = -3\left[\frac{f''(\theta)}{f'(\theta)}\right]^2 + 2\frac{f'''(\theta)}{f'(\theta)},$$

or, more precisely:

$$\left\{\left[f'(\theta)\right]^2 - 1\right\}\left\{3!\frac{\left(1+R^2\right)^2}{2R^2}H(R,R) - 1\right\} = 2\left[\frac{f''(\theta)}{f'(\theta)}\right]' - \left[\frac{f''(\theta)}{f'(\theta)}\right]^2. \tag{2.14}$$

Let us denote $\lambda = \lambda(R)$, the expression

$$2\lambda(R) = 3!\frac{\left(1+R^2\right)^2}{2R^2}H(R,R) - 1. \tag{2.15}$$

Then, it, follows that

$$2\lambda(R) = \frac{4!}{2}\left\{\sum_{n=1}^{\infty} b_n - \frac{R^2}{\left(1+R^2\right)^2}\right\} - 1, \tag{2.16}$$

where

$$b_n = n\left\{\frac{-3(-1)^n + \frac{1}{R^{2n}}}{R^{2n} + (-1)^n}\right\}.$$

From equations (2.14) and (2.15) we have that

$$\left\{\left[f'(\theta)\right]^2 - 1\right\} 2\lambda(R) = 2\left[\frac{f''(\theta)}{f'(\theta)}\right]' - \left[\frac{f''(\theta)}{f'(\theta)}\right]^2, \tag{2.17}$$

which implies:

$$\left\{\left[f'(\theta)\right]^2 - 1\right\} \lambda(R) = \left[\frac{f''(\theta)}{f'(\theta)}\right]' - \frac{1}{2}\left[\frac{f''(\theta)}{f'(\theta)}\right]^2 = \frac{f'''(\theta)}{f'(\theta)} - \frac{3}{2}\left[\frac{f''(\theta)}{f'(\theta)}\right]^2. \tag{2.18}$$

The right-hand side of (2.18) is called the Schwarzian Derivative $S(f)$ of $f$ (Navas 2007). □

REMARK 2.10. The numerical study of $\lambda(R)$ defined in (2.16) is done in Mendoza et al. 2009.

COROLLARY 2.11. *The solution of the equation* (2.18) *for* $\lambda(R) \geq 0$ *is* $f'(\theta) = 1$.

PROOF. Making the change of variables: $y(\theta) = \ln(f'(\theta))$, the equation (2.18) becomes

$$\left\{e^{2y(\theta)} - 1\right\}\lambda(R) = y''(\theta) - \frac{1}{2}\left[y'(\theta)\right]^2, \tag{2.19}$$

that is,

$$y'' = \frac{1}{2}\left[y'\right]^2 + \lambda\left\{e^{2y(\theta)} - 1\right\}. \tag{2.20}$$

Since $f(\theta + 2\pi) = f(\theta) + 2\pi$, we have that $f''$ and $f'$ are periodic of period $2\pi$. Then, integrating (2.20) between 0 and $2\pi$ we obtain

$$0 = \frac{1}{2}\int_0^{2\pi} \left[y'\right]^2 d\theta + \lambda\left\{\int_0^{2\pi} e^{2y} d\theta - 2\pi\right\}. \tag{2.21}$$

On the other hand,

$$0 \leq \int_0^{2\pi} 1 \cdot f' d\theta \leq \left(\int_0^{2\pi} 1 d\theta\right)^{\frac{1}{2}} \cdot \left(\int_0^{2\pi} [f']^2 d\theta\right)^{\frac{1}{2}}$$

that is,

$$2\pi \leq \int_0^{2\pi} \left[f'\right]^2 d\theta = \int_0^{2\pi} e^{2y} d\theta,$$

which implies that $y' = 0$. Because there is $0 \leq \theta_0 \leq 2\pi$ such that $f'(\theta_0) = 1$, we get $y = 0$. Therefore, $f' = 1$. □

It follows that $F$ restricted to the exterior boundary is a rotation and $\varphi$ equals zero there. The same conclusion holds for the restriction of $F$ to the interior boundary.

The general solution of the equations (2.18) can be obtained using the formulas of Chuaqui et al. 2003, page 1.

**ACKNOWLEDGMENTS**

We thank Gustavo Tamm and Henrique Araújo for helping us to solve and to understand the non-linear ordinary differential equation (2.18).

**RESUMO**

Neste artigo mostramos a relação entre os núcleos de Schwartz dos operadores Dirichlet-to-Neumann associados à métrica $g_0$ e $h = F^*(e^{2\phi}g_0)$, no anel circular $A_R$, e a Derivada Schwarziana da função argumento $f$, da restrição do difeomorfismo $F$ à fronteira de $A_R$.

**Palavras-chave:** anel, Funcional Dirichlet-Neumann, Derivada Schwarziana.

**REFERENCES**

BÂRZA I AND GUISA D. 1998. Explicit Formulas for Green's Functions on the Annulus and on the Möbius Strip. Acta Appl Math 54: 289–302.

CALDERÓN AP. 1980. On an inverse boundary value problem. Seminar on Numerical Analysis and its applications to continuum physics. Soc Bras Mat, Rio de Janeiro, RJ, Brasil, p. 65–73.

CHUAQUI M, DUREN P AND OSGOOD B. 2003. The Schwarzian derivative for harmonic mappings. J d'Analyse Math 91(1): 329–351.

GÓMEZ P AND MENDOZA R. 2006. Uniqueness for the Dirichlet-Neumann elliptic functional. Inverse Problem 22: 1575–1578.

GUILLARMOU C AND SÁ BARRETO A. 2009. Inverse Problems for Einstein Manifolds. Inverse probl Imag 3: 1–15.

LASSAS M AND UHLMANN G. 2001. On Determining a Riemannian manifold from the Dirichlet-to-Neumann map. Ann Sci École Norm Sup 34(5): 771–787.

LASSAS M, TAYLOR M AND UHLMANN G. 2003. The Dirichlet-Neumann map for complete Riemannian manifold with boundary. Comm Anal Geom 11: 207–221.

LEE J AND UHLMANN G. 1989. Determining anisotropic real analytic condutivities by boundary measurements. Comm Pure Appl Math XLII: 1097–1112.

MENDOZA R, ROJAS J, CAMPOS M AND BARRETO R. 2009. Injetividade do Funcional de Dirichlet-Neumann Elíptico via Matemática Intervalar. Temas, SBMAC 10: 31–40.

NAVAS A. 2007. Grupos de difeomorfismos del círculo. Ensaios Matemáticos 13, SBM.

POLYAKOV AM. 1987. Gauge Fields and Strings. Harwood Academic Publishers, Chur.