Bianchi Type I Tilted Cosmological Model for Barotropic Perfect Fluid Distribution with Heat Conduction in General Relativity

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Bianchi Type I tilted cosmological model for barotropic perfect fluid distribution with heat conduction is investigated.To get the deterministic solution, we have assumed barotropic condition p = $\gamma\in$,$0 \le \gamma \le 1$, p being isotropic pressure, $\in$ the matter density and a supplementary condition between metric potentials A, B, C as A = (BC)n where n is the constant. To get the model in terms of cosmic time, we have also discussed some special cases. The physical aspects of the model are also discussed.

Keywords: Tilted, barotropic perfect fluid, heat conduction, Bianchi I.

1. INTRODUCTION

Homogeneous and anisotropic cosmological models have been widely studied in the frame work of General Relativity in the search of realistic picture of the universe in the early stages of the evolution of universe. These models are of two types: (i) orthogonal models in which matter moves orthogonally to the hyper-surface of homogeneity (ii) the tilted models in which the fluid flow vector is not normal to the hyper-surface of homogeneity. The tilted models are more complicated than those of non-tilted one. The general dynamics of tilted cosmological models have been studied by King and Ellis [1], Ellis and King [2], Collins and Ellis [3]. Bradley and Sviestins [4] have investigated that heat flow is expected for tilted cosmological model. Mukherjee [5] has investigated tilted Bianchi Type I cosmological model with heat flux in General Relativity. He has shown that the universe assumes a Pan cake shape. The velocity vector is not geodesic and heat flux is comparable to the energy density. The cosmological models with heat flow have also been studied by number of researchers like Novello and Reboucas [6], Ray [7], Roy and Banerjee [8], Coley and Tupper [9], Deng [10]. Mukherjee [11], Banerjee and Santos [12], Coley [13], Roy and Prasad [14]. Bali and Sharma [15] have investigated tilted Bianchi Type I dust fluid cosmological model for perfect fluid distribution using the special condition $A = B^n$ between metric potential where n is the constant. Bali and Meena [16] have investigated Bianchi Type I tilted cosmological model for disordered radiation of perfect fluid using the supplementary condition $A = (BC)n$ between metric potentials, n being a constant.

In this paper, we have investigated Bianchi type I tilted cosmological model for barotropic perfect fluid distribution (p = $\gamma \in$) using the special condition $A = (BC)^n$ between metric potentials, n being a constant where p is the isotropic pressure, $\in$ the matter density, $0 \le \gamma \le 1$.

For complete solutions of equations (6) – (10),we need two extra conditions. An obvious one is equation of state p = $\gamma \in$ ($0 \le \gamma \le 1$) given by (11), is general condition for barotropic equation of state, p being isotropic pressure and $\in$ the matter density. This includes radiation for $\gamma = \frac{1}{3}$, dust filled universe p = 0 (Friedmann model) for γ = 0, stiff fluid universe $\in$ = p (Zel'dovich fluid) for γ = 1. These are physically valid conditions for the description of the universe.

The second condition $A = (BC)^n$ given by (12) is obtained by assuming $\sigma_1^1 \alpha \theta$ for non-tilt model i.e. for $\lambda = 0$ where σ_1^1 is the eigen value of shear tensor σ_i^j and θ the expansion in the model where

$$\sigma_1^1 = \left(\frac{2}{3}\frac{A_4}{A} - \frac{1}{3}\frac{B_4}{B} - \frac{1}{3}\frac{C_4}{C}\right)$$

and

$$\theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}$$

The motivation for assuming this condition is explained as : Referring to the Thorne [17], the observations of the velocity – redshift relation for extra galactic sources suggest that the Hubble expansion of the universe is isotropic today to within 30% [18,19]. More precisely, the redshift studies place the link $\frac{\sigma}{H} \le 0.30$ where σ is the shear and H is a Hubble constant. Collins et al. [20] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous hyper surface satisfies the condition $\frac{\sigma}{\theta}$ = constant. The condition $\frac{\sigma_1^1}{\theta}$ =constant for the metric (1) leads to $A = (BC)^n$
where n is the constant.

Some special cases for different values of n and γ are discussed. The physical aspects of the model and singularities in the model are also discussed.

2. METRIC AND FIELD EQUATIONS

We consider the Bianchi Type I metric in the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \ldots \quad (1)$$

where A, B, C are functions of t alone.
The energy momentum tensor for perfect fluid distribution with heat conduction is taken into form given by Ellis [21] as

$$T_i^j = (\varepsilon + p)\, v_i v^j + p g_i^j + q_i v^j + v_i q^j \ldots \quad (2)$$

together with

$$g_{ij} v^i v^j = -1 \ldots \quad (3)$$

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$$q_i q^i > 0 \ldots \quad (4)$$

$$q_i v^i = 0 \ldots \quad (5)$$

where p is the isotropic pressure, ε the matter density and qi the heat conduction vector orthogonal to vi. The fluid flow vector vi has the components

$$\left(\frac{\sinh\lambda}{A}, 0, 0, \cosh\lambda\right)$$

satisfying (3), λ being the tilt angle.
The Einstein's field equation $R_i^j - \frac{R}{2} g_i^j = -8\pi T_i^j$, (In the generalized unit where $c = 1$, $G = 1$ and taking $\Lambda = 0$) for the line-element (1) leads to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = -8\pi\left[(\varepsilon + p)\sinh^2\lambda + p + 2q_1 \frac{\sinh\lambda}{A}\right] \ldots \quad (6)$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} = -8\pi p \ldots \quad (7)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -8\pi p \ldots \quad (8)$$

$$\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} + \frac{B_4 C_4}{BC} = -8\pi\left[-(\varepsilon + p)\cosh^2\lambda + p - 2q_1 \frac{\sinh\lambda}{A}\right] \ldots \quad (9)$$

$$(\varepsilon + p)\, A \sinh\lambda \cosh\lambda + q_1 \cosh\lambda + q_1 \frac{\sinh^2\lambda}{\cosh\lambda} = 0 \ldots \quad (10)$$

where the subscript '4' denotes the ordinary differentiation with respect to 't'.

3. SOLUTION OF FIELD EQUATIONS

Equations from (6) to (10) are five equations in seven unknowns, A, B, C, ε, p, λ and q1. For the complete determination of these quantities, we assume that the model is filled with barotropic perfect fluid which leads to

$$p = \gamma\varepsilon \ldots \quad (11)$$

where $0 \leq \gamma \leq 1$
To get the deterministic solution, we also assume a supplementary condition between metric potentials A, B and C as

$$A = (BC)^n \ldots \quad (12)$$

where n is the constant.

For complete solutions of equations (6) – (10),we need two extra conditions. An obvious one is equation of state p = γ ∈ $(0 \leq \gamma \leq 1)$ given by (11), is general condition for barotropic equation of state, p being isotropic pressure and ∈ the matter density. This includes radiation for $\gamma = \frac{1}{3}$, dust filled universe p = 0 (Friedmann model) for γ = 0, stiff fluid universe ∈ = p (Zel'dovich fluid) for γ = 1. These are physically valid conditions for the description of the universe.

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where n is the constant.
Equations (6) and (9) lead to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{2B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = 8\pi(\varepsilon - p) \ldots \quad (13)$$

Using the barotropoic condition p = γε given by (11) in (13),we have

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{2B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = -8\pi p\left(1 - \frac{1}{\gamma}\right) \ldots \quad (14)$$

Using (8) in (14), we have

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{2B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = \left(\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB}\right)\left(1 - \frac{1}{\gamma}\right) \ldots \quad (15)$$

Equations (7) and (8) lead to

$$\left(\frac{B_{44}}{B} - \frac{C_{44}}{C}\right) + \frac{A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right) = 0 \ldots \quad (16)$$

Equation (12) leads to

$$\frac{A_4}{A} = n\left(\frac{B_4}{B} + \frac{C_4}{C}\right) \ldots \quad (17)$$

Thus equation (16) becomes

$$\frac{(C B_{44} - B C_{44})}{(CB_4 - BC_4)} + n\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = 0 \ldots \quad (18)$$

which on integration leads to

$$C^2\left(\frac{B}{C}\right)_4 = b\,(BC)^{-n}\ldots \quad (19)$$

where b is constant of integration.

Let

$$BC = \mu \text{ and } \frac{B}{C} = \nu\ldots \quad (20)$$

Using (20) in (19), we have

$$\frac{\nu_4}{\nu} = b\,\bar{\mu}^{(n+1)}\ldots \quad (21)$$

Using the assumptions (20) and (12) in (15), we have

$$\frac{\mu_{44}}{\mu} + n\frac{\mu_4^2}{\mu^2} = \left[\left(n+\frac{1}{2}\right)\frac{\mu_{44}}{\mu} + \frac{1}{2}\frac{\nu_{44}}{\nu} + \left(n^2-\frac{n}{2}-\frac{1}{4}\right)\frac{\mu_4^2}{\mu^2}\right.$$

$$\left. -\frac{1}{4}\frac{\nu_4^2}{\nu^2} + \left(\frac{n+1}{2}\right)\frac{\mu_4\nu_4}{\mu\nu}\right]\left(1-\frac{1}{\gamma}\right)\ldots \quad (22)$$

Equations (21) and (22) leads to

$$\frac{\mu_{44}}{\mu} + a\frac{\mu_4^2}{\mu^2} = \ell\mu^{-2(n+1)}\ldots \quad (23)$$

where

$$a = \frac{\left[n - \left(n^2-\frac{n}{2}-\frac{1}{4}\right)\left(1-\frac{1}{\gamma}\right)\right]}{\left[1-\left(n+\frac{1}{2}\right)\left(1-\frac{1}{\gamma}\right)\right]}\ldots \quad (24)$$

$$\ell = \frac{\frac{b^2}{4}\left(1-\frac{1}{\gamma}\right)}{\left[1-\left(n+\frac{1}{2}\right)\left(1-\frac{1}{\gamma}\right)\right]}\ldots \quad (25)$$

Let us assume that

$$\mu_4 = f(\mu)\ldots \quad (26)$$

Thus

$$\mu_{44} = \frac{d\mu_4}{dt} = f f'\ldots \quad (27)$$

Therefore equation (23) leads to

$$\frac{df^2}{d\mu} + \frac{2a}{\mu}f^2 = 2\ell\mu^{-2n-1}\ldots \quad (28)$$

Equation (28) leads to

$$f^2 = \frac{\ell}{(a-n)}\mu^{-2n} + L\mu^{-2a}\ldots \quad (29)$$

where L is constant of integration and

$$(a-n) = \frac{\left(1-\frac{1}{\gamma}\right)\left(\frac{4n+1}{4}\right)}{\left[1-\left(1-\frac{1}{\gamma}\right)\left(n+\frac{1}{2}\right)\right]}\ldots \quad (30)$$

$$\frac{\ell}{(a-n)} = \frac{b^2}{(4n+1)}\ldots \quad (31)$$

Using (31) in (29), we have

$$f^2 = \frac{b^2}{(4n+1)}\mu^{-2n} + L\mu^{-2a}\ldots \quad (32)$$

Equation (21) leads to

$$\frac{d\nu}{\nu} = \frac{b}{\mu^{(n+1)}}\frac{dt}{d\mu}d\mu$$

Thus, we have

$$\log\nu = \int\frac{b}{\mu^{(n+1)}}\frac{d\mu}{\sqrt{\frac{b^2}{(4n+1)}\mu^{-2n} + L\mu^{-2a}}}\ldots \quad (33)$$

Hence the metric (1) reduces to the form

$$ds^2 = -\left(\frac{dt}{d\mu}\right)^2 d\mu^2 + \mu^{2n}dx^2 + \mu(\nu dy^2 + \nu^{-1}dz^2)\ldots \quad (34)$$

which leads to

$$ds^2 = -\frac{dT^2}{\left\{\frac{b^2}{(4n+1)}T^{-2n} + L\,T^{-2a}\right\}} + T^{2n}dX^2 + T(\nu dY^2 + \nu^{-1}dZ^2)\ldots \quad (35)$$

where ν is determined by (33) and $\mu = T$.

4. SOME PHYSICAL AND GEOMETRICAL FEATURES

The isotropic pressure (p), the matter density (ε), the expansion (θ), cosh λ, v1, v4, q1, q4, σ11, σ14 are given by

$$8\pi p = (4an + 2a - 4n^2 + 2n + 1)\frac{L}{4T^{2a+2}}\ldots \quad (36)$$

$$8\pi\varepsilon = \frac{1}{\gamma}[4an + 2a - 4n^2 + 2n + 1]\frac{L}{4T^{2a+2}}\ldots \quad (37)$$

$$\cosh\lambda = \frac{\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2\,(2a+2n+1)\right]^{1/2}}{2\,(2a+2n+1)^{1/2}} \quad \ldots \tag{38}$$

$$\theta = \frac{\partial}{\partial t}\cosh\lambda + \cosh\lambda\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)$$

which leads to

$$\theta = \frac{(n+1)\left\{\frac{b^2}{(4n+1)}\frac{1}{T^{2n+2}}+\frac{L}{T^{2a+2}}\right\}^{1/2}\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2\,(2a+2n+1)\right]^{1/2}}{2(2a+2n+1)^{1/2}} \quad \ldots \tag{39}$$

$$v^1 = \frac{\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)-2(2a+2n+1)\right]^{1/2}}{2(2a+2n+1)^{1/2}\,T^n} \quad \ldots \tag{40}$$

$$v^4 = \frac{\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2(2a+2n+1)\right]^{1/2}}{2(2a+2n+1)^{1/2}} \quad \ldots \tag{41}$$

$$\sigma_{11} = \frac{(2n-1)\,T^{2n}\left\{\frac{b^2}{(4n+1)T^{2n+2}}+\frac{L}{T^{2a+2}}\right\}^{1/2}\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2(2a+2n+1)\right]^{1/2}\left[2(2a+2n+1)+(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)\right]}{24\,(2a+2n+1)^{3/2}} \quad \ldots \tag{42}$$

$$\sigma_{14} = \frac{-(2n-1)\,T^{n}\left\{\frac{b^2}{(4n+1)T^{2n+2}}+\frac{L}{T^{2a+2}}\right\}^{1/2}\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)-2(2a+2n+1)\right]^{1/2}\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2(2a+2n+1)\right]}{24\,(2a+2n+1)^{3/2}} \quad \ldots \tag{43}$$

Now

$$\sigma_{11}v^1+\sigma_{14}v^4 = \frac{(2n-1)T^n\left\{\frac{b^2}{(4n+1)T^{2n+2}}+\frac{L}{T^{2a+2}}\right\}^{1/2}\left[\left\{(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)\right\}^2-4\,(2a+2n+1)^2\right]^{1/2}\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2(2a+2n+1)\right](1-1)}{48\,(2a+2n+1)^2} = 0$$

$$\ldots \quad (44)$$

Similarly

$$\omega_{11}v^1 + \omega_{14}v^4 = 0 \ldots \quad (45)$$

$$q^1 = \frac{-L\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)-2(2a+2n+1)\right]^{1/2}\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2(2a+2n+1)\right]}{128\pi(2a+2n+1)^{1/2}T^{2a+2+n}} \ldots \quad (46)$$

$$q^4 = \frac{-L\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)-2(2a+2n+1)\right]\left[(4an+2a-4n^2+2n+1)\left(\frac{1}{\gamma}+1\right)+2(2a+2n+1)\right]^{1/2}}{128\pi(2a+2n+1)^{1/2}T^{2a+2}} \ldots \quad (47)$$

5. DISCUSSION

The reality conditions
(i) $\rho + p > 0$ (ii) $\rho + 3p > 0$
given by Ellis [22], lead to

$$4an + 2a > 4n^2 - 2n - 1 \ldots \quad (48)$$

The matter density $\in \to \infty$ when $T \to 0$ and $\in \to 0$ when $T \to \infty$. The model (35) starts with a big-bang at T = 0 and the expansion in the model decreases as time increases. $v1 \to 0$ at T= 0 when $n < 0$. q1 = 0, q4 = 0 when L = 0. $\cosh\lambda > 1$ implies that

$$(4an+2a-4n^2+2n+1) > \frac{2\gamma}{1+\gamma}(2a+2n+1) \ldots \quad (49)$$

$$\sigma_{ij}\, v^j = 0 \text{ and } w_{ij}\, v^j = 0 \ldots \quad (50)$$

are satisfied as shown in (44) and (45) where σij and wij are shear tensor and vorticity tensor respectively. Since $\lim_{T\to\infty} \frac{\sigma}{\theta} \neq 0$. Hence the model does not approach isotropy for large values of T. There is a Point Type singularity in the model (35) at T = 0 (MacCallum [23]). The spatial volume $R^3 = \sqrt{-g} = ABC = T^{n+1}$. Thus spatial (R3) increases as time T increases where n+1 > 0.

Special Cases

We have also investigated the following cases:

(i) $n = 1, \gamma = \frac{1}{2}, a = \frac{1}{2}$ lead to barotropic perfect fluid non-tilted cosmological model as in this case $\cosh\lambda = \frac{7}{8}$ which is not defined as $\cosh\lambda > 1$ for tilted model.

(ii) $n = \frac{1}{2}, a = 0$ leads to $\gamma = \frac{1}{3}$ (disordered radiation condition) and $\cosh\lambda = 1$.

(iii) $n = -\frac{1}{2}, a = -\frac{1}{2}$ leads to stiff fluid case $\gamma = 1$ and $\cosh\lambda = 1$.

(iv) $n = \frac{1}{2}, a = -\frac{1}{4}$ leads to $\gamma = 0$ (dust distribution) but $\cosh\lambda$ is not defined.

Thus in all the above mentioned cases, no tilted cosmological models are possible because for tilted model $\cosh\lambda > 1$.

[1] A.R. King and G.F.R. Ellis, Comm. Math. Phys., 31, 209 (1973).
[2] G.F.R. Ellis and A.R. King, Comm. Math. Phys., 38, 119 (1974).
[3] C.B. Collins and G.F.R. Ellis, Phys. Rep. 56, 65 (1979).
[4] J.M. Bradley and E. Sviestins, Gen. Relativ. Grav. 16, 1119 (1984).
[5] G. Mukherjee, J. Astrophys. Astron. 4, 295 (1983).
[6] M. Novello and M.J. Reboucas, Astrophys. J. 225, 719 (1978).
[7] D.Ray, J. Math. Phys. 21, 2797 (1980).
[8] S.R. Roy and S.K. Banerjee, Astrophys & Space-Science, 150, 213 (1988).
[9] A.A. Coley and B.O.J. Tupper, Phys. Lett. A 95, 357 (1983).
[10] Y. Deng, Gen. Relativ. Grav. 21, 503 (1989).
[11] G. Mukherjee, J. Astrophys. Astron. 7, 259 (1986).
[12] A. Banerjee and N.O. Santos, Gen. Relativ. Grav. 18, 1251 (1986).
[13] A.A. Coley, Gen. Relativ. Grav. 22, 3 (1990).
[14] S.R. Roy and A Prasad, Gen. Relativ. Grav. 26, 10, 939 (1994).
[15] R. Bali and K. Sharma, Pramana – J. Phys., 58, 457 (2002).
[16] R. Bali and B L Meena, Astrophys & Space-Science, 281, 565 (2002).
[17] K.S. Thorne, Astrophys & Space-Science, 148, 51 (1967).
[18] R. Kantowski and R.K. Sachs, J. Math. Phys. 7, 433 (1966).
[19] J. Kristian and R.K. Sachs, Astrophys. J. 143, 379 (1966)
[20] C.B. Collins, E.N. Glass and D.A. Wilkinson, Gen. Relativ.

Gravit., 12, 805 (1980).
[21] G.F.R. Ellis, General Relativity and Cosmology, ed R K Sachs, Academic Press, New York, p.116 (1971).
[22] G.F.R.Ellis, General Relativity and Cosmology ed R.K.Sachs, Academic Press, New York, p. 117 (1971).
[23] M.A.H.MacCallum, Comm. Math. Phys. 20, 57 (1971).