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## **When I have to teach what I have learned: the case of the concept of solution of a system of linear equations**

### **Cuando tengo que enseñar lo que aprendí: el caso del concepto de solución de un sistema de ecuaciones lineales**

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#### **Abstract**

This work presents the case of a mathematics student teacher who reveals, through the writing of an imaginary lesson in script form, her understanding of the concept of solution of a system of linear equations. The lesson is analyzed from the perspective of Anna Sierpinska's modes of thinking in linear algebra, and provides evidence of the difficulties of the prospective teacher regarding the mathematical knowledge that she seeks to teach. We reflect on the impact that a teacher who does not fully understand the topic being taught could have on mathematics teaching and this reflection is put into dialogue with research results that have not yet been taken into account in mathematics education in the secondary level.

**Keywords:** System of linear equations. Mathematical Knowledge for Teaching. Modes of thinking. Narrative.

#### **Resumen**

En este trabajo se presenta el caso de una estudiante de profesorado de matemática que revela, a través de la escritura de una clase imaginaria en forma de guion, la comprensión que posee sobre el concepto de solución de un sistema de ecuaciones lineales. Se analiza la clase desde la óptica de los modos de pensamiento en álgebra lineal de Anna Sierpinska, y se aporta evidencia de las dificultades de la futura profesora en torno al conocimiento matemático que procura enseñar. Se reflexiona sobre el impacto que podría tener, en la enseñanza de la matemática, un profesor que no comprende cabalmente el tema que está enseñando, y se pone en diálogo esta reflexión con resultados de investigación que, todavía, no han sido tenidos en cuenta en la educación matemática en el nivel secundario.

**Palabras clave:** Sistema de ecuaciones lineales. Conocimiento matemático para la enseñanza. Modos de pensamiento. Narrativa.

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## 1 Introduction

When teaching mathematics, teachers draw on a range of knowledge and beliefs whose nature has been extensively investigated and continues to be a subject of inquiry within the field of Mathematics Education (Ball; Thames; Phelps, 2008; Carrillo *et al.*, 2013; Rowland; Huckstep; Thwaites, 2005; Shulman, 1986). This paper focuses on the mathematical knowledge mobilised by a prospective mathematics teacher (PMT) in designing a teaching proposal for the concept of the solution to a system of two linear equations in two variables.

Prospective teachers enter teacher education with a conceptual repertoire acquired through their prior educational trajectory, spanning primary and secondary schooling and, in Uruguay, not requiring previous university-level study in mathematics. It is within teacher education itself that prospective teachers deepen their understanding of advanced mathematical content, mathematics education, and teaching practice, as well as of content from the educational sciences. These formative pillars, which develop in parallel over the course of four years, contribute to the development of the professional competences required for teaching curricular content at secondary level.

The problem identified by research is that the mathematical knowledge teachers develop during teacher education is not always aligned with formal mathematical knowledge; nevertheless, it is mobilised in teaching processes. It has been documented that prospective teachers' conceptions exert a decisive influence on their teaching practice (Chhabra; Baveja, 2012; Ryan; Williams, 2007). This implies that what is taught in classrooms does not necessarily correspond to canonical disciplinary knowledge, but rather to the personal synthesis each teacher has constructed. This reality justifies the relevance of studying the ways in which prospective teachers understand mathematical objects and their relationships during initial teacher education, in order to address their needs and to act effectively so as to influence teaching.

In particular, this paper focuses on the teaching of systems of linear equations in two unknowns (square or non-square) in lower secondary education (ages 14–15), and aims to explore the modes of thinking in linear algebra (Sierpinska, 2000) mobilised by a second-year PMT when constructing a narrative on the teaching of this topic.

## 2 The problem under study

Difficulties in understanding the concept of the solution to a system of linear equations, and of systems of linear equations more broadly, have been reported across all educational levels, from lower secondary education to university (Campos; Parraguez, 2019; Oktaç, 2018; Rodríguez *et al.*, 2022; among others). One of the central findings is the misconception that associates the solution to a system of two linear equations in two unknowns with intersection points that do not belong to all the lines in the system, or with points at which the lines of the system intersect the coordinate axes, rather than recognising it as the point common to all the lines associated with the equations of the system.

This phenomenon is evident in studies such as those by Campos and Parraguez (2019), Eslava and Villegas (1998), and Alcocer (2007), in which students at secondary, upper-secondary, and tertiary levels, respectively, interpreted a system of three equations in two unknowns as having three solutions when the lines intersected pairwise to form a triangle. In addition, Alcocer (2007) found that students associated the number of solutions with the number of unknowns, thereby revealing a limited understanding of the concept. Another recurrent difficulty is the inability to conceive of a linear equation in two variables as a mathematical object that defines an infinite set of solutions.

Panizza, Sadovsky, and Sessa (1999) showed that, when faced with a single linear equation in two variables, students were unable to interpret the letters as variables in a dependent relationship, instead viewing them solely as unknowns to be determined with the help of another equation. Only one participant in their study was able to approach this idea through the notion of linear function, whereas the others limited themselves to generating unique solutions within the context of systems of equations in which one equation was kept fixed.

The transition between analytic and geometric representations also emerges as a significant obstacle (Eslava; Villegas, 1998; Cutz, 2005). Cutz (2005), in particular, highlighted that the engineering students who participated in her study experienced difficulties in relating the algebraic solution to its geometric interpretation, especially in systems of linear equations with two and three unknowns. This disconnection becomes even more pronounced in the case of non-square inconsistent systems, where students, both at secondary level and among prospective teachers, tend to reduce the range of possible configurations to cases involving parallel lines only (Ochoviet, 2009; Magallanes, 2019).

In this vein, Ponciano, Segura, and Ávila (2020) analysed the difficulties experienced by undergraduate mathematics students with the concept of the solution set of a system of linear

equations in two and three unknowns. The researchers concluded that all three participants experienced difficulties in coordinating the algebraic representation of a system with its graphical representation in order to interpret the solution set geometrically. They also showed difficulties in identifying whether a given equation corresponds to a line or to a plane; that is, they were unable to visualise the solution set of the given equations. A similar situation was identified by Rodríguez *et al.* (2022) in a study conducted with students of Mathematics Teacher Education and undergraduate engineering students. The researchers found a lack of coordination among the three modes of thinking, particularly when parametric systems were addressed. The studies reviewed suggest that difficulties in understanding the notion of a system of linear equations, solution, and solution set cut across secondary and tertiary education, including among prospective teachers and undergraduate students in mathematics and engineering who have studied, in particular, university-level linear algebra courses.

In response to this problem, the authors consulted recommend the implementation of pedagogical strategies centred on the diversification of examples and the strengthening of connections between representations. Campos and Parraguez (2019) recommend proposing tasks that articulate at least two modes of thinking, since working within a single mode leads students to make errors. Oktaç (2018) suggests incorporating non-conventional systems, such as  $3 \times 2$  systems, in order to broaden the notion of solution beyond the  $2 \times 2$  case.

Likewise, Cutz (2005) and Alcocer (2007) emphasise the need to propose activities that promote interactions between the analytic and the geometric, while Alcocer (2007), in particular, warns that mere exposure to isolated examples is not sufficient to correct misconceptions. Although multiple pedagogical recommendations have been made to address the difficulties identified in teaching (Ochoviet, 2009; Oktaç, 2018), they would appear not yet to have been adequately addressed within the prevailing school mathematical discourse. This underscores the importance of attending to the teaching implemented by PMTs in their first classroom interventions, given that difficulties in understanding mathematical objects may be reflected in their pedagogical proposals.

The aim of this study, then, is to account for a PMT's understanding of mathematical knowledge through a narrative written by the student. In other words, it seeks to provide evidence of how a problem that has been extensively investigated, yet insufficiently addressed in terms of effective changes to prevailing school mathematical discourse, becomes visible when a prospective teacher is required to think about teaching; this may seriously compromise the learning of secondary-school students. The research questions guiding the study are as follows: (1) What modes of thinking are brought into play in an imaginary lesson written by a

mathematics PMT undertaking her first teaching practicum? and (2) What difficulties in understanding the concept of the solution to a non-square system of linear equations become evident?

The novelty of this study lies in the type of material analysed to obtain information. All the studies reported conducted their inquiry through the administration of questionnaires or interviews in which participants were asked to solve mathematical tasks. In this study, the material analysed is a narrative of a teaching scenario. This narrative consisted of the writing of an imaginary lesson (IL) (Chapman, 2008) in script format (Zazkis; Marmur, 2018), addressing aspects related to the teaching of systems of linear equations in lower secondary education (ages 14–15). This means that the participant's mathematical knowledge is not mobilised with the aim of solving mathematical tasks, but rather of teaching mathematics through the solving of tasks. This creates a context in which it becomes possible to explore whether the difficulties outlined above emerge in action in response to the intention to teach.

### 3 Conceptual aspects: modes of thinking in linear algebra

Sierpinska (2000, p. 2) identifies three modes of thinking in linear algebra, corresponding to three languages that operate interactively: “the ‘visual-geometric’ language, the ‘arithmetic’ language of vectors and matrices as lists and tables of numbers, and the ‘structural’ language of vector spaces and linear transformations”. The author thus distinguishes three modes of thinking, which she terms, respectively, synthetic-geometric (SG), analytic-arithmetic (AA), and analytic-structural (AS). These languages were identified by Sierpinska through an analysis of the history of mathematics and a study of the different languages used in the theoretical developments of the field of linear algebra. Although these three modes of thinking emerged sequentially in the history of mathematics, none of them completely displaced the others; rather, the three continued to coexist. Sierpinska considers that these three modes of thinking “are equally useful, each in its own context, and for specific purposes, and especially when they are in interaction” (2000, p. 27).

According to the author, the distinction between the synthetic mode (SG) and the analytic modes (AA and AS) lies in the fact that, in the synthetic mode, objects are presented directly to the mind, which then seeks to describe them on the basis of their representations, whereas in the analytic modes mathematical objects are given indirectly and are constructed by means of their definitions or properties. Sierpinska (2000, p. 28) illustrates this as follows:

If somebody is thinking about the possible solutions of a system of three linear

equations in three variables by visualizing the possible relative positions of three planes in the space, he or she is in the synthetic-geometric mode (or mood). If one thinks about the same problem in terms of the possible results of a row reduction of a  $3 \times 4$  matrix, one is operating in the analytic-arithmetic mode. Thinking in terms of singular and non-singular matrices would be a symptom of the analytic-structural mode.

In this study, we focus on the teaching of systems of linear equations in two unknowns (square or non-square) at lower secondary level (ages 14–15). The modes of thinking that the PMT might promote through the questions she poses, the explanations she develops, or the tasks she proposes in the IL should be appropriate to the knowledge and competences of students at that level. For example, the PMT might promote the SG or AA modes of thinking by interpreting the number of solutions of a system of equations through the relative positions of two lines in the plane, or by solving it through an analytic method, such as substitution or elimination. By contrast, if, when addressing the problem of investigating the number of solutions of a system, the PMT were to intervene so as to guide or suggest an analysis of the proportionality among the coefficients of the equations, she would be applying a property and promoting an AS mode of thinking. Since the construction of the IL also includes the interventions of the fictitious students throughout the lesson, what these students say or do will allow us to glimpse the modes of thinking the PMT attributes to them; that is, the way in which the PMT understands the modes of thinking of lower secondary students.

#### 4 Methodological aspects

The study of teacher narratives has become established as a valuable approach for understanding processes of pedagogical thinking in the preparation of mathematics teachers (Chapman, 2008). This study adopts a qualitative perspective through a single-case study centred on the analysis of narratives produced by a PMT. The work is situated within a line of inquiry that conceives of narratives not only as instruments for data collection, but also as spaces for the construction of and reflection on teaching practice (Zazkis; Marmur, 2018). The particular context of the participant is presented below, followed by a detailed account of the methodology employed for the production and analysis of her teacher narratives.

##### *Participant and context*

Initial teacher education in Uruguay is a four-year university-level degree programme. The participant in this study was a second-year student undertaking her first Didactics of Mathematics/Teaching Practice course. In this course, her Didactics lecturer had worked on the development of lesson scripts as a resource for planning teaching. In parallel with the Didactics

course, the student was also taking Geometry and Linear Algebra, as well as the first Calculus course.

### *Data collection*

Chapman (2008) proposes the use of narratives with prospective teachers with the aim of promoting processes of personal reflection that usually require inquiry into the nature of mathematics, its teaching, and its learning. She also suggests their use for constructing or extending pedagogical knowledge. Chapman (2008, p. 23) has used narratives as “object and a process of inquiry”. In this study, they are used as an object of inquiry into a PMT’s thinking. Chapman (2008) identifies two types of narrative: stories about the teaching of mathematics and stories about problem solving. In this study, a story about teaching was considered; more specifically, the participant was asked to write an IL: “This is a narrative equivalent of a lesson plan that allows them to imagine the lesson as they perceive or desire it to actually unfold.” (Chapman, 2008, p. 24). This lesson plan was presented in the form of a script, that is, “a task in which participants script interaction between an imaginary teacher-character and student-character(s)” (Zazkis; Marmur, 2018, p. 294). According to these authors, such scripts constitute a tool for data collection. The examination of these scripts provides “images of teaching and insights into the scriptwriters’ understanding of mathematics” (Zazkis; Marmur, 2018, p. 294).

### *Narrative production*

The PMT was given a response produced by a lower secondary student, taken from the transcript of a lesson observed in Ochoviet (2009): *Because there is no numerical equality*. On the basis of this response, the following prompt was proposed:

*a. You will be given a real response produced by a lower secondary student in class. You are required to indicate what, in your view, was the question posed by the teacher, and to justify your answer.*

*b. Then, on the basis of that information, you are required to develop an imaginary lesson that responds to the question: How did the lesson continue? The lesson should last approximately 40 minutes, with the final 5–8 minutes devoted to a recap of what was addressed during the lesson, highlighting the key ideas.*

The PMT was given two weeks to complete the task, and the narrative she submitted is the one analysed in this study. The intention underlying the prompt was to encourage the PMT to engage in retrospective thinking, since the information provided consisted of a student’s response to an unknown question. The PMT was thus required to construct that question by attempting to understand what issue the student was responding to. This involved a considerable

challenge, as effective teacher intervention requires an understanding of what the student is responding to or asking (Ball; Thames; Phelps, 2008), while at the same time placing the participant in a hypothetical situation of contingency (Rowland; Huckstep; Thwaites, 2005). The prompt did not explicitly require the planning of a lesson on a given topic, and the response provided to the prospective teacher could have been directed towards different scenarios; it was the PMT who chose to orient her work towards the context of systems of linear equations. It is assumed that she selected this topic because, in the teaching practicum she was undertaking, her mentor teacher was working on that topic with her students.

### *Narrative analysis*

In the narrative produced by the participant, in script form, attention is focused on the moments in which the modes of thinking in linear algebra are implicated; this entails considering all the episodes in which the PMT mobilises mathematical knowledge for teaching (Ball; Thames; Phelps, 2008) in order to intervene, question, propose, or solve mathematical tasks.

## **5 Results and discussion**

As noted above, the response given to the participant was: *Because there is no numerical equality*. On this basis, the PMT formulates a possible question which, in her view, could have been posed by the teacher in the IL she develops: *Why is the ordered pair (4, -7/5) not a solution to the second equation?* She justifies this choice as follows:

*I believe that this response may have been given in a context related to the topic of systems of equations. One possible interpretation is that the students may have solved the system incorrectly and that the ordered pair did not satisfy one of the equations; therefore, when these values were substituted into one of the equations, no numerical equality was obtained (PMT's response to part (a) of the prompt, 2023).*

The PMT justifies the question she proposes in terms of an AA mode of thinking, as she interprets the solution as an ordered pair of numbers and in terms of arithmetical work to verify that the pair is not a solution to an equation. The response can be associated with a situation in which the solution to an equation is under discussion (whether in one or two variables, linear or quadratic, as these are the types studied in lower secondary education in Uruguay). In the development of the IL, the PMT explains that she imagines the lesson beginning with the following task:

*In a ninth-grade lesson, students were asked to solve the following system over  $R$ :*  
$$2x + 5y = 1$$

$-x + y = 3$   
*Agustín solved it as follows:*  
 $2x + 5y = 1$   
 $2(4) + 5y = 1$   
 $8 + 5y = 1$   
 $5y = -7$   
 $y = (-7)/5 \quad \{(4, -7/5)\}$

*Paula solved it as follows:*  
 $-x + y = 3$   
 $y = 3 + x$   
 $2x + 5y = 1$   
 $2x + 5(3 + x) = 1$   
 $7x = 16$   
 $x = 16/7$   
 $y = 37/7 \quad \{(17/7, 37/7)\}$

*Do you agree with the students' solution? Why or why not? If the solution is not correct, solve the system* (Excerpt from the IL produced by the PMT, 2023).

The task proposed by the PMT is also presented in an AA mode of thinking: a system of linear equations in two unknowns is given, and the solutions offered by two hypothetical students refer to arithmetical manipulations. Although the PMT could have considered the same idea in the context of a single equation in two variables, she chooses instead to work with a system of equations, referring to the *second* equation. This may be because systems of linear equations in two unknowns were being addressed in the teaching practicum she was undertaking, or because she prefers that context in order to make sense of the concept of solution. This would be consistent with Panizza, Sadovsky, and Sessa (1999), who point to the difficulty experienced by the students in their study in making sense of the object linear equation. When they were presented with a single equation and asked to provide solutions to it, the students tended to introduce a second equation and seek the solution to the system.

The IL then unfolds as follows:

*Teacher (T): Why is the ordered pair (4, -7/5) not a solution to the second equation?*  
*Kiara: Because there is no numerical equality.*  
*T: Axel, would you like to say in your own words what Kiara said?*  
*Axel: I didn't understand, Miss.*  
*T: Kiara, could you say it again so that everyone can understand?*  
*Kiara: Erm, well, I think the pair (4, -7/5) does not work because, if we substitute these values into the second equation, there is no numerical equality, that is, it does not satisfy the equality.*  
*Sebastián: OK, but it satisfies the first one.*  
*T: What does everyone else think? Fiorella?*  
*Fiorella: The point is that, for it to be a solution to the system, it has to satisfy both equations, but since it does not satisfy the second one, it is no use to us.*  
*T: So, where did Agustín go wrong?*  
*Fiamma: In using only one equation. That "solution" he found does not satisfy the other one* (Excerpt from the IL produced by the PMT, 2023).

The responses of the fictitious students, as imagined by the PMT, reveal an AA mode of thinking in which, in order to determine whether or not the pair is a solution, the values of the

unknowns must be substituted and it must then be observed whether a numerical equality is obtained. There are also indications of an AS mode of thinking in Fiorella's response, when she draws on theoretical knowledge in stating that a solution to the system *has to satisfy both equations*.

The IL continues with the analysis of Paula's solution.

*Teacher (T): Right, let's continue. What did you think about Paula's work?*

*(Alejandro raises his hand.)*

*T: Go ahead, Alejandro.*

*Alejandro: We think the idea is right, but she made a mistake in the calculations.*

*T: Micaela, do you agree with what Alejandro said?*

*Micaela: Yes, we think the same; the mistake was in the penultimate step.*

*T: Why?*

*Tiago: Because she should have subtracted 15 from both sides, but on one side she subtracted it and on the other she added it.*

*(...)*

*T: You were saying that Paula's idea is right. Why?*

*Candela: Because she used one of the methods for solving systems of equations.*

*T: Which method was used?*

*Maximiliano: The substitution method (Excerpt from the IL produced by the PMT, 2023).*

In the dialogue proposed by the PMT, two noteworthy issues become evident: the imaginary students identify the error made in isolating the unknown in Paula's reasoning and, in addition, recognise that there is a correct idea underlying that reasoning, namely, the application of one of the methods studied, in this case substitution, to solve a system. The application of a known method forms part of the argument used to validate Paula's reasoning; it reveals an AA mode of thinking, since it involves the manipulation of equations and, therefore, the interpretation of symbols as equations.

Later in the script, the teacher asks the students in what other way the system could have been solved, and the students respond: the equating method, the elimination method, and the graphical method. The teacher proposes applying the latter and asks the students what is required in order to graph the line associated with each of the equations. One student replies that two points are needed, and the teacher then asks which points belong to the line in order to relate solution to an ordered pair that satisfies the equation. In this case, a flexible articulation between the AA and SG modes of thinking becomes apparent.

*Teacher (T): What can we do in order to draw the lines associated with the equations? What do we need in order to draw a line?*

*Micaela: Two points.*

*T: Which points will belong to the line associated with the first equation?*

*Lucio: (-2, 1).*

*T: Camila, do you agree with what Lucio said?*

*Camila: Yes, we had seen that the points belonging to the line are the solutions to the equation, and if we substitute  $x$  with minus two and  $y$  with one, it satisfies it (Excerpt from the IL produced by the PMT, 2023).*

The IL continues with the search for the coordinates of two points on each of the lines associated with the equations of the system, consistently relating an ordered pair that satisfies the equation to a point lying on the line, until the graphical representation of the lines is produced on the board.

*Teacher (T): Now that we have the lines associated with the equations, what do we do?*

*Ana Clara: Since they intersect at one point, the system has only one solution.*

*T: That's right. Since the lines intersect, the system has a unique solution. Do you remember what systems with a unique solution are called?*

*Alejandro: A consistent independent system, Miss.*

*T: Good. Would anyone like to add anything else?*

*Eliana: Me, Miss. If they have no solution, they are called inconsistent, and if they have infinitely many solutions, they are called consistent dependent.*

*T: Exactly. In the case of an inconsistent system, what are the lines associated with those equations like?*

*Mateo: They never touch.*

*T: When lines do not meet, we say they are distinct parallel lines, but yes, that is the idea. And when is the system dependent?*

*Kiara: It is because they are the same line.*

*T: Exactly. In that case, we say the lines are coincident. Returning to the task, what coordinates should the point of intersection of the two lines have?*

*Rafael: (-2, 1).*

*T: Why?*

*Julieta: Because where the lines meet is the solution to the equation, and we had got (-2, 1).*

*T: If you look, they intersect more or less at the point with coordinates (-2, 1). More or less, because the lines were drawn by hand on the board. We are observing that, both by an algebraic method and by the graphical method, the solution is the same (Excerpt from the IL produced by the PMT, 2023).*

In this excerpt, the teacher can be seen expressing herself in a way that reveals an SG mode of thinking when she refers to the configuration of intersecting lines as the graphical representation of a system of linear equations with a unique solution; indeed, her intervention may be interpreted as support aimed at helping the student Ana Clara, not to focus attention on the point of intersection of the two lines, but rather on the relative position of the two lines in the plane. Nevertheless, in the expressions of the fictitious students there appear familiar ideas associated with the concept of solution, such as *they intersect at one point* or *where the lines meet*, which are very close to interpreting the solution as the point of intersection of two lines (Ochoviet, 2009; Oktaç, 2018).

The teacher also reveals an AS mode of thinking when she refers to the classification of systems of linear equations according to the number of solutions. In turn, she promotes interactions between the AS and SG modes when she seeks to connect the type of system with the graphical configuration of the lines associated with the equations of the system, and between the SG and AA modes when she asks about the coordinates of the point of intersection of the

two lines. The teacher both displays and reinforces, in the students of the IL, some of the difficulties reported in the literature. That is, she fosters the idea of the solution as the point of intersection of two lines, contrary to what has been suggested, for example, by Oktaç (2018).

Once the discussion of the initial issue previously reported in the IL comes to an end, the PMT includes a second activity to bring the lesson to a close:

*Teacher (T): Right, now I am going to give you a short task to do with the person next to you. You have 6 minutes.*

*A system of three equations in two unknowns:*

- a. Can it have a unique solution?*
- b. Can it have exactly two solutions?*
- c. And exactly three?*
- d. Can it have infinitely many solutions?*
- e. And none? (Excerpt from the IL produced by the PMT, 2023).*

This task belongs to the questionnaire proposed in Ochoviet (2009) and was included in the textbook *Matemática 3* by Ochoviet and Olave (2009); it was from the latter source that the PMT took it. The plenary discussion that follows the task, as recorded in the IL, is presented below.

*Teacher (T): Right, let us go over the task. We have a system of three equations, and we are asked whether that system can have a unique solution. What did you think?*

*Valentina: We thought that, if each equation has a line, then we will have three lines.*

*T: Well, it has to do with that. Let us try to make sure everyone understands you: each linear equation in two unknowns can be associated with a line, yes, so then what?*

*Valentina: So, since there are three equations, each one corresponds to a line.*

*T: OK, so?*

*Valentina: So, if the three lines pass through one point, it would have a unique solution.*

*T: What would have a unique solution?*

*Mateo: The system (Excerpt from the IL produced by the PMT, 2023).*

Clearly, the case of a unique solution is associated with three concurrent lines. This interpretation is framed within the SG mode of thinking. The discussion could have ended with the preceding exchange, since proposing a single case was sufficient to answer question (a) of the task. However, the PMT suggests continuing the discussion in the following way:

*Teacher (T): Fiorela, do you agree with what your classmates are saying?*

*Fiorela: Yes, it would be like in the previous task, where we had two intersecting lines and the point of intersection was the solution. Now we have three lines.*

*T: What are these three lines like in relation to one another? Would anyone like to come and draw them on the board?*

*Axel: I will.*

*(Axel comes to the board and draws three distinct intersecting lines.)*

*T: What does everyone else think?*

*(The students nod.)*

*(The teacher draws two intersecting lines.)*

*T: And what if I told you that these two lines are associated with three equations? Discuss it for a minute with the person next to you.*

*(After a minute)*

*T: What did you come up with?*

*Kiara: Could it be that two of them are equivalent? Because we had seen that equivalent equations have the same line.*

*T: That is right, equivalent equations are associated with the same line.*

*Tiago: So, if there are three equations but only two lines, does that mean that two of the equations must be equivalent?*

*T: Luca, what do you think about what your classmate said?*

*Luca: It is right.*

*T: Well, yes, what Tiago says is correct. So, a system of three equations in two unknowns can have a unique solution. Can it have two solutions? (Excerpt from the IL produced by the PMT, 2023).*

In this excerpt, we highlight the way in which the fictitious student Fiorela expresses herself: *the point of intersection was the solution*. The teacher does not intervene, even though earlier she had directed attention towards the graphical configuration by referring to intersecting lines. It should be noted that, after the fictitious student Axel draws the three concurrent lines and the students nod their heads, thereby indicating that they all agree with that graphical representation of the unique-solution case, the teacher draws two intersecting lines and asks the students whether they could be associated with the three equations. Although it is correctly concluded that two of the equations could be equivalent, it would appear that the teacher needs to consider this case of two intersecting lines in order to conclude that, indeed, a system of three linear equations in two unknowns can have a unique solution. Note that it is only after this example that the teacher concludes:

*So, a system of three equations in two unknowns can have a unique solution (Excerpt from the IL produced by the PMT, 2023).*

This conclusion was not emphasised after the three concurrent lines had been drawn. Although it is highly important to analyse the different relative positions of three lines in the plane that lead to a system with a unique solution, the teacher would appear to need to display the graphical configuration of two intersecting lines in order to answer question (a). This would be consistent both with what the PMT has the fictitious student Fiorela say and with the point in the lesson at which it is concluded that a unique solution may exist, namely, only after two intersecting lines have been drawn on the board. This reveals a strong tendency to model the unique-solution case through that representation and may reinforce the difficulties reported in Oktaç (2018).

The IL continues with the discussion of parts (b) and (c) of the task.

*Camila: No, because lines only intersect once.*

*Teacher (T): Let us look at that more closely. What do you mean?*

*Camila: Well, the solution to the system is the intersection of the lines, and lines cannot cross more than once.*

*T: Intersect more than once.*

*Camila: Yes, that.*

*T: Good, so they cannot have two solutions. And three?*

*Mateo: No, for the same reason. If two do not intersect more than once, then three will not either* (Excerpt from the IL produced by the PMT, 2023).

The teacher corrects the fictitious student Camila's intervention, pointing out that the appropriate term is *intersect* rather than *cross*. The mode of thinking evidenced by both the teacher and the students is SG. The teacher does not ask for drawings, nor does she place any emphasis on the word *exactly*, which is present in the wording of the task; instead, she asks whether the system can have two or three solutions, which is clearly not the same as asking whether it can have exactly two or exactly three. The IL then moves on to parts (d) and (e) of the task.

*Teacher (T): Can it have infinitely many solutions?*

*Fiamma: Well, in that case it would have to be only one line, so they would all have to be coincident, wouldn't they? so that there is a single associated line.*

*T: What does everyone else think?*

*E: Yes, Miss.*

*T: And could it have no solution at all?*

*Rafael: If none of them meet?*

*T: Do you mean if they have no points in common?*

*Rafael: Yes, if they are parallel.*

*T: Yes, if they are parallel, then they have no point in common and it would have no solution. So, for the system to have no solution, what must the lines be like?*

*Martina: Distinct parallel lines.*

*T: Good* (Excerpt from the IL produced by the PMT, 2023).

The case of infinitely many solutions is discussed correctly. It is associated, in the SG mode, with the case of three coincident lines. The discussion then moves to the final part, in which the question is whether the system can have no solution. The fictitious student Rafael suggests the case of parallel lines, after which the teacher confirms that in that case there would be no solution. The problem is that the teacher establishes that graphical configuration as a necessary condition: *So, for the system to have no solution, what must the lines be like?*, when the lines could in fact have other possible configurations, such as the following (Figure 1):

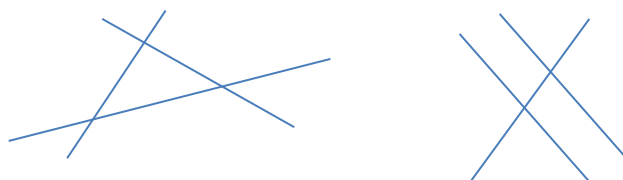


**Figure 1** – Other configurations associated with a system of three linear equations in two unknowns with an empty solution set  
Source: Own elaboration (2024)

This shows that the graphical configuration that prevails for a system of linear equations with no solution is that of parallel lines, which is the only possible one in the case of  $2 \times 2$

systems, even though the PMT was in the second year of initial mathematics teacher education and, in particular, was taking the Geometry and Linear Algebra course in that year of the programme. This is consistent with what was reported by Ochoviet (2009) and with Oktaç's (2018) observation that, specifically in the teaching of systems of linear equations, particular attention needs to be paid to the configurations associated with inconsistent systems so that students do not consolidate the case of parallel lines as the only possible graphical configuration for that type of system. The PMT's difficulty in establishing all the relative positions of three lines in the plane is also evident, which is consistent with what was reported by Eslava and Villegas (1998).

Further evidence is presented below of the graphical configurations that the PMT proposes to associate with each type of solution set. Following the last dialogue cited, the IL introduces a moment of recap of the first issue discussed in the lesson and then a reflection on the second task:

*Teacher (T): In the second task, we had a system of three equations in two unknowns, and we saw that this system could have a unique solution, infinitely many solutions, or no solution. If the lines associated with the linear equations intersect at a point, we are in the first case; if the lines associated with the equations are coincident, we have infinitely many solutions; and if the lines associated with the equations are distinct parallel lines, the system has no solution (Excerpt from the IL produced by the PMT, 2023).*

This evidence reinforces the earlier claim that the only graphical configuration the PMT associates with an inconsistent system is that of a set of parallel lines, and this idea, in a  $3 \times 2$  systems context, leads her to make an error. This mathematically incorrect idea is institutionalised in the IL as the lesson conclusion. The following diagram models the phenomenon observed (Figure 2).

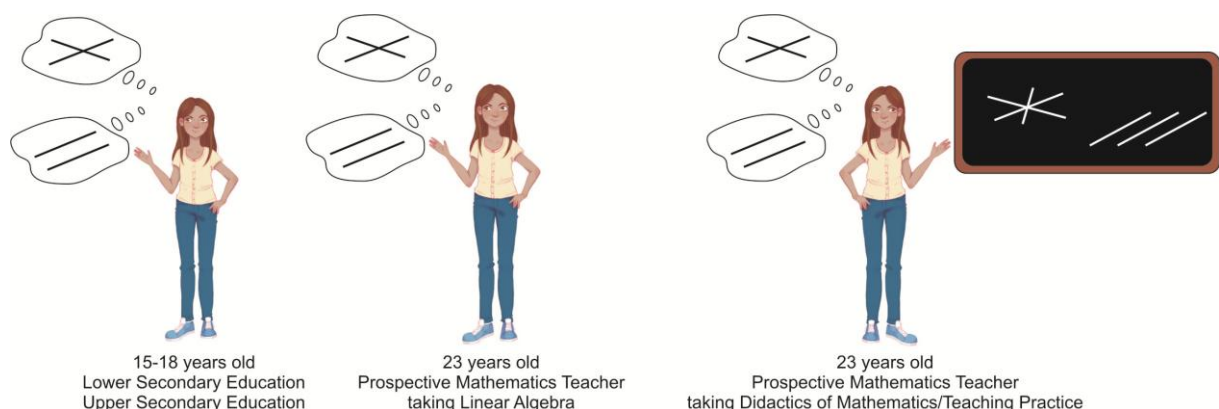


**Figure 2** – What is learned is transferred to teaching  
Source: Own elaboration (2024)

## 6 Conclusions

In this study, we set out to answer which modes of thinking are brought into play in an IL written by a PMT undertaking her first teaching practicum, and which difficulties become evident. The IL initially unfolds within an AA mode of thinking because the PMT was asked to formulate a possible teacher question in response to an answer that had itself been given in that mode: *Because there is no numerical equality*. Thus, the PMT proposes as a possible teacher question: *Why is  $(4, -7/5)$  not a solution to the second equation?* She then indicates the context in which this question arises, namely that of systems of two linear equations in two unknowns, and develops the discussion around the task being solved and which gave rise to that question. Once the plenary discussion has been completed, she proposes a second task that can be answered using different modes of thinking and situates the reflection in the characteristics of the solution set of a system of linear equations: it has one element, infinitely many, or none.

In the development of the IL, connections are established between the AA and SG modes, for example, when the teacher asks about the graphical representations associated with systems with different types of solution set. The AS mode is also evident in the teacher's interventions that refer to theoretical aspects, for example, when she explains the classification of systems according to the number of elements in their solution set, or when the discussion is proceeding in an AA mode of thinking and a fictitious student states that a solution to the system must satisfy both equations of the system, thereby drawing on the definition of the solution to a system of linear equations with two equations.

Although a possible question corresponding to the response *Because there is no numerical equality* could have been formulated in the context of a linear equation in one variable or of a single linear equation in two variables, the PMT chooses instead to refer to a second equation, marking, through the word *second*, the existence of a system of at least two equations. This may be because that topic was being addressed in the teaching practicum she was undertaking or, alternatively, because of a personal need to work in that context owing to the difficulty of conceiving an equation in two variables in isolation from systems of equations. This is consistent with the findings of Panizza, Sadovsky, and Sessa (1999), who report that one student interpreted the linear equation as a linear function, from which he was able to construct solutions by assigning values to one of the variables.

This same procedure is the one proposed by the PMT in the first task, in which the value 4 is assigned to  $x$  (without any explanation as to why that number was chosen) and, for that value, the corresponding value of  $y$  is then obtained by substitution. In other words, it is not the case that the system was solved incorrectly and an ordered pair was reached that is not a solution

to the system; rather, an arbitrary value is assigned to  $x$ , in this case 4, and the value of  $y$  is obtained, which will evidently not yield a solution pair for the system, since that 4 appears to have been selected at random. Although the teacher in the IL explicitly states that the graphical configuration for a system of two equations with a unique solution is that of intersecting lines and later, in the case of three equations, refers to three lines sharing the same common point, there is a noticeable tendency to associate the solution of a system with a point of intersection and with the configuration of two intersecting lines. This becomes clear when, while working with a system of three equations, the teacher in the IL draws two intersecting lines on the board, asks whether that graphical representation would be possible, correctly arrives at the conclusion that two of the equations of the system could be equivalent, and only after this institutionalises the idea that a system with three equations may have a unique solution. In other words, she would appear to need to visualise two intersecting lines in order to reach that conclusion, when she could have stated it much earlier, at the point when three lines sharing a single common point were being discussed.

Additionally, in the IL produced by the PMT, a system of three linear equations in two unknowns with no solution is associated exclusively with the graphical configuration of three parallel lines, and this case is institutionalised as the only possible one, thereby overlooking two other possible cases. This problem is consistent with what was reported by Ochoviet (2009) and Oktaç (2018), where the persistence of the parallel-lines model as the sole graphical representation of an inconsistent system of linear equations is documented. This situation is attributed to the strong influence of the model introduced when  $2 \times 2$  systems are addressed, which tends to constitute the first approach to the study of linear systems in secondary and upper-secondary education, thereby generating a negative impact on the learning of linear algebra. It is also evident that the teacher does not command the full range of relative positions of three lines in the plane (Eslava; Villegas, 1998), or else knows them but associates them with a different type of solution set. Likewise, the case of three parallel lines is institutionalised as the only possible case for a system of three equations with no solution.

The work presented suggests that initial conceptions constructed on the basis of  $2 \times 2$  systems are not easily extended, even when students go on to study linear algebra at tertiary level, in line with what was identified by Rodríguez *et al.* (2022) and Magallanes (2019). Through the IL produced by the PMT, we were able to appreciate the way in which she understands mathematical objects and their relationships in a teaching situation while imagining herself in the role of teacher (Zazkis; Marmur, 2018).

This raises a serious dilemma, since we are no longer dealing solely with difficulties in

learning linear algebra, but with these difficulties becoming mathematically incorrect forms of knowledge that emerge as content for teaching in lower secondary classrooms, thereby intensifying the spiral of difficulties. That is, to the difficulties reported in the literature regarding the learning of the concepts of system, solution to a system, and solution set in the linear algebra classroom, there must be added the implications of such learning for teaching. It is time for those who teach linear algebra at any educational level to take account of research findings in order to reformulate, as a matter of urgency, prevailing school mathematical discourse.

### Authorship Contributions

All authors contributed substantially to the conception and planning of the study; to the acquisition, analysis, and interpretation of the data; to the writing and critical revision; and approved the final version to be published.

### Data Availability

The data generated or analyzed during this study are included in the published article.

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**Submitted on:** April 07, 2024.  
**Approved on:** October 22, 2025.



**Editor-in-Chief:** Prof. Dr. Roger Miarka  
**Associate Editor:** Prof. Dr. Vicenç Font