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Exact solutions for drying with coupled phase-change in a porous medium with a heat flux condition on the surface

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Abstract. Exact solutions for the problem of drying with coupled phase change in a porous medium with a heat flux condition on $x = 0$ of the type $-q_0/\sqrt{\tau}$, with $q_0 > 0$, for any value of the Luikov number L_u is obtained. This solution can be only obtained when q_0 verifies a certain inequality. Besides, for large Luikov number $\left(\text{more precisely, } L_u > \dfrac{1}{\varepsilon K_0 + 1}\right)$, we obtain that the temperature distribution t_2 reaches to a minimum value which is smaller than its initial temperature or limit value reached at $+\infty$.

Mathematical subject classification: 35R35, 80A22, 35C05.

Key words: free boundary, Stefan problem, phase change, drying, heat conduction, mass transfer, porous medium.

1 Introduction

Heat and mass transfer with phase change problems, taking place in a porous medium, such as evaporation, condensation, freezing, melting, sublimation and desublimation, have wide application in separation processes, food technology, heat and mixture migration in soils and grounds, etc. Due to the non-linearity of the problem, solutions usually involve mathematical difficulties. Only a few exact solutions have been found for idealized cases. Mathematical formulation of the heat and mass transfer in capillary porous bodies has been established by Luikov [13], [14], [15], [16], [17]. Other problems in this direction are [3], [6], [7], [9], [20], [22].

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A large bibliography on free and moving boundary problems for the heat-diffusion equation was given in [23]. Gupta [10] presented an approximate solution to a coupled heat and mass transfer problem involving evaporation. The problem Gupta [10] treated has analytical solution, which is presented in Cho [5].

Heat and mass transfer during drying from an homogeneous point of view are also considered in [1], [2], [4], [8], [11], [18], and [19].

In the following, we study a similar problem as that of [5]. A semi-infinite porous medium is dried by maintaining a heat flux condition at $x = 0$ of the type $-q_0/\sqrt{t}$, with $q_0 > 0$, which was firstly considered in [21]. Initially, the whole body is at uniform temperature t_0 and uniform moisture potential u_0. The moisture is assumed to evaporate completely at a constant temperature, evaporation point t_v. It is also assumed that the moisture potential in the first region, $0 < x < s(\tau)$, is constant at u_v, where $x = s(\tau)$ locates the evaporation front at time $\tau > 0$. It is further assumed that the moisture in vapor form does not take away any appreciable amount of heat from the system. Neglecting mass diffusion due to temperature variation, the problem can be expressed as:

$$\frac{\partial t_1}{\partial \tau}(x,\tau) = a_1 \frac{\partial^2 t_1}{\partial x^2}(x,\tau), \qquad 0 < x < s(\tau), \tau > 0 \text{ (region 1)} \tag{1.1}$$

$$u_1 = u_v, \qquad 0 < x < s(\tau), \tau > 0 \text{ (region 1)} \tag{1.2}$$

$$\frac{\partial t_2}{\partial \tau}(x,\tau) = a_2 \frac{\partial^2 t_2}{\partial x^2} + \frac{\varepsilon L c_m}{c_2}\frac{\partial u_2}{\partial \tau}, \qquad x > s(\tau), \tau > 0 \text{ (region 2)} \tag{1.3}$$

$$\frac{\partial u_2}{\partial \tau}(x,\tau) = a_m \frac{\partial^2 u_2}{\partial x^2}(x,\tau), \qquad x > s(\tau), \tau > 0 \text{ (region 2)} \tag{1.4}$$

The initial and boundary conditions are:

$$k_1 \frac{\partial t_1}{\partial x} = -\frac{q_0}{\sqrt{\tau}} \quad \text{at} \quad x = 0, \tau > 0 \tag{1.5}$$

$$t_2 = t_0 \quad \text{in} \quad x > 0, \tau = 0 \tag{1.6}$$

$$u_2 = u_0 \quad \text{in} \quad x > 0, \tau = 0 \tag{1.7}$$

$$t_1(s(\tau),\tau) = t_2(s(\tau),\tau) = t_v > t_0 \quad \text{at} \quad x = s(\tau) \tag{1.8}$$

$$u_1(s(\tau), \tau) = u_2(s(\tau), \tau) = u_v < u_0 \quad \text{at} \quad x = s(\tau) \tag{1.9}$$

$$-k_1 \frac{\partial t_1}{\partial x}(s(\tau), \tau) + k_2 \frac{\partial t_2}{\partial x}(s(\tau), \tau) = (1 - \varepsilon)\rho_m L \frac{ds}{dt} \quad \text{at} \quad x = s(\tau) \tag{1.10}$$

Symbols are given in the nomenclature. We clarify that t_1 is the temperature of the dried porous medium, t_2 is the temperature of the humid porous medium and u_2 is the mass-transfer potential of the humid porous medium.

In paragraph 2, we find a solution of this problem, depending on the value of the Luikov number L_u, then in paragraphs 3 and 4 we discuss the equation that determines the dimensionless constant which characterizes the evaporation front when the Luikov number L_u equals to one and L_u is different to one. Finally, in paragraph 5 we give some illustrative results and a sufficient condition (5.4) for the Luikov number L_u in order to obtain when the temperature distribution has a minimum value less than its initial temperature.

This study was motivated by the following mathematical and physical analysis. Taking into account (1.1), (1.5) and (1.8), and (1.4), (1.7) and (1.9), by the maximum principle, we have $t_1(x, \tau) > t_v$ for region 1 and $u_v < u_2(x, \tau) < u_0$ for region 2 respectively. We expect from a physical point of view that the phase change front $s(\tau)$ should be an increasing function. In this case, thanks again to the maximum principle, we should obtain that $\frac{\partial u_2}{\partial \tau}(x, \tau) < 0$ for region 2, then the heat equation (1.3) has a heat sink within the corresponding region 2. Due to the maximum principle, we have $t_2(x, \tau) < t_v$ for region 2 and we can say anything about where the temperature has an absolute minimum value. One of the goals of this paper is to obtain a sufficient condition for the data in order to have a minimum value for the temperature within its corresponding domain. Moreover, we can characterize the coordinate of this point when the dimensionless variable $\eta = \dfrac{x}{2\sqrt{a_1 \tau}}$ takes the value (5.7) as a function of the data.

2 Solution of the problem

Let be the following dimensionless variables and parameters:

$$U_i = \frac{u_i - u_0}{u_v - u_0}, \text{ for } i = 1, 2 \tag{2.1}$$

$$T_i = \frac{t_i - t_0}{t_v - t_0}, \text{ for } i = 1, 2 \tag{2.2}$$

$$\eta = \frac{x}{2\sqrt{a_1 \tau}} \tag{2.3}$$

$$L_u = \frac{a_m}{a_1} > 0 \tag{2.4}$$

$$Ko = \frac{Lc_m (u_0 - u_v)}{c_2 (t_v - t_0)} > 0 \tag{2.5}$$

$$\nu = \frac{(1-\varepsilon)\, \rho_m L a_1}{k_1 (t_v - t_0)} > 0 \tag{2.6}$$

$$k_{21} = \frac{k_2}{k_1} > 0. \tag{2.7}$$

Assuming U and T are only functions of the variable η, the conditions (1.1)-(1.9) imply us that

$$s(\tau) = 2\lambda \sqrt{a_1 \tau} \tag{2.8}$$

where λ is a positive constant to be determined later. Therefore, equations (1.1)-(1.4) are transformed to the following dimensionless ordinary differential equations of the form:

$$T_1''(\eta) + 2\eta T_1'(\eta) = 0, \quad 0 < \eta < \lambda \tag{2.9}$$

$$U_1 = 1, \quad 0 < \eta < \lambda \tag{2.10}$$

$$T_2''(\eta) + 2\eta T_2'(\eta) - 2\varepsilon K_o \eta U_2'(\eta) = 0, \quad \eta > \lambda \tag{2.11}$$

$$L_u U_2''(\eta) + 2\eta U_2'(\eta) = 0, \quad \eta > \lambda. \tag{2.12}$$

The boundary conditions (1.5)-(1.10) become:

$$T_1' = -\frac{2q_0\sqrt{a_1}}{k_1\,(t_v - t_0)} \quad \text{at} \quad \eta = 0, \tag{2.13}$$

$$T_2 = 0 \quad \text{as} \quad \eta \to \infty, \tag{2.14}$$

$$U_2 = 0 \quad \text{as} \quad \eta \to \infty, \tag{2.15}$$

$$T_1 = T_2 = 1 \quad \text{at} \quad \eta = \lambda, \tag{2.16}$$

$$U_1 = U_2 = 1 \quad \text{at} \quad \eta = \lambda, \tag{2.17}$$

$$T_1' - k_{21} T_2' = -2\nu\lambda \quad \text{at} \quad \eta = \lambda, \tag{2.18}$$

Solutions of the equations (2.9) and (2.12), which satisfy boundary conditions (2.13), (2.15), (2.16) and (2.17), are easily obtained as follows

$$T_1\,(\eta) = 1 + \frac{q_0\sqrt{\pi\,a_1}}{k_1\,(t_v - t_0)}\,(\operatorname{erf}\lambda - \operatorname{erf}\eta)\,, \quad 0 < \eta < \lambda \tag{2.19}$$

$$U_2\,(\eta) = \frac{1 - \operatorname{erf}\left(\dfrac{\eta}{\sqrt{L_u}}\right)}{1 - \operatorname{erf}\left(\dfrac{\lambda}{\sqrt{L_u}}\right)}, \quad \eta > \lambda. \tag{2.20}$$

Substituting expression (2.20) into equation (2.11), and solving the resulting non-homogeneous ordinary differential equation with boundary conditions (2.14) and (2.16), we obtain the following results, depending on $L_u = 1$ or $L_u \neq 1$, i.e.:

$$\begin{aligned} T_2\,(\eta) \;&=\; \frac{\varepsilon K_o}{\sqrt{\pi}\,(1 - \operatorname{erf}(\lambda))}\left[\lambda\, e^{-\lambda^2}\frac{1 - \operatorname{erf}(\eta)}{1 - \operatorname{erf}(\lambda)} - \eta\, e^{-\eta^2}\right] \\ &\quad + \frac{1 - \operatorname{erf}(\eta)}{1 - \operatorname{erf}(\lambda)}, \quad \text{if } \; L_u = 1, \eta > \lambda \end{aligned} \tag{2.21}$$

or

$$\begin{aligned} T_2(\eta) &= \frac{\varepsilon K_o L_u}{L_u - 1}\left[-\frac{1-\operatorname{erf}\left(\frac{\eta}{\sqrt{L_u}}\right)}{1-\operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)} + \frac{1-\operatorname{erf}(\eta)}{1-\operatorname{erf}(\lambda)}\right] \\ &\quad + \frac{1-\ \operatorname{erf}\eta}{1-\operatorname{erf}\lambda}, \quad \text{if } L_u \neq 1, \eta > \lambda. \end{aligned} \tag{2.22}$$

Functions (2.19), (2.20) and (2.21) or (2.22) satisfy all boundary conditions except condition (2.18). Substituting these expressions into condition (2.18), the positive constant λ is determined from the following equation, depending on the value of L_u, as follows:

$$\begin{aligned} &\frac{k_{21}}{\sqrt{\pi}}\frac{e^{-\lambda^2}}{1-\operatorname{erf}(\lambda)}\left[-\frac{2\varepsilon K_0}{\sqrt{\pi}}\lambda\frac{e^{-\lambda^2}}{1-\operatorname{erf}(\lambda)} + 2\varepsilon K_0\lambda^2 - \varepsilon K_0 - 2\right] \\ &\quad + \frac{2\sqrt{a_1}q_0}{k_1(t_v - t_0)}e^{-\lambda^2} = 2\nu\lambda, \qquad \lambda > 0 \qquad\qquad \text{if } L_u = 1, \end{aligned} \tag{2.23}$$

or

$$\begin{aligned} &\frac{\sqrt{\pi a_1}q_0}{(t_v - t_0)}e^{-\lambda^2} + \frac{L_u\varepsilon K_0}{L_u - 1}k_2\left[\frac{1}{\sqrt{L_u}}F_1\left(\frac{\lambda}{\sqrt{L_u}}\right) - F_1(\lambda)\right] \\ &\quad = k_2F_1(\lambda) + \sqrt{\pi}k_1\nu\lambda, \qquad \lambda > 0 \qquad\qquad \text{if } L_u \neq 1. \end{aligned} \tag{2.24}$$

3 Discussion of the equation that determines λ, considering the case when the Luikov number equals to one

Now let's study in detail the equation (2.23), vinculated to the case $L_u = 1$, that is to say, when $a_m = a_1$. We define the following real functions:

$$\begin{aligned} \alpha(x) &= \frac{k_{21}}{\sqrt{\pi}}\frac{e^{-x^2}}{1-\operatorname{erf}(x)}\left[-\frac{2\varepsilon K_0}{\sqrt{\pi}}x\frac{e^{-x^2}}{1-\operatorname{erf}(x)} + 2\varepsilon K_0x^2 - \varepsilon K_0 - 2\right] \\ &\quad + \frac{2\sqrt{a_1}q_0}{k_1(t_v - t_0)}e^{-x^2} \end{aligned} \tag{3.1}$$

$$\chi(x) = 2\nu x \tag{3.2}$$

Then, equation (2.23) can be expressed saying that λ must be the solution of the following equation

$$\alpha(x) = \chi(x), \qquad x > 0. \tag{3.3}$$

We shall see the characteristics of each one of the functions α and χ which appears in equation (3.3).

Firstly, we have that χ is a strictly increasing function, with the properties:

$$\chi(0) = 0; \quad \chi(+\infty) = +\infty; \quad \chi'(x) = 2\nu > 0,\ x > 0.$$

Before we study the function α, let's define the following real functions:

$$Q(x) = \sqrt{\pi}\, x e^{-x^2}\left(1 - \operatorname{erf}(x)\right), \qquad x > 0$$
$$W(x) = \frac{x}{\sqrt{\pi}} \frac{e^{-x^2}}{1 - \operatorname{erf}(x)} - x^2 = x^2\left(\frac{1}{Q(x)} - 1\right), \qquad x > 0.$$

Function Q has the following properties:

$$Q\left(0^+\right) = 0; \quad Q(+\infty) = 1; \quad Q'(x) > 0,\ x > 0.$$

Function W is a positive valued function, with the following properties [12]:

$$W\left(0^+\right) = 0; \quad W(+\infty) = \frac{1}{2}; \quad W'(x) > 0$$

then W is a strictly increasing function. Now we take care about α. Taking into account W, we can put α in the following way:

$$\alpha(x) = \frac{2\sqrt{a_1} q_0}{k_1(t_v - t_0)} e^{-x^2} - \frac{k_{21}}{\sqrt{\pi}} F_1(x)\left[2\varepsilon K_0 W(x) + \varepsilon K_0 + 2\right]$$

where function F_1 is defined by

$$F_1(x) = \frac{e^{-x^2}}{1 - \operatorname{erf}(x)} \tag{3.4}$$

which has the following properties

$$F_1\left(0^+\right) = 1; \quad F_1(+\infty) = +\infty; \quad F_1'(x) > 0,\ x > 0; \quad F_1''(x) > 0,\ x > 0$$

$$\lim_{x \to +\infty} \frac{F_1(x)}{x} = \frac{\sqrt{\pi}}{Q(+\infty)} = \sqrt{\pi}.$$

Then α is written as the sum of two strictly decreasing functions, therefore it results that α is also a strictly decreasing one. Besides, it has the following properties:

$$\alpha(0) = \frac{2\sqrt{a_1} q_0}{k_1(t_v - t_0)} - \frac{k_{21}}{\sqrt{\pi}} [\varepsilon K_0 + 2] ; \quad \alpha(+\infty) = -\infty$$

$$\begin{aligned}\alpha'(x) &= \frac{-4x\sqrt{a_1} q_0}{k_1(t_v - t_0)} e^{-x^2} - \frac{k_{21}}{\sqrt{\pi}} \frac{e^{-x^2}}{1 - \operatorname{erf}(x)} \left[2\varepsilon K_0 W'(x) \right] \\ &\quad + \frac{(-k_{21})}{\sqrt{\pi}} F_1'(x) \left[2\varepsilon K_0 W(x) + \varepsilon K_0 + 2 \right] < 0, \qquad x > 0.\end{aligned}$$

Next, to assure that the two functions α and χ have an intersection point, we need to assume that

$$\alpha(0) > \chi(0),$$

that is to say, $\dfrac{2\sqrt{a_1} q_0}{k_1(t_v - t_0)} > \dfrac{k_{21}}{\sqrt{\pi}} [\varepsilon K_0 + 2]$, which is equivalent to the condition

$$q_0 > \frac{k_2(t_v - t_0)}{2\sqrt{\pi a_1}} \left[\varepsilon K_0 + 2 \right], \tag{3.5}$$

and we can finally give the following:

Theorem 3.1. *If the Luikov number is equals to one, and the coefficient* q_0 *verifies the condition* (3.5) *then there exists one and only one solution* $\lambda > 0$ *of the equation* (2.23). *Furthermore, the solution of the problem* (1.1)-(1.10) *is given by* (2.19)-(2.21), *where* λ *is the unique solution of the equation* (2.23), *that is:*

$$u_1(x, \tau) = u_v, \qquad 0 < x < s(\tau), \ \tau > 0 \tag{3.6}$$

$$\begin{aligned} t_1(x, \tau) &= 1 + \frac{q_0 \sqrt{\pi a_1}}{k_1(t_v - t_0)} \left(\operatorname{erf} \lambda - \operatorname{erf}\left(\frac{x}{2\sqrt{a_1 \tau}} \right) \right), \\ & 0 < x < s(\tau), \ \tau > 0 \end{aligned} \tag{3.7}$$

$$u_2(x,\tau) = \frac{1-\operatorname{erf}\left(\dfrac{x}{2\sqrt{a_m\tau}}\right)}{1-\operatorname{erf}(\lambda)}, \qquad x > s(\tau),\ \tau > 0 \tag{3.8}$$

$$\begin{aligned} t_2(\eta) &= \frac{\varepsilon K_0}{\sqrt{\pi}\,(1-\operatorname{erf}(\lambda))}\left[\lambda\, e^{-\lambda^2}\frac{1-\operatorname{erf}\left(\dfrac{x}{2\sqrt{a_1\tau}}\right)}{1-\operatorname{erf}(\lambda)} - \frac{x}{2\sqrt{a_1\tau}}e^{-\frac{x^2}{4a_1\tau}}\right] \\ &+ \frac{1-\operatorname{erf}\left(\dfrac{x}{2\sqrt{a_1\tau}}\right)}{1-\operatorname{erf}\lambda}, \qquad x > s(\tau),\ \tau > 0 \end{aligned} \tag{3.9}$$

$$s(\tau) = 2\lambda\sqrt{a_1\tau}. \tag{3.10}$$

4 Discussion of the equation that determines λ, considering the case when the Luikov number is different to one

In this paragraph we will study in detail the equation (2.24), which determines the unknown λ for the case $L_u \neq 1$, that is to say, $a_m \neq a_1$. For this propose, we define the following functions:

$$\phi(x) = \frac{\sqrt{\pi a_1}q_0}{(t_v - t_0)}e^{-x^2} + P(x) \tag{4.1}$$

$$\varphi(x) = k_2F_1(x) + \sqrt{\pi}k_1\nu x. \tag{4.2}$$

where

$$P(x) = \frac{L_u\ \varepsilon K_0}{L_u - 1}k_2\left(\frac{1}{\sqrt{L_u}}F_1\left(\frac{x}{\sqrt{L_u}}\right) - F_1(x)\right), \qquad x > 0. \tag{4.3}$$

Then, equation (2.24) can be written saying that λ must be the solution of the equation

$$\phi(x) = \varphi(x), \qquad x > 0. \tag{4.4}$$

Therefore, we can see the characteristics of each one of these functions ϕ and φ.

Firstly, let's see that $\varphi(x)$ is a strictly increasing function with the properties:

$$\varphi(0) = k_2\,; \quad \varphi(+\infty) = +\infty\,; \quad \varphi'(x) > 0,\ x > 0.$$

Before studying ϕ, we need to analyse the function P.

Obviously, function P when $x = 0^+$ is equal to $-\frac{\sqrt{L_u}\,\varepsilon K_0}{\sqrt{L_u}+1}k_2 < 0$, and when x tends to $+\infty$ its behaviour may depends on the value of L_u. Well, it doesn't happen:

We have

$$\lim_{x\to\infty}\left(\frac{F_1\left(\frac{x}{\sqrt{L_u}}\right)}{F_1(x)} - \sqrt{L_u}\right) = \frac{1}{\sqrt{L_u}} - \sqrt{L_u} = \frac{1-L_u}{\sqrt{L_u}},$$

so, we can verify that:

i) If $L_u > 1$, we have $\lim_{x\to\infty}\left(\frac{1}{\sqrt{L_u}}F_1\left(\frac{x}{\sqrt{L_u}}\right) - F_1(x)\right) = -\infty$, then $P(+\infty) = -\infty$.

ii) If $L_u < 1$, we have $\lim_{x\to\infty}\left(\frac{1}{\sqrt{L_u}}F_1\left(\frac{x}{\sqrt{L_u}}\right) - F_1(x)\right) = +\infty$, then $P(+\infty) = -\infty$.

Therefore, it doesn't matter whether L_u is less or greater than 1, $P(x)$ always tends to $-\infty$ when $x \to +\infty$. Then, the properties of $\phi(x)$ are:

$$\phi(0) = \frac{\sqrt{\pi a_1}q_0}{(t_v - t_0)} + P\left(0^+\right) = \frac{\sqrt{\pi a_1}q_0}{(t_v - t_0)} - \frac{\sqrt{L_u}\varepsilon K_0}{\sqrt{L_u}+1}k_2;\ \phi(+\infty) = -\infty$$

$$\phi'(x) = -\frac{2\sqrt{\pi a_1}q_0}{(t_v - t_0)}xe^{-x^2} + \frac{L_u\varepsilon K_0}{L_u - 1}k_2\left[\frac{1}{L_u}F_1'\left(\frac{x}{\sqrt{L_u}}\right) - F_1'(x)\right] < 0,\ x > 0.$$

Concluding, to assure an intersection point between the two functions ϕ and φ, we impose the condition $\phi(0) > \varphi(0)$, that is to say

$$\frac{\sqrt{\pi a_1}q_0}{(t_v - t_0)} - \frac{\sqrt{L_u}\varepsilon K_0}{\sqrt{L_u}+1}k_2 > k_2,$$

which is equivalent to

$$q_0 > k_2\left(1 + \frac{\sqrt{L_u}\varepsilon K_0}{1+\sqrt{L_u}}\right)\frac{t_v - t_0}{\sqrt{\pi a_1}}, \tag{4.5}$$

and we can finally give the following theorem:

Theorem 4.1. *If the Luikov number is different than one, and the coefficient* q_0 *verifies the condition* (4.5) *then there exists one and only one solution* $\lambda > 0$ *of the equation* (2.24). *Furthermore, the solution of the problem* (1.1)-(1.10) *is given by* (2.19)-(2.21), *where* λ *is the solution of the equation* (2.24), *that is:* (3.6), (3.7), (3.8), (3.10) *and*

$$t_2(\eta) = \frac{\varepsilon K_o L_u}{L_u - 1}\left[-\frac{1-\operatorname{erf}\left(\dfrac{x}{2\sqrt{a_m \tau}}\right)}{1-\operatorname{erf}\left(\dfrac{\lambda}{\sqrt{L_u}}\right)} + \frac{1-\operatorname{erf}\left(\dfrac{x}{2\sqrt{a_1 \tau}}\right)}{1-\operatorname{erf}(\lambda)}\right] + \frac{1-\operatorname{erf}\left(\dfrac{x}{2\sqrt{a_1 \tau}}\right)}{1-\operatorname{erf}\lambda}, \qquad x > s(\tau),\ \tau > 0. \tag{4.6}$$

Remark 1. The right side member of the inequality (4.5) goes to the right side member of the inequality (3.5) when L_u tends to 1, that is to say, we can study the case $L_u = 1$ considering the limit $L_u \to 1$ in the case $L_u \neq 1$, then we can resume both results in the following one:

Theorem 4.2. *Let be consider the coefficient* q_0 *verifying the condition* (4.5), *then, for any positive value of* L_u, *there exists one and only one solution* $\lambda > 0$ *of the equation* (2.23) *or* (2.24) *depending on what value takes* L_u. *Furthermore, the solution of the problem* (1.1)-(1.10) *is given by:*

a) (3.7)-(3.8), (3.9) *and* (3.10), *if* $L_u = 1$,

b) (3.7)-(3.8), (4.6) *and* (3.10), *if* $L_u \neq 1$.

5 Some illustrative results and a sufficient condition for the Luikov number in order to obtain the minimum value of the temperature distribution

Some results of sample calculations are shown here. In this examples we take $\varepsilon K_0 = 2$, $a_1 = 1$, $k_2 = 1$, and $(t_v - t_0) = 1$. Figure 1 shows the behaviour of λ as a function of q_0. Figure 2, 3 and 4 shows the behaviour of the dimensionless temperature with respect to the dimensionless variable η, taking L_u equals to 0.1, 1 and 4 respectively.

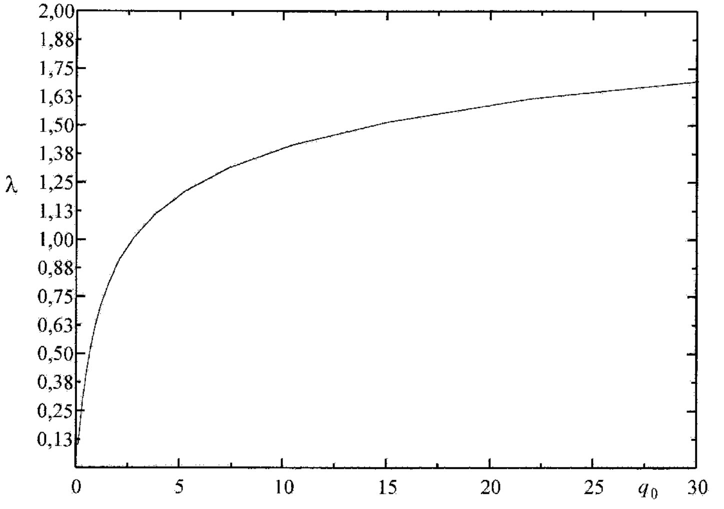

Figure 1 – Behavior of λ as a function of q_0.

Looking at Figures 2, 3 and 4, we see that the temperature distribution t_2 reaches to a minimum value which is smaller than the limit value t_0 that the function reaches at $+\infty$, i.e. the initial temperature, although in Figure 2 the function has no such minimum value. We shall find the values of the coefficient L_u for which the function T_2 has a minimum value which is smaller than its limit value when $\eta \to +\infty$.

For $L_u \neq 1$, we take $T_2(\eta)$ for any $\eta > \lambda$, and we have

$$T_2'(\eta) = \frac{\varepsilon K_o L_u}{L_u - 1}\left[\frac{\frac{2}{\sqrt{\pi Lu}}\, e^{\left(-\frac{\eta^2}{L_u}\right)}}{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)} - \frac{\frac{2}{\sqrt{\pi}}\, e^{(-\eta^2)}}{1 - \operatorname{erf}(\lambda)}\right] - \frac{\frac{2}{\sqrt{\pi}}\, e^{(-\eta^2)}}{1 - \operatorname{erf}\lambda}$$

and we get that

$$T_2'(\eta) = 0 \Leftrightarrow \frac{\varepsilon K_o \sqrt{L_u}}{L_u - 1}\,\frac{e^{\left(-\frac{\eta^2}{L_u}\right)}}{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)} = \left(\frac{\varepsilon K_o L_u}{L_u - 1} + 1\right)\frac{e^{(-\eta^2)}}{1 - \operatorname{erf}\lambda}$$

$\Leftrightarrow$ η is the solution of the following equation:

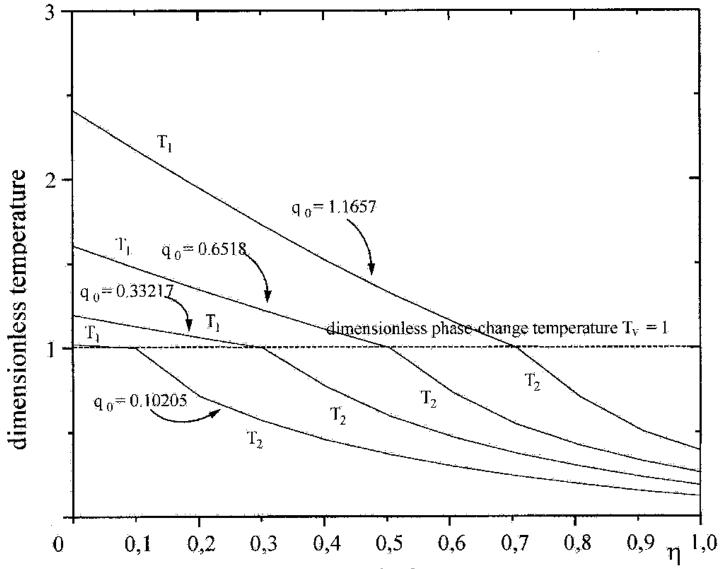

Figure 2 – Behavior of the temperature with respect to the dimensionless variable η considering $Lu = 0.1 < 1$.

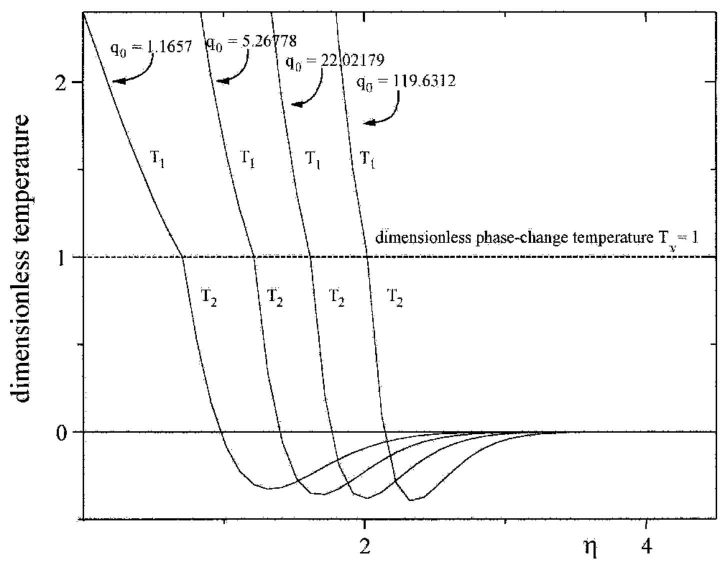

Figure 3 – Behavior of the temperature with respect to the dimensionless variable η considering $Lu = 1$.

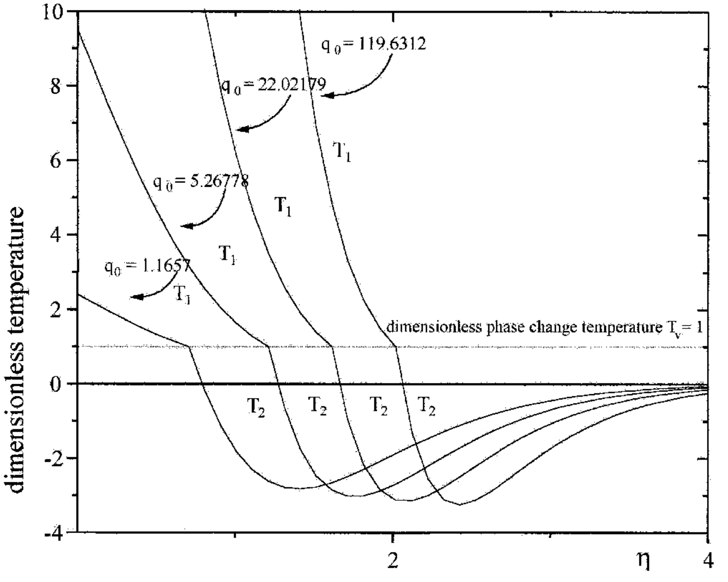

Figure 4 – Behavior of the temperature with respect to the dimensionless variable η considering $Lu = 4 > 1$.

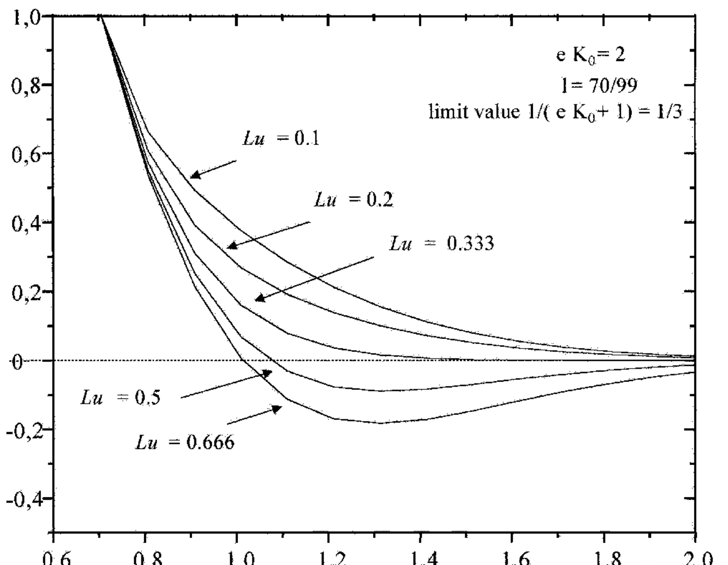

Figure 5 – Behavior of the temperature T_2 with respect to the dimensionless variable η varying the values of Lu.

$$S(x) = Z(x), \ x > \lambda, \tag{5.1}$$

where S and Z are defined by:

$$S(x) = \frac{\varepsilon K_o \sqrt{L_u}}{L_u - 1} \frac{e^{\left(-\frac{x^2}{L_u}\right)}}{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)} \tag{5.2}$$

$$Z(x) = \left(\frac{\varepsilon K_o L_u}{L_u - 1} + 1\right) \frac{e^{(-x^2)}}{1 - \operatorname{erf} \lambda} \tag{5.3}$$

Obviously, both S and Z are strictly decreasing (increasing) functions for any $x > 0$ when $L_u > 1$ $(0 < L_u < 1)$. Moreover, we have

$$S(x) = Z(x) \Leftrightarrow \ \varepsilon K_o \sqrt{L_u} \frac{e^{\left(-\frac{x^2}{L_u}\right)}}{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)} = ((\varepsilon K_0 + 1) L_u - 1) \frac{e^{(-x^2)}}{1 - \operatorname{erf} \lambda}$$

$$\Leftrightarrow \ e^{\left(1 - \frac{1}{L_u}\right) x^2} = \frac{((\varepsilon K_0 + 1) L_u - 1)}{\varepsilon K_o \sqrt{L_u}} \frac{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)}{1 - \operatorname{erf} \lambda}$$

which implies that in order to solve the equation (5.1), firstly we must to assume that $((\varepsilon K_0 + 1) L_u - 1) > 0$, that is

$$L_u > \frac{1}{\varepsilon K_0 + 1} \tag{5.4}$$

Secondly, if $L_u > 1$ we must to impose that

$$\frac{((\varepsilon K_0 + 1) L_u - 1)}{\varepsilon K_o \sqrt{L_u}} \frac{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)}{1 - \operatorname{erf} \lambda} > 1 \tag{5.5}$$

which is always satisfied taking into account that the error function is a strictly increasing function. Moreover, if $L_u < 1$ we must to impose that

$$\frac{((\varepsilon K_0 + 1) L_u - 1)}{\varepsilon K_o \sqrt{L_u}} \frac{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)}{1 - \operatorname{erf} \lambda} < 1 \tag{5.6}$$

which is satisfied for all $0 < L_u < 1$. Therefore, if the Luikov number L_u verifies the condition (5.4) we obtain that the solution of the equation $S(x) = Z(x)$ is given by

$$\eta = \sqrt{\left(\frac{L_u}{L_u - 1}\right) \log\left(\frac{((\varepsilon K_0 + 1)\, L_u - 1)}{\varepsilon K_o \sqrt{L_u}} \frac{1 - \operatorname{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)}{1 - \operatorname{erf} \lambda}\right)} \tag{5.7}$$

Then we have obtained the following result:

Theorem 5.1. *If the Luikov number L_u verifies the condition* (5.4) *the temperature distribution t_2 reaches to a minimum value which is smaller than the initial temperature or its limit value at $+\infty$. The minimum value is attained when the dimensionless variable η takes the value* (5.7).

Remark 2. For large Luikov number the temperature distribution $t_2 = t_2(\eta)$ has an absolute minimum value less than its initial temperature. Moreover, the minimum value for the Luikov number in order to have that property is given explicitely by the coefficient $\frac{1}{\varepsilon K_0 + 1}$, which is not an intuitive result.

6 Conclusion

Exact solutions for the problem of drying with coupled phase change in a porous medium with a heat flux condition on $x = 0$ of the type $-\frac{q_0}{\sqrt{\tau}}$, with $q_0 > 0$, for any value of L_u is obtained. This solution is only obtained when q_0 verifies a certain explicited inequality. The temperatures of the two phases and the mass-transfer potential were obtained by using the similarity method. Some illustrative results are shown. Finally, for large Luikov number $\left(\text{more precisely, } L_u > \frac{1}{\varepsilon K_0 + 1}\right)$ we obtain that the temperature distribution t_2 reaches to an absolute minimum value which is smaller than the initial temperature (or its limit value at $+\infty$), and we characterize the coordinate of this point when the dimensionless variable $\eta = \frac{x}{2\sqrt{a_1 \tau}}$ takes the value (5.7) as a function of the data.

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Nomenclature:

$a_i, i = 1, 2$	thermal diffusivity of the phase-i.
a_{12}	ratio of thermal diffusivities from phase 1 to phase 2
a_m	moisture diffusivity
c_m	specific mass capacity
c_2	specific heat capacity
$k_i,\ i = 1, 2$	thermal conductivity of the phase-i.
k_{21}	ratio of thermal conductivity from phase 2 to phase 1
$K_0 = \dfrac{Lc_m (u_0 - u_v)}{c_2 (t_v - t_0)}$	Kossovitch number
L	latent heat of evaporation of liquid per unit mass
$L_u = a_m / a_1$	Luikov number
q_0	coefficient that characterizes the heat flux at $x = 0$
$s(\tau)$	position of the evaporation front
$t_i(x, \tau), i = 1, 2$	temperature of the phase-i.
t_0	initial temperature
t_v	temperature at the phase-change state
$T_i,\ i = 1, 2$	non-dimensional temperature of the phase-i
u	mass-transfer potential
u_0	initial mass-transfer potential
$U_i, i = 1, 2$	dimensionless mass-transfer potential of the phase-i
x	space coordinate
X	dimensionless length

Greek symbols

ε	coefficient of internal evaporation
η	dimensionless variable
λ	dimensionless constant which characterizes the evaporation front
ρ_m	density of moisture
τ	time

Subscripts

0	at initial time, $t = 0$
1	dried porous medium, $0 < x < s(\tau)$
2	humid porous medium, $x > s(\tau)$
v	at evaporation front, $x = s(\tau)$

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