

## Partial sums of analytic functions of bounded turning with applications

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**Abstract.** In this article, we determine certain conditions under which the partial sums of the multiplier integral operators of analytic univalent functions of bounded turning are also of bounded turning.

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**Key words:** multiplier integral operator, partial sums, bounded turning.

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### 1 Introduction

Let  $\mathcal{H}$  be the class of functions analytic in the open unit disk  $U = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$  and  $\mathcal{H}[a, n]$  be the subclass of  $\mathcal{H}$  consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

Let  $\mathcal{A}$  be the subclass of  $\mathcal{H}$  consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \quad (1)$$

For  $0 \leq \mu < 1$ , let  $B(\mu)$  denote the class of functions  $f$  of the form (1) so that  $\Re\{f'\} > \mu \in U$ . The functions in  $B(\mu)$  are called functions of bounded turning

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(c.f. [1, Vol. II]). By the Nashiro-Warschowski Theorem (see e.g. [1, Vol. I]) the functions in  $B(\mu)$  are univalent and also close-to-convex in  $U$ .

For  $f$  of the form (1), several interesting families of integral operators, which have been investigated rather extensively in analytic function theory, including each of the following integral operators (see [2-10]),

$$\begin{aligned} P_b^a f(z) &= \frac{(b+1)^a}{z^b \Gamma(a)} \int_0^z t^{b-1} \left(\log \frac{z}{t}\right)^{a-1} f(t) dt \\ &= z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^a a_n z^n, \quad (z \in U) \\ &\quad (a > 0, \quad b > -1, \quad z \in U, \quad f \in \mathcal{A}) \end{aligned} \tag{2}$$

and

$$\begin{aligned} J_c f(z) &= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \\ &= z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n, \quad (z \in U) \\ &\quad (c > -1, \quad z \in U, \quad f \in \mathcal{A}). \end{aligned} \tag{3}$$

Also, we define a general integral operator as the following:

$$\begin{aligned} J_{\lambda, \delta}^k f(z) &= z + \sum_{n=2}^{\infty} \frac{a_n}{[1 + (n-1)\lambda]^k C(\delta, n)} z^n, \\ &\quad (k \in \mathbb{N}_0, \quad \lambda \geq 0, \quad \delta \geq 0, \quad z \in U, \quad f \in \mathcal{A}), \end{aligned} \tag{4}$$

where

$$C(\delta, n) = \binom{n + \delta - 1}{\delta} = \frac{\Gamma(n + \delta)}{\Gamma(n)\Gamma(\delta + 1)}.$$

**Remark 1.1.** When  $\lambda = 0$ , operator (4) gives Noor integral operator (see [11, 12]).

The  $m$ -th partial sums of the operators (2-4) are respectively given by

$$P_m(z) = z + \sum_{n=2}^m \left(\frac{b+1}{b+n}\right)^a a_n z^n, \quad (z \in U), \tag{5}$$

$$J_m(z) = z + \sum_{n=2}^m \left( \frac{c+1}{c+n} \right) a_n z^n, \quad (z \in U) \tag{6}$$

and

$$F_m(z) = z + \sum_{n=2}^m \frac{a_n}{[1 + (n-1)\lambda]^k C(\delta, n)} z^n, \quad (z \in U). \tag{7}$$

It was shown that for a normalized univalent function  $f$  of the form (1) the partial sums of the Libera integral operator of functions is starlike in  $|z| < \frac{3}{8}$ . The number  $\frac{3}{8}$  is sharp ([13]). In [14], it was also shown that the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning. We determine conditions under which the partial sums (5-7) of the multiplier integral operators (2-4) of analytic univalent functions of bounded turning are also of bounded turning. In the sequel we need to the following results.

**Lemma 1.1** [14]. *For  $z \in U$  we have*

$$\Re \left\{ \sum_{n=1}^j \frac{z^n}{n+2} \right\} > -\frac{1}{3}, \quad (z \in U).$$

**Lemma 1.2** [1, Vol. I]. *Let  $P(z)$  be analytic in  $U$ , such that  $P(0) = 1$ , and  $\Re(P(z)) > \frac{1}{2}$  in  $U$ . For functions  $Q$  analytic in  $U$  the convolution function  $P * Q$  takes values in the convex hull of the image on  $U$  under  $Q$ .*

The operator  $(*)$  stands for the Hadamard product or convolution of two power series in  $\mathcal{A}$ ,

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U).$$

## 2 Main Results

By making use of Lemma 1.1 and Lemma 1.2, we illustrate the conditions under which the  $m$ -th partial sums (5-7) of the multiplier integral operators (2-4) of analytic univalent functions of bounded turning are also of bounded turning.

**Theorem 2.1.** Let  $f \in \mathcal{A}$ . If  $\frac{1}{2} < \mu < 1$  and  $f(z) \in B(\mu)$ , then

$$P_m(z) \in B\left(\frac{3 - (b+1)^a(1-\mu)}{3}\right), \quad 0 < a \leq 1 \text{ and } -1 < b \leq 1.$$

**Proof.** Let  $f$  be of the form (1) and  $f(z) \in B(\mu)$  that is

$$\Re\{f'(z)\} > \mu, \quad \left(\frac{1}{2} < \mu < 1, \quad z \in U\right).$$

This implies

$$\Re\left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\} > \mu > \frac{1}{2}.$$

Now for  $\frac{1}{2} < \mu < 1$  we have

$$\Re\left\{1 + \sum_{n=2}^{\infty} a_n \frac{n}{1-\mu} z^{n-1}\right\} > \Re\left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\}$$

then

$$\Re\left\{1 + \sum_{n=2}^{\infty} \frac{n}{1-\mu} a_n z^{n-1}\right\} > \frac{1}{2}. \quad (8)$$

Applying the convolution properties of power series to  $P'_m(z)$  we may write

$$\begin{aligned} P'_m(z) &= 1 + \sum_{n=2}^m \left(\frac{b+1}{b+n}\right)^a na_n z^{n-1} \\ &= \left[1 + \sum_{n=2}^m \frac{n}{(1-\mu)} a_n z^{n-1}\right] * \left[1 + \sum_{n=2}^m \left(\frac{b+1}{b+n}\right)^a (1-\mu)z^{n-1}\right] \quad (9) \\ &:= P(z) * Q(z). \end{aligned}$$

In virtue of Lemma 1.1 and for  $j = m - 1$ , we receive

$$\Re\left\{\sum_{n=2}^m \frac{z^{n-1}}{n+1}\right\} > -\frac{1}{3}. \quad (10)$$

Thus for  $0 < a \leq 1$  and  $-1 < b \leq 1$  yields

$$\Re\left\{\sum_{n=2}^m \frac{z^{n-1}}{(b+n)^a}\right\} \geq \Re\left\{\sum_{n=2}^m \frac{z^{n-1}}{n+1}\right\}. \quad (11)$$

Hence

$$\Re \left\{ \sum_{n=2}^m \frac{z^{n-1}}{(b+n)^a} \right\} > -\frac{1}{3}. \tag{12}$$

A computation gives

$$\Re \{Q(z)\} = \Re \left\{ 1 + \sum_{n=2}^m \left(\frac{b+1}{b+n}\right)^a (1-\mu)z^{n-1} \right\} > \frac{3 - (b+1)^a(1-\mu)}{3}.$$

On the other hand, the power series

$$P(z) = \left[ 1 + \sum_{n=2}^m \frac{n}{(1-\mu)} a_n z^{n-1} \right], \quad (z \in U)$$

satisfies:  $P(0) = 1$  and

$$\Re \{P(z)\} = \Re \left\{ 1 + \sum_{n=2}^m \frac{n}{(1-\mu)} a_n z^{n-1} \right\} > \frac{1}{2}, \quad (z \in U).$$

Therefore, by Lemma 1.2, we have

$$\Re \left\{ P'_m(z) \right\} > \frac{3 - (b+1)^a(1-\mu)}{3}, \quad (z \in U).$$

This completes the proof of Theorem 2.1. □

In the next corollary, we establish the conditions of the partial sums of the operator (3) to be of bounded turning when  $f$  is of bounded turning.

**Corollary 2.1.** *Let  $f \in \mathcal{A}$ . If  $\frac{1}{2} < \mu < 1$  and  $f(z) \in B(\mu)$ , then  $J_m(z) \in B\left(\frac{3-(c+1)(1-\mu)}{3}\right)$ .*

**Proof.** Setting  $a = 1$  and  $b = c$  in Theorem 2.1 leads to Corollary 2.1.

**Corollary 2.2.** *Let  $f \in \mathcal{A}$ . If  $\frac{1}{2} < \mu < 1$  and  $f(z) \in B(\mu)$ , then  $L_m(z) \in B\left(\frac{1+2\mu}{3}\right)$ , where  $L(z)$  denotes the Libera integral operator:*

$$L(z) = \frac{2}{z} \int_0^z f(\zeta) d\zeta = z + \sum_{n=2}^{\infty} \left[ \frac{2}{n+1} \right] a_n z^n, \quad (z \in U)$$

and its  $m$ -th partial sums are given by

$$L_m(z) = z + \sum_{n=2}^m \left[ \frac{2}{n+1} \right] a_n z^n, \quad (z \in U).$$

**Proof.** Setting  $a = b = 1$  in Theorem 2.1 leads to Corollary 2.2.

**Corollary 2.3.** Let  $f \in \mathcal{A}$ . If  $\frac{1}{2} < \mu < 1$  and  $f(z) \in B(\mu)$ , then  $S_m(z) \in B\left(\frac{2+\mu}{3}\right)$ , where  $S^k(z)$  denotes the integral operator which analogous to one defined by Sălăgean (see [15]):

$$S^k(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{n^k} z^n, \quad k \in \mathbb{N}_0, \quad (z \in U) \quad (13)$$

and its  $m$ -th partial sums are given by

$$S_m(z) = z + \sum_{n=2}^m \frac{a_n}{n^k} z^n, \quad k \in \mathbb{N}_0, \quad (z \in U).$$

**Proof.** Setting  $a = k$ ,  $b = 0$  in Theorem 2.1 leads to Corollary 2.3.

**Theorem 2.2.** Let  $f \in \mathcal{A}$ . If  $\frac{1}{2} < \mu < 1$  and  $f(z) \in B(\mu)$ , then

$$F_m(z) \in B\left(\frac{2+\mu}{3}\right), \quad \delta = 0, \quad k = 1, \quad \text{and} \quad 0 \leq \lambda \leq 1.$$

**Proof.** By the hypotheses of the theorem we have

$$\Re \left\{ 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right\} > \mu > \frac{1}{2}.$$

This implies, for  $\frac{1}{2} < \mu < 1$ ,

$$\Re \left\{ 1 + \sum_{n=2}^{\infty} a_n \frac{n}{1-\mu} z^{n-1} \right\} > \Re \left\{ 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right\}$$

then

$$\Re \left\{ 1 + \sum_{n=2}^{\infty} \frac{n}{1-\mu} a_n z^{n-1} \right\} > \frac{1}{2}. \quad (14)$$

Applying the convolution properties of power series to  $F'_m(z)$  we have

$$\begin{aligned}
 F'_m(z) &= 1 + \sum_{n=2}^m \frac{na_n}{[1 + (n - 1)\lambda]^k C(\delta, n)} z^{n-1} \\
 &= \left[ 1 + \sum_{n=2}^m \frac{na_n}{(1 - \mu)} z^{n-1} \right] \\
 &\quad \times \left[ 1 + \sum_{n=2}^m \frac{(1 - \mu)}{[1 + (n - 1)\lambda]^k C(\delta, n)} z^{n-1} \right] \\
 &:= P(z) * Q(z).
 \end{aligned}
 \tag{15}$$

In view of Lemma 1.1 with

$$j = m - 1, \quad \delta = 0, \quad k = 1, \quad \text{and} \quad 0 \leq \lambda \leq 1,
 \tag{16}$$

yields

$$\Re \left\{ \sum_{n=2}^m \frac{z^{n-1}}{[1 + (n - 1)\lambda]} \right\} > \Re \left\{ \sum_{n=2}^m \frac{z^{n-1}}{n + 1} \right\}.
 \tag{17}$$

Hence

$$\Re \left\{ \sum_{n=2}^m \frac{z^{n-1}}{[1 + (n - 1)\lambda]} \right\} > -\frac{1}{3}.
 \tag{18}$$

Under the conditions given in (16) we obtain

$$\Re \{Q(z)\} = \Re \left\{ 1 + \sum_{n=2}^m \frac{(1 - \mu)}{[1 + (n - 1)\lambda]} z^{n-1} \right\} > \frac{2 + \mu}{3}.$$

On the other hand, the power series

$$P(z) = \left[ 1 + \sum_{n=2}^m \frac{n}{(1 - \mu)} a_n z^{n-1} \right], \quad (z \in U)$$

satisfies:  $P(0) = 1$  and

$$\Re \{P(z)\} = \Re \left\{ 1 + \sum_{n=2}^m \frac{n}{(1 - \mu)} a_n z^{n-1} \right\} > \frac{1}{2}, \quad (z \in U).$$

Therefore, by Lemma 1.2, we have

$$\Re \{P'_m(z)\} > \frac{2 + \mu}{3}, \quad \left( \frac{1}{2} < \mu < 1 \right).$$

The proof of Theorem 2.2 is complete. □

**Corollary 2.4.** *Let  $f \in \mathcal{A}$ . If  $\frac{1}{2} < \mu < 1$  and  $f(z) \in B(\mu)$ , then  $S_m(z) \in B\left(\frac{2+\mu}{3}\right)$ , where  $S(z)$  defined in (13) of order one.*

**Proof.** Setting  $\lambda = 1$  in Theorem 2.2 leads to Corollary 2.4.

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