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Solutions to the recurrence relation $u_{n+1} = v_{n+1} + u_n \otimes v_n$ in terms of Bell polynomials

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Abstract. Motivated by time series analysis, we consider the problem of solving the recurrence relation $u_{n+1} = v_{n+1} + u_n \otimes v_n$ for $n \neq 0$ and u_n, given the sequence v_n. A solution is given as a Bell polynomial. When v_n can be written as a weighted sum of nth powers, then the solution u_n also takes this form.

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1 Introduction and summary

We define the convolution of sequences $\{a_n, b_n : n \geq m\}$ as

$$a_n \otimes b_n = \sum_{j=0}^{n} a_j b_{n-j}$$

for $n \geq 0$. We consider the recurrence equation for u_n,

$$u_{n+1} = v_{n+1} + u_n \otimes v_n \tag{1.1}$$

for $n \geq 0$, where $u_0 = v_0$, and v_n is a given sequence.

The need for solutions to (1.1) arises with respect to the distribution of the maximum of first order autoregressive processes and more generally to that of

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autoregressive processes of any order, see Withers and Nadarajah (2010). There are many papers studying the distribution of the maximum of autoregressive processes, see, for example, Chernick and Davis (1982), McCormick and Mathew (1989), McCormick and Park (1992), Borkovec (2000), Ol'shanskii (2004) and Elek and Zempléni (2008). However, the results either give the limiting extreme value distributions or assume that the errors come from a specific class (for example, uniform distributed errors, negative binomial errors, $ARCH(1)$ errors, etc). We are aware of no work giving the *exact* distribution of the maximum of autoregressive processes. As explained in Withers and Nadarajah (2010), solutions to (1.1) will lead to the exact distribution of the maximum of autoregressive processes of any order.

The aim of this note to provide solutions for (1.1) accessible for all scientists, not just mathematicians. These solutions are given in terms of Bell polynomials. In-built routines for Bell polynomials are available in most computer algebra packages. For example, see BellB in Mathematica and IncompleteBellPoly in Matlab. So, the solutions given will be accessible to most practitioners.

In Section 2, a solution for (1.1) is given as a Bell polynomial. In order to investigate the behavior of u_n for large n, an assumption needs to made on the behavior of v_n for large n. In Section 3, we show that when v_n can be written as a weighted sum of nth powers, then the solution u_n also has this form. This assumption is extended in Section 4 to the case when the weights are not constants, but polynomials in n. Some conclusions and future work are noted in Section 5.

2 The solution as a Bell polynomial

Theorem 2.1 provides an explicit solution of the recurrence relation (1.1) in terms of the *complete ordinary Bell polynomial*, $\widehat{B}_n(\mathbf{w})$, a function of $(w_1, \ldots, w_n)$, defined for any sequence $\mathbf{w} = (w_1, w_2, \ldots)$ by the formal generating function

$$(1 - W(t))^{-1} = \sum_{n=0}^{\infty} \widehat{B}_n(\mathbf{w}) t^n,$$

where

$$W(t) = \sum_{n=1}^{\infty} w_n t^n.$$

The solution given by Theorem 2.1 can be computed by a single call to the in-built routine, BellB, in Mathematica or some other equivalent computer package. We believe that this is the most direct and the most efficient way to calculate a solution for the recurrence relation (1.1). However, in the absence of computer packages, the complete ordinary Bell polynomial can be calculated using the recurrence relation derived by Theorem 2.2.

Theorem 2.1. *The recurrence relation* (1.1) *has the solution:*

$$u_n = \widehat{B}_{n+1}(\mathbf{w}) \tag{2.1}$$

for $n \geq 0$*, where* $w_n = v_{n-1}$.

Proof. Multiply (1.1) by t^n and sum from $n = 0$ to obtain

$$(U(t) - V(t))/t = U(t)V(t),$$

where

$$U(t) = \sum_{n=0}^{\infty} u_n t^n, \;\; V(t) = \sum_{n=0}^{\infty} v_n t^n,$$

$$U(t) = (1 - W(t))^{-1} V(t)$$

since $W(t) = tV(t)$, and

$$1 + tU(t) = 1 + (1 - W(t))^{-1} W(t) = (1 - tV(t))^{-1}. \tag{2.2}$$

Taking the coefficient of t^n in the last line gives the explicit solution (2.1). □

Theorem 2.2. *A recurrence relation for* $b_n = \widehat{B}_n(\mathbf{w})$ *is given by*

$$b_n = w_n \otimes b_n$$

for $n \geq 1$*, where* $w_0 = 0$ *and* $b_0 = 1$*. For example,*

$$\begin{aligned} b_1 &= w_0 b_1 + w_1 b_0 = w_1, \\ b_2 &= w_0 b_2 + w_1 b_1 + w_2 b_0 = w_1 b_1 + w_2 = w_1^2 + w_2, \\ b_3 &= w_0 b_3 + w_1 b_2 + w_2 b_1 + w_3 b_0 = w_1 b_2 + w_2 b_1 + w_3 \\ &= w_1^3 + 2 w_1 w_2 + w_3, \end{aligned}$$

and so on.

Proof. Follows by taking the coefficient of t^n in $(1-x)^{-1} - 1 = x(1-x)^{-1}$, where $x = W(t)$. □

An alternative to the complete ordinary Bell polynomial is the *partial ordinary Bell polynomial*, $\widehat{B}_{nj}(\mathbf{w})$, defined by

$$W(t)^j = \sum_{n=j}^{\infty} \widehat{B}_{nj}(\mathbf{w}) t^n$$

for $0 \le j \le n$. For example,

$$\widehat{B}_{n0}(\mathbf{w}) = \delta_{n0}, \ \widehat{B}_{n1}(\mathbf{w}) = w_n, \ \widehat{B}_{nn}(\mathbf{w}) = w_1^n,$$

where $\delta_{00} = 1$ and $\delta_{n0} = 0$ for $n \ge 0$. These polynomials are tabled on page 309 of Comtet (1974) for $1 \le n \le 10$. Recurrence formulas for them are also given by Comtet (1974). They can be computed by a single call to the in-built routine, IncompleteBellPoly, in Matlab.

Theorem 2.3 states the relationship between the complete ordinary Bell polynomial and the partial ordinary Bell polynomial. It also provides a recurrence relation for the latter. Corollary 2.1 derives the relationship between u_n and v_n. Corollary 2.2 derives the reciprocal relationship between v_n and u_n.

Theorem 2.3. *We have*

$$\widehat{B}_n(\mathbf{w}) = \sum_{j=0}^{n} \widehat{B}_{nj}(\mathbf{w}). \tag{2.3}$$

A recurrence relation for $b_{nj} = \widehat{B}_{nj}(\mathbf{w})$ *is*

$$b_{n,j_1+j_2} = \sum_{n_1+n_2=n}^{n} b_{n_1 j_1} b_{n_2 j_2} \tag{2.4}$$

for $j_1, j_2 \ge 0$. *For example,*

$$b_{n,j+1} = \sum_{n_1+n_2=n} b_{n_1 j} w_{n_2} = \sum_{k=1}^{n-1} b_{kj} w_{n-k}$$

for $j \ge 1$.

Proof. Take the coefficient of t^n in the expansion for $(1 - W(t))^{-1}$ to obtain (2.3). The recurrence relation (2.4) follows by taking the coefficient of t^n in $W(t)^{j_1+j_2} = W(t)^{j_1} W(t)^{j_2}$. The proof is complete. □

Corollary 2.1. *We have*

$$\begin{aligned}
u_0 &= v_0,\\
u_1 &= v_1 + v_0^2,\\
u_2 &= v_2 + 2v_0v_1 + v_0^3,\\
u_3 &= v_3 + \left(2v_0v_2 + v_1^2\right) + 3v_0^2v_1 + v_0^4,\\
u_4 &= v_4 + (2v_0v_3 + 2v_1v_2) + \left(3v_0^2v_2 + 3v_0v_1^2\right) + 4v_0^3v_1 + v_0^5,\\
u_5 &= v_5 + \left(2v_0v_4 + 2v_1v_3 + v_2^2\right) + \left(3v_0^2v_3 + 6v_0v_1v_2 + v_1^3\right)\\
&\quad + \left(4v_0^3v_2 + 6v_0^2v_1^2\right) + 5v_0^4v_1 + v_0^6,\\
u_6 &= v_6 + (2v_0v_5 + 2v_1v_4 + 2v_2v_3)\\
&\quad + \left(3v_0^2v_4 + 6v_0v_1v_3 + 3v_0v_2^2 + 3v_1^2v_2\right)\\
&\quad + \left(4v_0^3v_3 + 12v_0^2v_1v_2 + 4v_0v_1^3\right) + \left(5v_0^4v_2 + 10v_0^3v_1^2\right) + 6v_0^5v_1 + v_0^7,\\
u_7 &= v_7 + \left(2v_0v_6 + 2v_1v_5 + 2v_2v_4 + v_3^2\right)\\
&\quad + \left(3v_0^2v_5 + 6v_0v_1v_4 + 6v_0v_2v_3 + 3v_1^2v_3 + 3v_1v_2^2\right)\\
&\quad + \left(4v_0^3v_4 + 12v_0^2v_1v_3 + 6v_0^2v_2^2 + 12v_0v_1^2v_2 + v_1^4\right)\\
&\quad + \left(5v_0^4v_3 + 20v_0^3v_1v_2 + 10v_0^2v_1^3\right)\\
&\quad + \left(6v_0^5v_2 + 15v_0^4v_1^2\right) + 7v_0^6v_1 + v_0^8,\\
u_8 &= v_8 + (2v_0v_7 + 2v_1v_6 + 2v_2v_5 + v_3v_4)\\
&\quad + \left(3v_0^2v_6 + 6v_0v_1v_5 + 6v_0v_2v_4 + 3v_0v_3^2 + 3v_1^2v_4 + 6v_1v_2v_3 + v_2^3\right)\\
&\quad + \left(4v_0^3v_5 + 12v_0^2v_1v_4 + 12v_0^2v_2v_3 + 12v_0v_1^2v_3 + 12v_0v_1v_2^2 + 4v_1^3v_2\right)\\
&\quad + \left(5v_0^4v_4 + 20v_0^3v_1v_3 + 10v_0^3v_2^2 + 30v_0^2v_1^2v_2 + 5v_0v_1^4\right)\\
&\quad + \left(6v_0^5v_3 + 30v_0^4v_1v_2 + 20v_0^3v_1^3\right) + \left(7v_0^6v_2 + 21v_0^5v_1^2\right)\\
&\quad + 8v_0^7v_1 + v_0^9,
\end{aligned}$$

$$
\begin{aligned}
u_9 \;=\; & v_9 + \left(2v_0v_8 + 2v_1v_7 + 2v_2v_6 + 2v_3v_5 + v_4^2\right) \\
& + \left(3v_0^2v_7 + 6v_0v_1v_6 + 6v_0v_2v_5 + 6v_0v_3v_4 + 3v_1^2v_5\right. \\
& \left. + 6v_1v_2v_4 + 3v_1v_3^2 + 3v_2^2v_3\right) + \left(4v_0^3v_6 + 12v_0^2v_1v_5 + 12v_0^2v_2v_4\right. \\
& \left. + 6v_0^2v_3^2 + 12v_0v_1^2v_4 + 24v_0v_1v_2v_3 + 4v_0v_2^3 + 4v_1^3v_3 + 6v_1^2v_2^2\right) \\
& + \left(5v_0^4v_5 + 20v_0^3v_1v_4 + 20v_0^3v_2v_3 + 30v_0^2v_1^2v_3 + 30v_0^2v_1v_2^2\right. \\
& \left. + 20v_0v_1^3v_2 + v_1^5\right) + \left(6v_0^5v_4 + 30v_0^4v_1v_3 + 15v_0^4v_2^2\right. \\
& \left. + 60v_0^3v_1^2v_2 + 15v_0^2v_1^4\right) + \left(7v_0^6v_3 + 42v_0^5v_1v_2 + 35v_0^4v_1^3\right) \\
& + \left(8v_0^7v_2 + 28v_0^6v_1^2\right) + 9v_0^8v_1 + v_0^{10}. \qquad (2.5)
\end{aligned}
$$

Proof. Applying (2.3) to (2.1), and reading the partial polynomials from Comtet's table, we obtain the results. □

Corollary 2.2. *We have* $v_n = -\widehat{B}_{n+1}(\mathbf{x})$ *for* $n \geq 0$*, where* $x_n = -u_{n-1}$.

Proof. This follows from $W(t) = 1 - (1 - X(t))^{-1}$, where $X(t) = -tU(t) = \sum_{n=1}^{\infty} x_n t^n$. Alternatively, it follows by writing (1.1) as $u'_{n+1} = v'_{n+1} + u'_n \otimes v'_n$, $n \geq 0$, $u'_0 = v'_0$, where $u'_n = -v_n$, $v'_n = -u_n$ and applying (2.1). □

3 The weighted sum of powers solution

The solution (2.1) gives no indication of the behavior of u_n for large n. To obtain this we need to make an assumption on the behavior of v_n for large n. Then if $V(t)$ is tractable, we can obtain u_{n-1}, $n \geq 1$, as the coefficient of t^n in (2.2). Theorems 3.1 and 3.2 illustrate this by two methods.

Theorem 3.1. *Suppose* v_n *is a weighted sum of powers, at least for large enough* n*, say*

$$v_n = \sum_{j=1}^{r} b_j \nu_j^n \qquad (3.1)$$

for $n \geq n_0$*, where* $1 \leq r \leq \infty$*,* $n_0 \geq 0$*. Suppose also* $\{\delta_j\}$ *are the roots of*

$$\sum_{k=1}^{r} b_k \nu_k^{n_0} / (\delta - \nu_k) = p_{n_0}(\delta), \qquad (3.2)$$

where $p_{n+1}(\delta) = \delta^{n+1} - v_n \otimes \delta^n$. *Assume that* $\{b_j\}$ *are all non-zero and that* $\{\nu_j\}$ *are all non-zero and distinct. Assume also that the* r *roots of* (3.2) *are all distinct. Then* u_n *has the form*

$$u_n = \sum_{j=1}^{J} \gamma_j \delta_j^n \tag{3.3}$$

for $n \geq m_0$, *where* $J \leq I' = r + n_0$, $m_0 = 2n_0 - 1$ *if* $n_0 \geq 1$ *and* $m_0 = 0$ *if* $n_0 = 0$.

Proof. Having found $\{\delta_j\}$, $\{\gamma_j\}$ are the roots of

$$\sum_{j=1}^{J} A_{jn_0}(\nu)\gamma_j = q_{n_0}(\nu) \tag{3.4}$$

for $\nu = \nu_1, \ldots, \nu_I$, where $A_{jn}(\nu) = \delta_j^n/(\delta_j - \nu)$ and $q_{n+1}(\nu) = \nu^{n+1} + u_n \otimes \delta^n$. Note (3.4) can be written

$$\mathbf{A}_{n_0}\boldsymbol{\gamma} = \mathbf{Q}_{n_0}, \tag{3.5}$$

where $(\mathbf{A}_n)_{kj} = A_{jn}(\nu_k)$, $\mathbf{Q}_n = (Q_{n1}, \ldots, Q_{nI})'$ and $Q_{nk} = q_n(\nu_k)$. So, if $J = r$, a solution is

$$\boldsymbol{\gamma} = \mathbf{A}_{n_0}^{-1}\mathbf{Q}_{n_0}.$$

If $r = \infty$, numerical solutions can be found by truncating the infinite matrix $\mathbf{A}_n$ and infinite vectors $(\mathbf{Q}_n, \boldsymbol{\gamma})$ to $N \times N$ matrix and N-vectors, then increasing N until the desired precision is reached. The proof, which is by substitution, assumes that $\{\delta_j, \nu_j\}$ are all distinct. The proof relies on the fact that if $\sum_{j=1}^{J} a_j r_j^n = 0$ for $1 \leq n \leq J$ and $r_1, \ldots, r_J$ are distinct, then $a_1 = \cdots = a_J = 0$, since $\det\left(r_j^n : 1 \leq n, j \leq J\right) \neq 0$.

If $J < r$, a solution is given by dropping $r - J$ rows of (3.5). If $J > r$, there are not enough equations for a solution by this method. If $n_0 \geq 2$, the values of u_n for $n < 2n_0 - 2$ can be found from (1.1) or (2.1) or the extension of (2.5).

Now suppose that $n_0 \geq 1$ and $n = 2n_0 - 1$. Then $s_2 = 0$ so that the result remains true. □

Set $f(\delta)$ equal to the left and right hand sides of (3.2). Then a sufficient condition that its roots are distinct, is that $f(\delta) = 0$ implies $\dot{f}(\delta) = 0$.

The second and more general method is to compute u_n from its generating function via the generating function of v_n and (2.2). The advantages of this method are: (i) it always applies, provided that the generating functions exist near $t = 0$; (ii) we shall see that it becomes clear how to extend the method when multiple roots exist.

Theorem 3.2. *Suppose* (3.1) *holds. Then u_n has the form*

$$u_{n-1} = \sum_{j=1}^{R} T_j^n p_j(n) \tag{3.6}$$

for some R, T_j and $p_j(n)$, a polynomial of some degree n_j.

Proof. Again we begin with (3.1). The generating function is $V(t) = V_0 + V_1$, where $V_0 = \sum_{n=0}^{n_0-1} v_n t^n$, $V_1 = \sum_{j=1}^{r} b_j s_j^{n_0}/(1 - s_j)$, $s_j = \nu_j t$. Set $D = \prod_{j=1}^{r}(1-s_j)$, $N = DV_1$. Then $D(1-tV(t)) = L$, where $L = D(1-tV_0)-tN$ is a polynomial of degree $J = r + n_0 > r$, assuming that $n_0 > 0$. So, we can write $L = \prod_{k=1}^{J}(1 - tt_k)$ say, and

$$1 + tU(t) = (1 - tV(t))^{-1} = D/L.$$

Suppose first that $\{t_k\}$ are distinct. Then by the usual partial fractions expansion,

$$D/L = \sum_{k=1}^{J} q_k(t_k) / (1 - tt_k),$$

where $q_k(t) = D/L_k$ and $L_k = L/(1 - tt_k)$. So,

$$u_{n-1} = \sum_{k=1}^{J} q_k(t_k)\, t_k^n$$

for $n \geq 1$. If $n_0 = 0$ then $V_0 = 0$, $U(t) = N/L = \sum_{k=1}^{r} m_k(t_k)/(1 - tt_k)$, where $m_k(t) = M/L_k$ so that

$$u_n = \sum_{k=1}^{r} m_k(t_k)\, t_k^n$$

for $n \geq 0$. Now suppose first that $\{t_k\}$ are not distinct. Then we can write $L = \prod_{j=1}^{R}(1 - tT_j)^{n_j}$, where $\prod_{j=1}^{R} n_j = J$ and $\{T_k\}$ are distinct. By the general partial fraction expansion, (compare Section 2.10 of Gradshteyn and Ryzhik (2007)),

$$D/L = \sum_{j=1}^{R} \sum_{k=1}^{n_j} c_{jk} / \left(1 - tT_j\right)^k,$$

where $c_{j,n_j-k+1} = Q_j^{(k-1)}(T_j^{-1})/(k-1)!$ and $Q_j(t) = (1 - tT_j)^{n_j} D/L$. So, (3.6) holds, where

$$p_j(n) = \sum_{k=1}^{n_j} c_{jk} \binom{k+n-1}{k-1}$$

for $n \geq 1$. So, $p_j(n)$ is a polynomial of degree n_j. If $n_0 = 0$ just replace D/L by N/L as for the case of distinct roots; then u_n is equal to the right hand side of (3.6) for $n \geq 0$. □

Corollaries 3.1 to 3.3 deduce the asymptotics of u_n and v_n as $n \to \infty$ when (3.1), (3.3) and (3.6) are satisfied. Further particular cases are considered by Corollaries 3.4 to 3.7.

Corollary 3.1. *If* (3.1) *holds then*

$$v_n \approx B_n r_1^n$$

as $n \to \infty$, where

$$B_n = \sum_j \left\{ b_j \exp\left(i\theta_j n\right) : \left|\nu_j\right| = r_1 \right\} = O(1),$$

$$r_1 = \max_{j=1}^{I} \left|\nu_j\right|, \ \nu_j = r_j \exp\left(i\theta_j\right).$$

Suppose in addition $r = 2$ and $n_0 = 1$. By (3.1), *$V(t) = \sum_{j=1}^{2} b_j (1 - \nu_j t)^{-1} = N(t)/D$ when $\max_{j=1}^{2} |\nu_j t| < 1$, where $N(t) = b_1(1 - \nu_2 t) + b_2(1 - \nu_1 t)$, $D = \sum_{i=0}^{2} d_i t^i$ for $d_0 = 1$, $d_1 = -\nu_1 - \nu_2$ and $d_2 = \nu_1 \nu_2$. So, $1 - tV(t) = M(t)/D$, where $M(t) = D - tN(t) = 1 - c_1 + c_2 t^2$, $c_1 = \nu_1 \nu_2 + b_1 + b_2$ and $c_2 = \nu_1 \nu_2 + b_1 \nu_2 + b_2 \nu_1$.*

Corollary 3.2. *If* (3.3) *holds then*

$$u_n \approx C_n r_2^n$$

as $n \to \infty$, where

$$C_n = \sum_j \left\{ b_j \exp\left(i\psi_j n\right) : \left|\delta_j\right| = R_1 \right\} = O(1),$$

$$R_1 = \max_{j=1}^{J} \left|\delta_j\right|, \ \delta_j = R_j \exp\left(i\psi_j\right).$$

Corollary 3.3. *If* (3.6) *holds, set $T_j = r_j \exp(i\psi_j)$, $r = \max_{j=1}^{r} r_j$ and $N = \max\{n_j : r_j = r\}$. Then*

$$u_{n-1} \approx r^n \sum_{r_j=r, n_j=n} \exp\left(in\psi_j\right) c_{jN} \binom{N+n-1}{N-1}$$

$$\approx r^n n^{N-1} D_n/(N-1)!$$

for

$$D_n = \sum_{r_j=r, n_j=n} \exp\left(in\psi_j\right) c_{jN} = O(1).$$

Corollary 3.4. *If $c_2 \neq 0$ and $M(t)$ has two distinct roots then the solution* (3.3) *holds with $J = 2$.*

Corollary 3.5. *Suppose $c_2 = 0 \neq c_1$. Then $M(t)$ has one root and* (3.3) *holds with $J = 1$. Alternatively, since the right hand side of* (2.2) *is equal to $D(1 - c_1 t)^{-1} - 1$, we obtain $u_{n-1} = c_1^n \sum_{i=0}^{2} d_i c_1^{-i}$ for $n \geq 1$.*

Corollary 3.6. *Suppose $c_1 = c_2 = 0$. Then $M(t) = 1$ and the right hand side of* (2.2) *is equal to D, giving $u_0 = d_1$, $u_1 = d_2$ and $u_n = 0$ for $n \geq 2$.*

Corollary 3.7. *Suppose $c_2 \neq 0$ and $M(t)$ has two equal roots, say $t = t_1$. The root satisfies $M(t) = \dot{M}(t) = 0$, giving $t_1 = c_1/(2c_2) = 2/c_1$ and $4c_2 = c_1^2$. So, $M(t) = (1 - t/t_1)^2$ and the right hand side of* (2.2) *is equal to*

$$D \sum_{j=0}^{\infty} (j+1)(t/t_1)^j - 1,$$

giving $u_{n-1} = t_1^n\{n + 1 + nd_1 t_1 + (n-1)d_2 t_1^2\}$ for $n \geq 1$. So, in this case, the solution is a weighted power with the weight linear in n.

4 An extension to polynomial weights

The weighted sum of powers assumption for v_n arose naturally in Withers and Nadarajah (2010) assuming that a certain matrix had diagonal Jordan form, or at least if the eigenvalues of the non-diagonal Jordan blocks are zero. When this is not the case, we showed in Withers and Nadarajah (2010) that

$$v_n = \sum_{i=1}^{r} \sum_{k=0}^{\min(n,m_i)-1} \binom{n-1}{k} w_{ik} \nu_i^{n-k-1} \tag{4.1}$$

for $n \geq 1$, so that $v_1 = \sum_{i=1}^{r} w_{i0}$. For this more general case, the method of obtaining u_n from v_n via its generating function, (2.2), still holds, as shown by Theorem 4.1.

Theorem 4.1. *Suppose* (4.1) *holds. Then* u_n *has the form*

$$u_{n-1} = \sum_{k=1}^{J} c_k t_k^n \tag{4.2}$$

for $n \geq 1$ *for some* J, c_k *and* t_k. *Note* (4.2) *is of the form* (3.3) *with* $n_0 = 1$.

Proof. Setting $s = \nu t$ and $D = d/ds$,

$$\sum_{n=1}^{\infty} t^n \binom{n-1}{k} \nu^{n-k-1} = t^{k+1} A_k,$$

where

$$A_k = \sum_{n=1}^{\infty} \binom{n-1}{k} s^{n-k-1} = \sum_{n=1}^{\infty} D^k s^{n-1}/k! = D^k (1-s)^{-1}/k! = (1-s)^{-k-1}$$

for $|s| < 1$. So, setting $s_i = \nu_i t$ for $\max_{i=1}^{r} |s_i| < 1$, (4.1) gives the generating function for $\{v_n\}$

$$V(t) = 1 + V_0(t) + V_1(t),$$

where

$$V_0(t) = \sum_{\nu_i = 0} \sum_{k=0}^{m_i - 1} w_{ik} t^{k+1}, \quad V_1(t) = \sum_{\nu_i \neq 0} \sum_{k=0}^{m_i - 1} w_{ik} t^{k+1} (1 - s_i)^{-k-1}.$$

Let $\{\theta_j, j = 1, \ldots, R\}$ be the distinct non-zero values of $\{\nu_i, i = 1, \ldots, r\}$. Set

$$n_0 = \max\{m_i : \nu_i = 0\}, \ J = M + 1 + n_0, \ M_j = \max\left\{m_i : \nu_i = \theta_j\right\},$$

$$m = \max_j M_j = \max_i m_i, \ M = \sum_{j=1}^{R} M_j, \ c_{jk} = \sum_i \left\{w_{ik} : \nu_i = \theta_j\right\},$$

$$N_j(t) = \sum_{k=0}^{M_j - 1} t^{k+1} \left(1 - t\theta_j\right)^{M_j - k - 1} c_{jk}, \ D(t) = \prod_{j=1}^{R} \left(1 - t\theta_j\right)^{M_j}.$$

Then $V_0(t)$ is a polynomial of degree n_0, $N_j(t)$ is a polynomial of degree M_j, $D(t)$ is a polynomial of degree M, and

$$V_1(t) = \sum_{j=1}^{R} N_j(t) / \left(1 - t\theta_j\right)^{M_j},$$

where

$$D(t) V_1(t) = \sum_{j=1}^{R} N_j(t) Q_j(t) = N(t)$$

say, $Q_j(t)$ is a polynomial of degree $M - M_j$, and $N(t)$ is a polynomial of degree M. So,

$$D(t)\left(1 - tV(t)\right) = D(t)\left(1 - v_0 t - tV_0(t)\right) - tN(t) = L(t) = \prod_{k=1}^{J} (1 - tt_k)$$

say, is a polynomial of degree J, giving

$$1 + tU(t) = (1 - tV(t))^{-1} = D(t) / \prod_{k=1}^{J} (1 - tt_k).$$

Expanding in partial fractions (see Section 2.10 of Gradshteyn and Ryzhik (2007)) and taking the coefficient of t^n gives u_{n-1} as a weighted sum nth powers of $\{t_k\}$ with the weights polynomials in n. If $\{t_k\}$ are all distinct, then the partial fraction expansion has the form

$$1 + tU(t) = (1 - tV(t))^{-1} = \sum_{k=1}^{J} c_k / (1 - tt_k)$$

so that (4.2) follows. If $\{t_k\}$ are not distinct, then proceed as for the case given in Section 3. □

Suppose that $m_1 = 2$, $\nu_1 \neq 0$, $(m_i, \nu_i) = (1, 0)$ for $i > 1$. Then $R = 1$, $\theta_1 = \nu_1$, $M = M_1 = m_1 = 2$, $J = 4$, $M_2 = m_2 = 1$ and $v_n = w_{10}\nu_1^{n-1} + (n-1)w_{11}\nu_1^{n-2}$ for $n \geq 2$. So, setting $s = \nu_1 t$ and $c_0 = \sum_{i=2}^{r} w_{i0}$,

$$V_0(t) = c_0 t, \ V_1(t) = w_{10}t(1-s)^{-1} + w_{11}t^2(1-s)^{-2},$$

$$V(t) = v_0 + c_0 t + N_1(t)(1-s)^{-2},$$

$$N(t) = N_1(t) = w_{10}t(1-s) + w_{11}t^2, \ Q_1(t) = 1,$$

$$D(t)\left(1 - tV(t)\right) = (1-s)^2\left(1 - tv_0 - t^2 c_0\right) - tN(t) = L(t) = \prod_{i=1}^{4}\left(1 - t_i t\right).$$

So, if $\{t_i\}$ are distinct, then

$$1 + tU(t) = \left(1 - tV(t)\right)^{-1} = (1-s)^2/L(t) = \sum_{k=1}^{4} c_k/\left(1 - tt_k\right)$$

say, giving (4.2) with $J = 4$. For analytic expressions for the roots of a quartic, see Section 3.8.3, page 17 of Abramowitz and Stegun (1964).

5 Conclusions

We have solved the recurrence relation $u_{n+1} = v_{n+1} + u_n \otimes v_n$, $n \neq 0$ for u_n, given the sequence v_n. The solution for u_n is in terms of Bell polynomials. This form is convenient since in-built routines for computing Bell polynomials are widely available. So, the solutions can be directly applied to derive the distribution of the maximum for autoregressive processes.

We have also established the behavior of u_n for large n by assuming some form for the behavior of v_n for large n. The assumed forms are (3.1), a weighted sum of powers, and (4.1). As shown in Withers and Nadarajah (2010), one of these assumptions always holds. So, the assumptions are not at all restrictive.

The work presented can be extended in several ways: 1) consider solving $u_{n+1} = v_{n+1} + \omega_{n+1} + u_n \otimes v_n + u_n \otimes \omega_n$, $n \neq 0$ for u_n, given the sequences v_n and ω_n; 2) consider solving $u_{n+1} = v_{n+1} + \omega_{n+1} + \mu_{n+1} + u_n \otimes v_n + u_n \otimes \omega_n + u_n \otimes \mu_n$, $n \neq 0$ for u_n, given the sequences v_n, ω_n and μ_n; and, 3) consider solving multivariate forms of (1.1) taking the form

$$\mathbf{u}_{n+1} = \mathbf{v}_{n+1} + \mathbf{u}_n \otimes \mathbf{v}_n,$$

where the equality and the convolution operator are interpreted element wise. We hope to address some of these problems in a future paper.

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