OTHER THEMES

Intermediate Representations in the Learning of Combinatorial Situations

Juliana Azevedo Montenegro[I]
Rute Elizabete de Souza Rosa Borba[I]
Marilena Bittar[II]

[I]Universidade Federal de Pernambuco (UFPE), Recife/PE – Brazil
[II]Universidade Federal do Mato Grosso do Sul (UFMS), Campo Grande/MS – Brazil

ABSTRACT – Intermediate Representations in the Learning of Combinatorial Situations. The article analyses identification, conversion and treatment of registers in combinatorial situations. Two studies were carried out: 1) identification of conversions and 2) with intermediate representations between registers in natural language and in numerical expression. Hypotheses of greater difficulties in the combination situation and in the conversion to numerical expression were confirmed. Intermediate representations (tree of possibilities or systematized listing) for the teaching of Combinatorics were indicated, however, working with trees resulted in better performance – because they have more congruence with numerical expressions. It is recommended, therefore, the intermediation of representations in the identification, conversion and treatment of combinatorial situations.
Keywords: **Identification. Conversion. Treatment. Combinatorial Situations. Intermediate representations.**

RESUMO – Representações Intermediárias na Aprendizagem de Situações Combinatórias. No artigo analisa-se identificação, conversão e tratamento de registros em situações combinatórias. Foram realizados dois estudos: 1) de identificação de conversões e 2) com representações intermediárias entre registros em língua natural e em expressão numérica. Confirmaram-se hipóteses de maiores dificuldades na situação de combinação e na conversão para expressão numérica. Indicou-se representações intermediárias (árvore de possibilidades ou listagem sistematizada) para o ensino da Combinatória, entretanto, trabalhar com árvores resultou em melhores desempenhos – por terem mais congruência com expressões numéricas. Recomenda-se, assim, a intermediação de representações na identificação, conversão e tratamento de situações combinatórias.
Palavras-chave: **Identificação. Conversão. Tratamento. Situações combinatórias. Representações intermediárias.**

Introduction

According to the National Curriculum Parameters – *Parâmetros Curriculares Nacionais*, PCN (Brasil, 1997), Mathematics enables students to arouse their curiosity in learning, and instigates their generalisation capacity. Combinatorics, specific content of this area of knowledge, which is characterized as a type of counting based on multiplicative reasoning, can promote these abilities listed by this document, because, as Batanero, Navarro-Pelayo and Godino claim, combinatorial problems can be used to stimulate "[...] the students in counting, making conjectures, generalisation and systematic thinking [...]" (Batanero; Navarro-Pelayo; Godino, 1997, p. 181).

Vergnaud (1986), author of the Theory of Conceptual Fields (TCF), considers Combinatorics part of the conceptual field of multiplicative structures and identifies problems of this nature as products of measures. These problems, according to Vergnaud, involve a ternary relation, that is, between three variables, of which one quantity is the product of the other two. Pessoa and Borba (2010) articulate these problems – also called Cartesian products – to others (arrangements, combinations and permutations) in a unique classification of combinatorial situations, and emphasize that in solving these situations there is a great variety of representations used by students, such as: drawings, listings, trees of possibilities, tables, diagrams, use of formulas and Fundamental Counting Principle[1], among others.

Vergnaud (1996, p. 191) points out that in the process of conceptualisation symbolic representations are as important as situations and their operative invariants, since, for him, language and mathematical symbols play a relevant role in conceptualisation, as without them, "[...] the schemes and the situations, remain empty of meaning".

Still on representations, Duval, in his Theory of Semiotic Representation Registers (TSRR), states that "It is not possible to study phenomena related to knowledge without resorting to the notion of representation [...]" (Duval, 2009, p. 29), because all knowledge must be mobilized by means of representation.

This author emphasizes, therefore, that representations are indispensable for the understanding of a concept. However, representations cannot be confused with the concept itself. In this case, he calls attention to a paradox that raises the need to work with many representations of the same concept, and thus to have *access* to the concept itself and not only to its representation (Duval, 2011).

In addition, the author emphasizes that representations may be more or less congruent with each other, depending on the degree of difficulty in the conversion between these representations, and with this, the need for transitional auxiliary representation registers arises. Such registers are characterized in this way, because, to the extent that students understand faster and more formal registers, they abandon these auxiliary registers (Duval, 2011).

The two theories discussed here detail the processes of identification, conversion and treatment through intermediary representations (TSRR) and emphasize how situations and their invariants must also be taken into account, besides the representations (TCF) in conceptualisation processes. Thus, it is necessary to analyse identification, conversion and treatment processes for each combinatorial situation, according to its invariants.

In this sense, the present study aims at discussing the use of different representation registers – such as natural language, listings, trees of possibilities, and numerical expressions, such as those applying the Fundamental Counting Principle (FCP) – in expanding knowledge of different combinatorial situations: Cartesian products, arrangements, combinations and permutations. In the research carried out, natural language was the starting register; trees and listings were worked on as intermediate registers; and numerical expressions were the arrival register.

The Use of Different Representations in the Teaching and Learning of Mathematical Situations and Relations

According to Vergnaud (1986), the teaching and learning of Mathematics presupposes mastery of situations over a long period of time, taking into account relational and operative invariants, as well as their connection with symbolic representations. Duval (2009) also indicates that for the development of cognitive functioning of thought, the diversity of semiotic representations is paramount.

Combinatorics, as a branch of Mathematics, is the study of the counting of the elements of a set, in which different types of representations can be used, from the most intuitive, such as listings and drawings, through tables, to the formal representations of Mathematics – as the generalisation[2] of possibilities by means of a multiplication, the Fundamental Counting Principle (FCP) and formulas, as emphasized previously.

Colombo, Flores and Moretti (2007, p. 183) point out that the idea of conceptual acquisition of Vergnaud (1986) assumes that "[...] a concept can only be defined by considering three sets that form a tripod: situations, invariants and representations". In this way, Vergnaud argues that for the formation of new concepts it is necessary to consider the three dimensions simultaneously, so as to consider the situations that give meaning to the concept, its relational and in-action invariants that elicit the use of different symbolic representations.

Gitirana, Campos, Magina and Spinillo (2014, p. 10) emphasize that situations "[...] make the concept meaningful", being the invariants "[...] objects, properties and relations that can be recognized by the subject to analyse and dominate situations", and the symbolic representations that can be used to represent invariants and situations.

In the Theory of Semiotic Representation Registers, Duval states that the conceptual grasp happens when the subject mobilizes the different registers of the same mathematical object in a way that he/she can differentiate the representative and what is represented (Colombo; Flores; Moretti, 2007). Thus, it is through semiotic representations that Duval believes a conceptual grasp is possible, since for him "[...] one cannot have understanding in mathematics if one does not distinguish an object from its representation" (Duval, 2009, p. 14).

Duval (2012) identifies a register of semiotic representation as a system endowed with rules. A semiotic system characterizes a semiotic representation register when it satisfies three conditions: 1) Identification – when the individual is able to identify the concept represented; 2) Transformation of Treatment – internal to the register itself; and 3) Conversion Transformation – passing from one register to another register.

In addition, Duval (2009) emphasizes that the conversions carried out can generate a difference in the understanding of the knowledge in question, due to the level of congruence between the registers. This author states that "[...] every task in which conversion is not congruent gives way to a more or less weak rate of success according to the degree of non-congruence" (Duval, 2009, p. 19). The level of congruence occurs when a situation takes into account three fundamental criteria: the first, relative to semantic correspondence – to each simple significant unit of one of the representations can be associated an elementary significant element; the second, related to the presence of terminal semantic univocity – at each elementary significant unit of the starting representation, it corresponds a single elementary significant unit in the register of arrival representation; and finally, the third, which concerns the ordering of the signifying units that make up each one of the representations – units in semantic correspondence in the same order in the two representations.

Duval (2011) also emphasizes the importance of intermediate representations. He points out that they are used mainly in situations whose conversion from the mother tongue to the resolution in numerical expression does not present congruence, that is, the more non-congruous the conversion, the greater the need for an intermediate representation. This is because, in this type of situation, the numerical expression is not clear enough for the students to make use of it, without any specific intervention. Thus, after instruction, students gradually use a mathematical representation that seems to them less slow and costly, making this auxiliary representation a transitional representation.

Combinatorial situations are characterized by non-congruence in the conversion between the natural language registers of the utterance and the formal mathematical register of its resolution, since they do not have a semantic correspondence or semantic univocity. This is because the numbers that appear in combinatorial problem statements are not always used directly in the numeric expressions that can be used to solve them. In this sense, an intermediate register that has an approximation with the starting and ending registers can help in the articulation between statements and numerical expressions of combinatorial problems.

The present research highlights the contributions of the TCF, drawing attention to the three dimensions necessary for the formation of a concept (situations, invariants and representations), as well as the TRRS, regarding the fundamental role of the identification, conversion and treatment of representations in conceptualisation.

Combinatorics in K-12 Education

Combinatorial Analysis[3] is an area of Mathematics related to the counting of discrete quantities. Morgado, Pitombeira de Carvalho, Carvalho and Fernandez (1991) point out that one of the first activities of children in schools is related to quantities of objects in a given set, enumerating them. During the following grades of basic schooling problems are dealt with another type of counting based on the multiplicative principle.

Several authors (Guirado; Cardoso, 2007; Pessoa; Borba, 2009; Azevedo; Borba, 2013) argue that even in the initial grades of Elementary School, it is necessary for teachers to work with their students situations that require combinatorial reasoning, so that they can develop systematic and generalized ways of thinking in the enumeration of elements combined one with each other.

According to Borba (2010, p. 3), combinatorial reasoning is understood as a way of thinking that is present in the analysis of situations in which, given certain sets, one must select elements from them, in order to meet specific criteria (choice and/or ordering the elements) and determine – directly or indirectly – the total number of possible groupings.

In this way, combinatorial thinking is characterized by a type of reasoning that enables the enumeration, systematisation, generalisation and abstraction of a situation that indicates certain conditions that need to be respected for its resolution.

Pessoa and Borba (2009) emphasize that the learning of Combinatorics must begin in the first grades of schooling, through different combinatorial situations. The authors argue that in this way new learning can be stimulated in the different levels of schooling, as well as the mistakes and the difficulties presented initially may be overcome, thus favouring the moment of systematic learning offered in High School.

Thus, even though Combinatorics is more intensively worked during High School, through the use of formulas, it is essential that combinatorial relations and properties be discussed from the earliest grades of Elementary School. In this direction, Guirado and Cardoso (2007) point out that by working on combinatorial problems from the earliest grades, students can be led to "[...] abstraction and generalisation, and the habit of guessing the right formula to solve a problem of combinatorics will be replaced by a work of analysis and synthesis" (Guirado; Cardoso, 2007, p. 1). Thus, it is understood that for this to be possible, it is necessary that, from the earliest grades of Elementary Education, students be given the opportunity to come into contact with different combinatorics problems, so that they use different ways of representing their combinatorial solutions and can discuss them with their peers and teachers, establishing greater articulation with the relations present in combinatorial reasoning.

Pessoa and Borba (2009) organize combinatorial problems into a single classification that indicates four different situations in the relations and properties of each problem: Cartesian product, arrangement, permutation and combination. The authors, based on the Theory of Conceptual Fields (Vergnaud, 1991), point out that these problems are different from the point of view of relational calculus, that is, from the point of view of understanding the logic of the problem. Borba (2010) also points out that these combinatorial situations differ according to the criteria of choice and ordering, which are characterized as the relations and properties of these situations. Thus, the problems involving the Cartesian product situation are those where the choice is made from two (or more) data sets in which one element of each set is grouped, so that the order in which those elements are grouped does not create new possibilities. In combination problems, from a single set, some elements are chosen so that their ordering does not generate new possibilities. In the problems involving the arrangement situation, the choice also happens from a single set given in which some elements of this set will be grouped, however, in this type of problem one must take into account the order of these elements. In permutation problems, one must take into consideration in the choice of elements that they will all be used only by modifying the order in which they will be grouped.

Pessoa and Borba (2009) analysed the comprehension of students from the 2nd grade of Elementary School to the 3rd grade of High School on situations involving combinatorial reasoning based on the Theory of Conceptual Field (Vergnaud, 1991). In this study, 412 students from public and private schools participated in the 11 grades of schooling studied. All students answered a test with eight combinatorial situations, two of each type of problem (arrangement, combination, permutation and cartesian product). The authors point out that:

> [...] students from the initial grades to the final grades of K-12 Education are able to understand problems of combinatorial reasoning, and their performance is influenced by the type of school they attend, the period of schooling, the type of combinatorial problem they are solving (and implicitly by the properties and relations involved in each type of problem), by the form of symbolic representation used to solve situations, as well as by the order of magnitude of the numbers involved (Pessoa; Borba, 2009, p. 11).

Thus, Pessoa and Borba (2009) observed that one of the factors that can influence the resolution of combinatorial problems are the symbolic representations used. It was found that many students – even those in High School who had already been formally instructed in Combinatorial Analysis – preferred to represent situations through more transparent register systems, such as listings, in which different possibilities were visible, rather than use of procedures in which the cases identified were not clearly shown – such as when using the Fundamental Counting Principle or formulas.

Moro and Soares (2006), in a study with 4th and 5th grade Elementary School children, investigated the development of combinatorial reasoning in Cartesian product situations. The authors emphasise in their results that there are different levels of combinatorial thinking, from the absence of a combinatorial solution to a more advanced level, when students indicate the correct answer through the use of tables, diagrams or multiplicative operations.

Duro and Becker (2015) conducted a study using the Piagetian clinical interview method with regular High School students and with Adults Education students. In this study, involving the four types of combinatorial situations, the authors state that there is a progression of combinatorial thinking from when students use random procedures (level I) to using systematic procedures for organizing possibilities (level III). The use of systematisation indicates a multiplicative generalisation, whereas random procedures derive from additive generalisation, which does not converge with combinatorial thinking.

Fischbein, Pampu and Minzat (1970) observed the effect of specific instructions on the ability to deal with permutations and arrangements through the tree of possibilities diagram with 10, 12 and 14-grade-old students. The authors point out that "[...] even 10-grade-olds have learned to use the tree diagram and the appropriate procedures for permutations and arrangements" (Fischbein; Pampu; Minzat, 1970, p. 261). Fischbein (1975) argued that only the development of logical-mathematical thinking will not be sufficient for the resolution of combinatorial problems, so a specific instruction is necessary, for example, with the use of a tree of possibilities, so that students can organize and systematize information and generalize possibilities.

Azevedo (2013) analysed the influence of the construction of trees of possibilities, with and without the use of an educational software aimed at the teaching and learning of Combinatorics, together with 5th grade students. The author points out that both groups (with and without the use of the software) significantly developed their combinatorial reasoning after the intervention, because, through an immediate post-test, applied immediately after the intervention processes, it was possible to perceive that they had improved their performance in solving combinatorial problems. Also, in a later post-test, applied nine weeks after the immediate post-test it was possible to verify that the students remained in development.

Azerêdo (2013) argues that the semiotic representations of the multiplication operation are instruments of pedagogical mediation in the teaching and learning process of this content. In her study, carried out with eight teachers who taught in the 2nd, 3rd, 4th and 5th grades of

Elementary School and their respective students, different multiplicative situations were approached, including a combinatorial situation of Cartesian product. Specifically about this combinatorial situation, with an illustration of inputs and outputs, students from early grades did not relate the illustration to problem solving, which, according to the interviewed teachers, came as a surprise, since students were expected to use the illustration as a strategy to convert natural language to solve the problem.

In the research reported in the present article, there is the intention to verify and corroborate with the work involving different types of representation, such as the tree of possibilities and the list, but also including the work with numerical expressions, since it constitutes an important strategy for the teaching and learning of Combinatorics. This idea converges with the thinking of the theories presented by Vergnaud (1986) and Duval (2009), since intermediate registers (trees and listings) will be used between the natural language and the numerical expression and that will take into account identification, conversion and treatment of different combinatorial situations.

Method

The purpose of the present study was to analyse the role that the identification and transformation of treatment and the conversion of registers have in the expansion of knowledge of different combinatorial situations by Elementary School students.

To reach this objective, two studies were carried out: one of probing (survey) of knowledge and another of intervention. The aim of the first study was to analyse the identification of conversions by students of initial grades of Elementary School in different combinatorial situations, so that it was required to identify two conversions: from natural language to tree of possibilities or to listing; and from tree or listing to numeric expression. The intervention study, based on the results of the survey, aimed to investigate the effect of pedagogical interventions, which mobilize identifications and transformations of registers, through different intermediate representations, which may be more or less congruent with numerical expressions, in the performance of Elementary School students in different combinatoric problems.

In the first study, a test was applied with 16 students from the 5th grade of a private school in Recife. The test consisted of eight combinatorial problems, two of each type of situation (arrangement, combination, permutation and Cartesian product). In each problem there was the written statement in natural language, two alternative answers represented in tree or listing, in which only one of them was true and then four alternatives with numerical expressions, in which only one of them was true. Thus, students should perform the identification of two conversions: the first one was related to the choice of the correct alternative for the tree representation or listing, and the second, choosing the correct alternative for the numerical expression that correctly answered the situation.

The tests varied so that, in one type, the first four situations were presented with the intermediate representation in tree and the last four in listing. In the other type, the order was changed. There was also a variation of the test in relation to the explanation of the repeated cases in the combination problems, in which there was explicitness with repeated cases crossed out, and there was no explicit explanation.

There were two hypotheses with respect to this study: in the first one, it was believed that combination problems would be the most difficult to identify conversions, since it is necessary to disregard repeated cases and the numerical expressions representing the situation need to take the repetitions into account, through division. The second hypothesis was related to a greater difficulty in identifying the second requested conversion (to numerical expression), since children may not easily recognize numerical expressions.

In the second study, based on the results of the first study, an intervention was proposed with 5th, 7th and 9th grade students, divided into two groups. In the first group (G1), after answering a pre-test asking what were all possibilities and which operation could be used to answer each of the eight problems, the students reworked the pre-test questions using the tree of possibilities as an intermediate representation, that is, between the starting representation (uttered in natural language) and the arrival representation (numerical expression). In the second group (G2), the intervention occurred in the same way, and the intermediate representation used was the systematized listing. In the first intervention session, the first four problems of the pre-test were worked out. In the second session, the last four problems were worked out. In the interventions, the combination situations were worked out making explicit all the cases, and then to crossed out the repeated cases, since the results of the first study indicated this need.

After the intervention sessions, the students solved a post-test also with eight combinatorial problems. Unlike the pre-test, when all questions did not exceed 24 possibilities, in the post-test only the first four problems had a low number of possibilities. The last four problems resulted between 56 and 120 possibilities, so that to answer to these situations, numerical expressions were the most viable option.

It was believed that both groups would have progress in their performance, but that for the group that had intervention using the tree of possibilities it would be easier to indicate a multiplication corresponding to the resolution of the combinatorial problem. This is because this representation seems to indicate more clearly the one-to-many relationship involved in combinatorial situations, since the organisation in branches indicating this multiplicative idea is apparently more congruent with the mathematical operation required for problem solving.

The results were analysed quantitatively and qualitatively. The Statistical Package for the Social Sciences – SPSS was used for a quantitative analysis. For the qualitative analysis the representations used by the students in the different stages of the research were examined. It was believed that, from these different modes of intervention, students would advance more in their combinatorial reasoning, as well as perceive the multiplicative reasoning implicit in combinatorial situations.

Results of the Survey Study

Table 1 shows the number of correct identifications for the conversions requested in the different test types (Test 1 – combination problem not considering repeated cases; Test 2 – combination problem with the presentation of repeated cases crossed out), with the first conversion from natural language to listing or tree and the second conversion from listing or tree to numeric expression.

Observing Table 1, it is possible to see that identifying which listing or which tree of possibilities represents the natural language-registered utterance was easier for the students when compared to the second requested conversion – from the listing or the tree to a corresponding numerical expression. In the conversion of natural language to listing 36 items (of possible 64) were answered correctly and in the conversion from natural language to tree, 33 items. In the second conversions, the score fell about 50% (16 items answered correctly, both from listings and from trees).

Conversion from natural language to listing and natural language to tree of possibilities practically did not present any difference, implying that the children equally understood the register in list and in tree. This result can be justified by the fact that there is a great congruence between listings and trees, since for each element present in the listing there is a corresponding one in the tree of possibilities (semantic matching criterion), which does not occur between listings/trees and numerical expression.

Table 1 – Correct identifications on each conversion by test type

Type of test	Conversion 1		Conversion 2	
	NL → TP	NL → TP	L → NE	TP → NE
1.1 (Test without repeated cases, first listings and then trees.)	9	6	1	1
1.2 (Test without repeated cases, first trees and then listings.)	5	8	1	2
Total Test 1	14/32 (43.75%)	14/32 (43.75%)	2/32 (6.25%)	3/32 (9.375%)
2.1 (Test with repeated cases crossed out, first listings and then trees.)	12	9	5	7
2.2 (Test with repeated cases crossed out, first trees and then listings.)	10	10	9	6
Total Test 2	22/32 (68.75%)	19/32 (59.375%)	14/32 (43.75%)	13/32 (40.625%)
Total Test 1 + Test 2	36/64 (56.25%)	33/64 (51.56%)	16/64 (25%)	16/64 (25%)

NL: Natural Language; L: Listing; TP: Tree of possibilities; NE: Numerical Expressions. Source: Montenegro (2018).

In Table 1 it can be observed that there were more correct answers, both for the first conversion and for the second one. However, the difference between the tests was only in the combination situations; in these situations, the repeated cases were crossed out, so that it made explicit that, for this situation, the order of choice of the elements does not generate new possibilities. It can be inferred that, possibly, making explicit the order invariant in the combination situation may have influenced the performance of the students also in the other types of combinatorial problems. Thus, the students paid more attention to the differences between the problems with regard to the ordering of the elements, constituting or not, different possibilities.

Table 2 shows the results by type of problem. It is perceived that performance is similar in Cartesian product, combination, and arrangement problems. Permutation problems, on the other hand, have a better performance. Nevertheless, when it is analysed the conversion to Numerical Expression (NE), it can be seen that in the Cartesian product situations, of the 14 correct answers in the first conversion, 9 also are correct in the second one. In the problems of permutation there are 22 correct answers in the first conversion and, of these, 11 correct answers in the second conversion.

Table 2 – Correct identifications in each conversion by combinatorial problem type

Problem type	Conversion 1		Conversion 2		
	NL → L	NL → TP	L → NE	TP → NE	Total
CP	7	7	4	5	23
C	9	6	4	2	21
A	8	10	2	4	24
P	12	10	6	5	33
	36/64	33/64	16/64	16/64	101/256

NL: Natural Language; L: Listing; TP: Tree of possibilities; NE: Numerical Expressions; CP: Cartesian Product; C: Combination; A: Arrangement; P: Permutation. Source: Montenegro (2018).

In the combination problems, in which there is a difference between the tests, there are 15 correct answers in the first conversion, of which 10 are in Test 2 in which repeated cases are crossed out. In the second conversion, to Numerical Expression, there are only 6 correct answers, of which, 5 are in Test 2, highlighting the importance of making explicit the repeated cases.

Thus, it is possible to perceive that identification of conversion, when performed from natural language to listing or to tree of possibilities, results in a higher success rate, while identification of conversion to numerical expression the success rate is weaker. The results seem to indicate, therefore, that the reason for greater success in the identification of conversions occurs when there is greater congruence, in the case of register from natural language to listing or a tree of possibilities, while from tree or listing to numerical expression, the congruence level is lower.

Table 3 shows that the number of incorrect or blank rationales (87.5%) for the answers given is much higher than the correct rationales (12.5%). In the incorrect rationales presented, in general, there is no explanation, in fact, of the reason why a numerical expression corresponds to the statement in natural language or to the listing and tree of possibilities. In some cases, they were related only to the isolated treatment of operations or, for example, difficulties in understanding tree representation, as it can be seen in Figure 1.

Table 3 – Quantitative of the response type according to each type of problem and the type of intermediate conversion performed

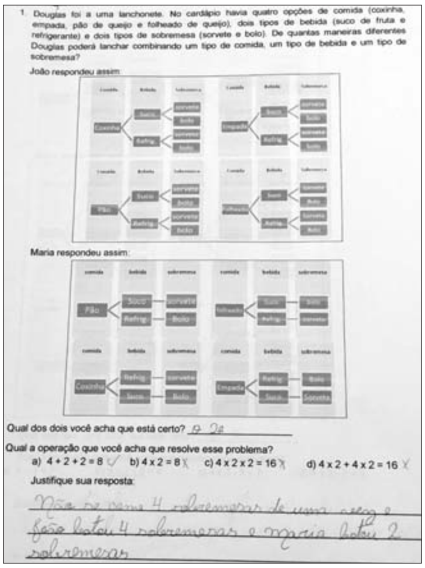
Type of problem	Intermediate conversion type	Correct rationale	Incorrect rationale	Blank rationale
CP	Listing	3	6	7
	Tree	2	7	7
C	Listing	1	9	6
	Tree	1	8	7
A	Listing	0	8	8
	Tree	2	7	7
P	Listing	4	6	6
	Tree	3	6	7
Total		16 (12,5%)	57 (44,5%)	55 (43%)

CP: Cartesian Product; C: Combination; A: Arrangement; P: Permutation. Source: Montenegro (2018).

In Figure 1 the student responded incorrectly both to the first requested conversion and to the second one. In the rationale, according to personal criteria, he spoke about the reasoning he used to respond to the situation, stating: “You do not eat 4 desserts at a time and John has 4 desserts and Mary has 2 desserts”. This statement leads to the conclusion that the student could not interpret the situation as a tree of possibilities, because there is no possibility with four desserts, but each dessert indicating a different possibility.

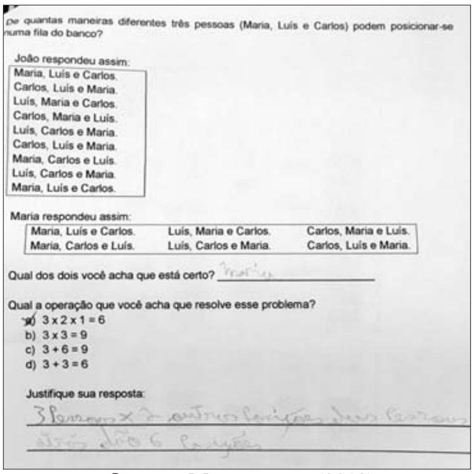
The correct rationales indicate textually why the chosen operation indicates the correct response, as it can be seen in Figure 2, below. In this example, the student replied that “3 people x (times) 2 other positions of the people behind results in 6 positions”, that is, there are three people to position themselves in a queue and when one is first in the queue, the other two people occupy either the second or the third place, thus, there are two possibilities for each occupying the first position in the queue. With the rationale it is possible to observe that the student in question perceived the multiplicative principle involved in the problem and coordinated in a satisfactory way the three registers of representations presented in the situation – the natural language statement, the tree of possibilities and the corresponding numerical expression.

Figure 1 – Cartesian product situation incorrectly answered by Student 14 with solution presented in tree



Source: Montenegro (2018).

Figure 2 – Permutation situation correctly answered by Student 2, with solution presented in list



Source: Montenegro (2018).

It is important to emphasize that reverse thinking is more complex and may be the main reason for the difficulties in presenting correct rationales for the identifications made. This activity is configured in a new conversion that indicates a return to the representation register used initially (natural language), a task identified as complex. Duval (2009, p. 109) states that:

> Here, the arrival record is a description of the situation presented by the intermediate representation and not by the output representation. The complexity of inverse conversion sticks to the fact that we are in the presence of a composition of two successive conversions.

The initial register, in natural language, went through an intermediate register (tree or listing) until the arrival record in numerical expression, however, in the end, a rationale was requested in which the students were expected to write in natural language. This activity, to present rationales in natural language, is characterized as a very difficult conversion, which presented a high number of blank responses, which ratifies its complexity.

Results of the Intervention Study

The tests applied before and after the interventions were analysed according to the possibilities survey and the numerical expression (operation) used to respond to the situation.

In the survey of possibilities, it was considered an error in cases where the answers were blank, or those that did not present a combinatorial reasoning in its resolution. In this case, the question did not receive any points. The answers with combinatorial reasoning in which there were less than half the possibilities that answered the situation were considered Partial Correctness 1 and received one point. Two points were given for those questions that presented what was considered Partial Correctness 2. In this score were the cases in which half or more of the number of possibilities were presented, but there was still no exhaustion of all possibilities. Three points were given to those that were able to correctly answer the problem with the exhaustion of all possibilities. Thus, in the test containing eight situations, each student could reach a total of 24 points (eight problems x three points in each problem).

The analysis for the numeric expression that responded to the situation was also performed with scores from 0 to 3 points, with no points received in the blank answers or that presented a calculation that did not correspond to the one used to respond to the situation. Students who wrote the type of operation that should be performed but did not indicate which numerical expression or numerical response were also scored with zero score. Partial Correctness 1 and 2, for numerical expression, are characterized as those that indicated the correct numerical expression, however they erred the numerical calculation itself, thus hitting what Vergnaud (1986) calls relational calculus, however, with difficulties in numerical calculation; or difficulty in the treatment within the registry itself, as pointed out by Duval (2012). Thus, when the student answered correctly the relational calculus but missed the numerical calculation indicating less than half the number of possibilities, it was characterized as Partial Correctness 1; and when he/she answered correctly the relational calculation but missed the numerical calculation indicating half or more of the number of possibilities, it was characterized as Partial Correctness 2. Also, partial correctness was found with incomplete generalisation of possibilities. The 3 points were assigned to those who indicated the correct numerical expression, either by means of a generalisation of possibilities or by the FCP.

In Table 4 it is possible to observe the mean of performances, both for the survey of possibilities and for the numerical expression. It can be noticed that the mean in the pre-test of all groups is low, considering that the total average could reach 24 points, mainly in the numerical expression. It is also possible to note that all groups advanced in their performance after the intervention, and, through the statistical analysis performed with the SPSS software, it is highlighted that this advance was significant[4] in all groups[5], revealing that both intermediate representations used are important strategies for working with combinatorial situations.

Table 4 – Mean pre-test and post-test performance by teaching grade and by intervention group

		Pre-test		Post-test	
		Survey	Expression	Survey	Expression
5th grade	G1 (Tree)	1.89	0.31	6.11	4.57
	G2 (Listing)	2.85	1.,2	5.15	3.30
7th grade	G1 (Tree)	1.38	0.57	6.76	4.76
	G2 (Listing)	1.77	0.46	6.23	3.46
9th grade	G1 (Tree)	6.74	3.68	9.52	7.89
	G2 (Listing)	6.25	2.56	8.43	5.93

Source: Montenegro (2018).

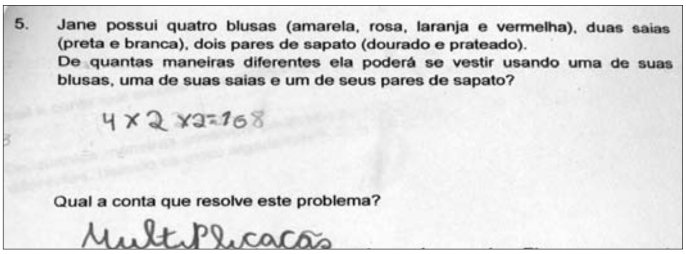
It is also important to highlight that the mean performance of the groups that worked with the tree of possibilities is higher in the post-test in all grades of schooling surveyed, what may be justified because this representation has more congruence with the numerical expression – the initial hypothesis of this study. However, when the statistical analysis was performed, in the comparison between the results of the post-test in G1 and the post-test of G2, it was verified that the higher advance of G1 was not significant in relation to G2[6]. In this way, both the tree of possibilities and the list are auxiliary representations that help in the development of combinatorial reasoning, including the presentation of corresponding numerical expressions.

The difference between the groups was also analysed taking into account only the results in situations where the number of possibilities in the response was high, and therefore the use of a numerical expression is recommended. The results indicate a significant[7] difference between groups. Thus, Group 1, which had intervention using the tree of possibilities, obtained better performance in situations where it was advisable to use a numerical expression. The explanation of this result may be due to the fact that this intermediate representation has a greater degree of congruence with the numerical expression necessary for the resolution of combinatorial problems.

The means that can be observed in Table 4 indicate that 5^{th} and 7^{th} grades have similar performances before and after the intervention process, indicating that there was probably no specific work with combinatorial situations between those grades of schooling. After the 7^{th} grade, probably this work happened, since the results in the 9th grade pre-test are matched with the 5^{th} and 7^{th} grade post-test results. It is important to point out that, although the 9^{th} grade already presented better results in the pre-test, it continued to advance in its combinatorial reasoning, since it advanced in its performance in the occasion of the post-test. Statistical analysis indicated that the 9^{th} grade presented significant differences with the 5^{th} and the 7^{th} grades, both for the survey of possibilities (9^{th} x 5^{th}: p = 0.003; 9^{th} x 7^{th}: p = 0.023) and for the indication of a corresponding numeric expression (9^{th} x 5^{th}: p = 0.007; 9^{th} x 7^{th}: p = 0.007). However, the 5^{th} and 7^{th} grades did not show any differences between them, neither in the survey of possibilities (p = 0.650) nor in the numerical expression (p = 0.991).

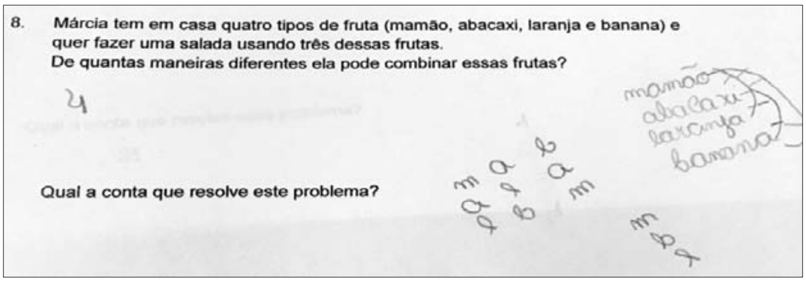
Regarding the representations used, all the school grades had preference for listings in the pre-test (Figure 4), or for a mathematical operation, which could be correct or incorrect for the situation (Figure 5). The FCP was used in the pre-test only in situations of Cartesian product, when the numbers of the statement are used to answer the problem (Figure 3). In the 9^{th} grade, the use of the generalisation of possibilities, by means of a systematic listing followed by a multiplication that correctly answers the situation (Figure 6), was observed already in the pre-test.

Figure 3 – Situation of Cartesian product with correct answer by numerical expression that answers the problem, used by a 5^{th} grade student



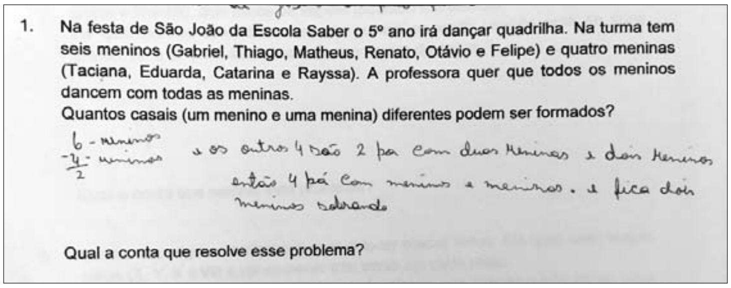
Source: Montenegro (2018).

Figure 4 – Combination situation with correct answer through listing and diagram, used by a 7th grade student



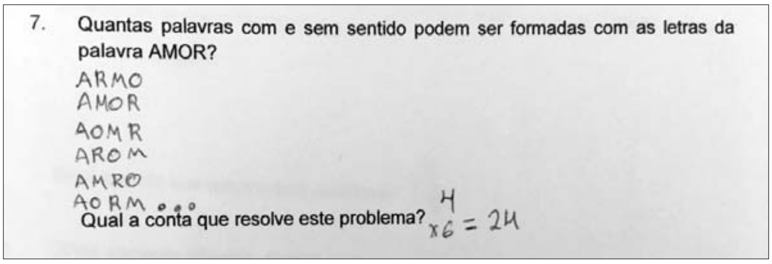
Source: Montenegro (2018).

Figure 5 – Cartesian product situation with partially correct response 1 with incorrect numerical expression, used by a 9th grade student



Source: Montenegro (2018).

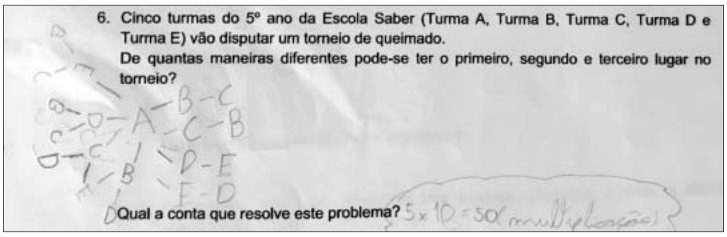
Figure 6 – Permutation situation with correct answer through generalisation of possibilities, used by 9th grade student



Source: Montenegro (2018).

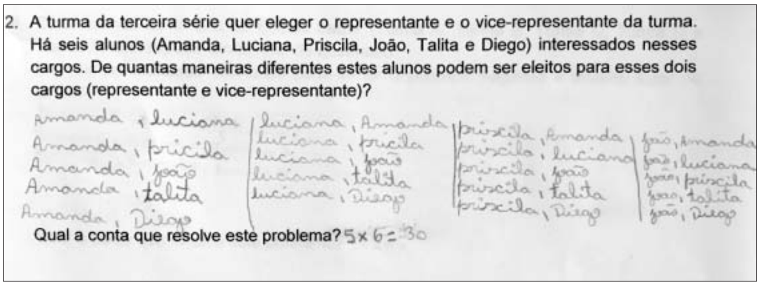
In the post-test, it was found lists and trees of possibilities used as intermediate representations for the generalisation of possibilities in all school grades (Figures 7, 8, 11 and 12), indicating the use of successive conversions to arrive at the numerical expression. The same happened with the use of the PFC, and this procedure was more used by the 7th and 9th grade students (Figures 9 and 14).

Figure 7 – Arrangement situation with partially correct type 2 response through the generalisation of possibilities using the tree and with numerical expression that partially responds to the situation, used by a 5th grade student



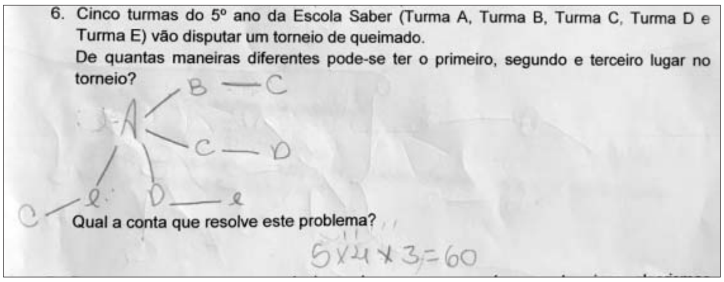
Source: Montenegro (2018).

Figure 8 – Arrangement situation with correct answer through the generalisation of possibilities with use of the list, used by a 5th grade student



Source: Montenegro (2018).

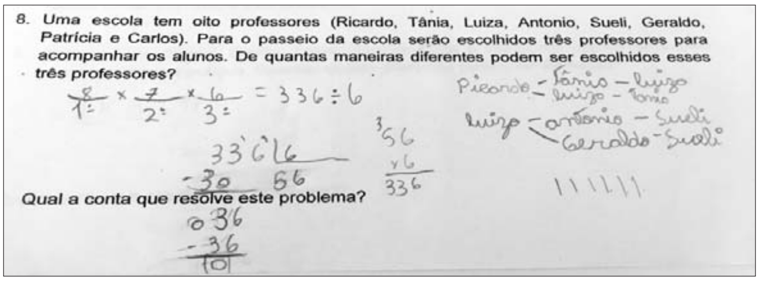
Figure 9 – Arrangement situation with correct response through the Fundamental Counting Principle, used by a 5th grade student



Source: Montenegro (2018).

In the combination situations, the 5th grade did not present any corresponding numerical expression, indicating that the conversion of the combination situation to a numerical expression is more difficult at this stage of Elementary School. In the 7th and 9th grade, students were able to develop a numerical expression for this situation (Figures 10 and 13), corroborating with the results of the first study that indicated the difficulty of the 5th grade students to recognize the numerical expressions for this type of situation. The conversion of natural language to the FCP without the need of an auxiliary representation happened in the 5th grade only in Cartesian product problems. In the 7th and 9th grades, the direct conversion to the use of the FCP also happened in other combinatorial situations.

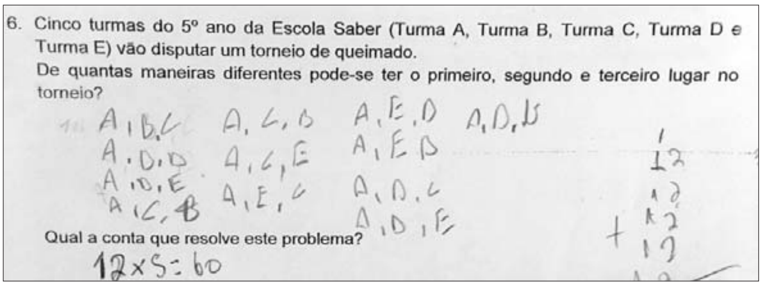
Figure 10 – Combination situation with correct answer through the Fundamental Counting Principle, used by a 7th grade student


8. Uma escola tem oito professores (Ricardo, Tânia, Luiza, Antonio, Sueli, Geraldo, Patrícia e Carlos). Para o passeio da escola serão escolhidos três professores para acompanhar os alunos. De quantas maneiras diferentes podem ser escolhidos esses três professores?

Qual a conta que resolve este problema?

Source: Montenegro (2018).

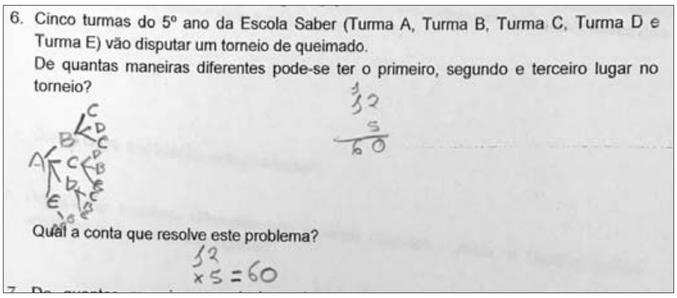
Figure 11 – Arrangement situation with correct answer through the generalisation of possibilities, used by a 7th grade student


6. Cinco turmas do 5º ano da Escola Saber (Turma A, Turma B, Turma C, Turma D e Turma E) vão disputar um torneio de queimado.
De quantas maneiras diferentes pode-se ter o primeiro, segundo e terceiro lugar no torneio?

Qual a conta que resolve este problema?
12x5=60

Source: Montenegro (2018).

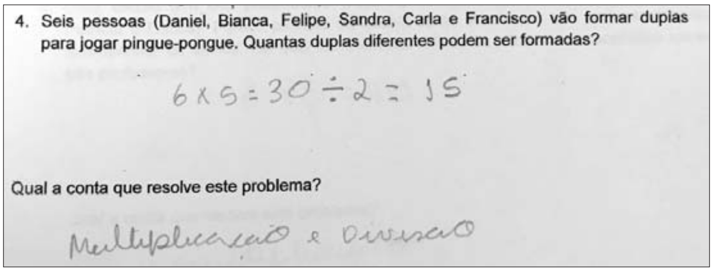
Figure 12 – Arrangement situation with correct answer through generalisation of possibilities, used by a 9th grade student


6. Cinco turmas do 5º ano da Escola Saber (Turma A, Turma B, Turma C, Turma D e Turma E) vão disputar um torneio de queimado.
De quantas maneiras diferentes pode-se ter o primeiro, segundo e terceiro lugar no torneio?

Qual a conta que resolve este problema?
12 x5 =60

Source: Montenegro (2018).

Figure 13 – Combination situation with correct answer using a numerical expression, used by a 9th grade student


4. Seis pessoas (Daniel, Bianca, Felipe, Sandra, Carla e Francisco) vão formar duplas para jogar pingue-pongue. Quantas duplas diferentes podem ser formadas?

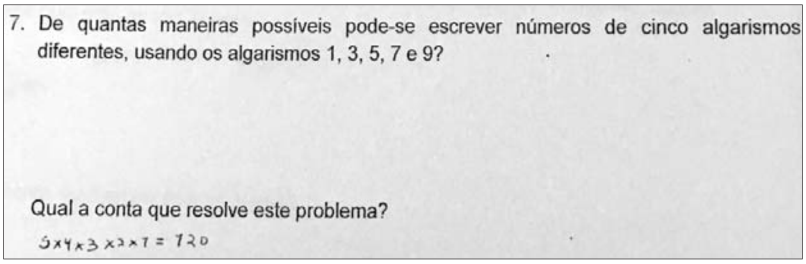
6x5=30÷2=15

Qual a conta que resolve este problema?

Multiplicação e Divisão

Source: Montenegro (2018).

Figure 14 – Permutation situation with correct response through the PFC, used by a 9th grade student


7. De quantas maneiras possíveis pode-se escrever números de cinco algarismos diferentes, usando os algarismos 1, 3, 5, 7 e 9?

Qual a conta que resolve este problema?

5x4x3x2x1 = 120

Source: Montenegro (2018).

Although results of low performance in the final test are still presented, interventional processes, that is, teaching with systematized listings or with trees of possibilities, resulted in significant improvements in performance in both groups and in the three school grades.

The group that worked with trees of possibilities had better performance than the group that worked with systematized listings, in the problems with greater number of possibilities. This shows that this register may be a better intermediate representation between statements in natural language and numerical expressions.

Teaching also served to broaden the repertoire of symbolic representations used in combinatorial problems. Initially, listings were the preferred register, but after the teaching, listings became more systematized and other registers were also used: trees of possibilities and numerical expressions, especially in the 7th and 9th grades, that used numerical expressions even in problems of combination or which requires, in addition to multiplication, also dividing by the number of equal possibilities among them.

Final Remarks

The present research had the objective of analysing the role that the identification and the transformations of conversion and treatment of registers have in the expansion of the knowledge of varied combinatorial situations. Two studies were carried out: the first one of probing (survey) with students of the 5^{th} grade of Elementary School; and the second study, based on the results of the first one, of intervention with the use of intermediate representations in the conversion of registers and in the treatment within the realized register itself and in the expansion of the combinatorial reasoning, so that the students perceived the multiplicative character in the solving of combinatorial problems.

In the first study, the hypotheses initially raised, the greater difficulty in the problems with the combination situation, were confirmed, as well as the greater difficulty in converting to a numerical expression corresponding to the resolution of the problem. This is because few students identified the conversions for the corresponding numerical expression, few students were able to consistently justify their responses, and only one student was able to justify the numerical expression for the combination problem. This happened possibly due to the non-congruence between the representations in natural language (utterance) and the numerical expression, especially in the situations of combination, where besides multiplication, it is necessary to divide by the number of repeated cases. Thus, the difficulties of the students are emphasized in the conversion of natural language from the utterance to the representation in systematic listing or tree of possibilities, as well as, more frequently, with the conversion to numerical expression, indicates the necessity of teaching processes so that students can identify these different conversions (from natural language to trees or listings and from those to numerical expression).

Due to these results, in Study 2, different interventions were proposed, not only in the 5^{th} grade of Elementary School, since the difficulty with the use of the FCP was highlighted in this first study, but also with classes from the 7^{th} and 9^{th} grade of schooling. In the 5^{th} grade, the objective was to verify if and how the difficulties to recognize the numerical expression as representation of arrival could be overcome. In the 7^{th} and 9^{th} grades, it was intended to observe how there is an expansion of combinatorial knowledge, particularly with regard to the use of numerical expressions/FCP.

In the second study, the results indicate that both the intermediate representations, tree of possibilities or systematized listing, are good paths for the teaching of Combinatorics, since the mean accuracy increased in the comparison between pre-test and post-test for both groups, in all grades of schooling, both for the survey of possibilities and for the numerical expression. In addition, this difference was significant in all school grades surveyed. This may indicate that the congruence between the starting (natural language), intermediate (listing or tree) and arrival (numerical expression) registers can be evidenced in the two types of intervention performed.

Although both groups progressed in their performance, it is noteworthy that the groups that worked with the tree of possibilities (G1) had better means in all the school grades surveyed. Also, it is highlighted that in these groups there were more correct answers in the second part of the post-test, with a greater number of possibilities of the post-test, indicating a significant difference with G2, since the correctness of these problems was directly related to the use of the FCP or a generalisation of possibilities.

With the present research, we sought to claim that, based on the Theory of Conceptual Fields and the Theory of Semiotic Representation Registers, an intermediate, transparent and systematized representation register is recommended to assist students in the interpretation and representation of problem expressions involving various combinatorial situations. It is also important that in the use of these representations the invariants of the different combinatorial situations are discussed. Another objective of this paper is to defend that among the intermediate registers of representation, the tree of possibilities can be more efficient, because it has more congruence with numerical expressions.

The discussions revealed how necessary and important is a discussion that articulates TCF and TSRR. It was observed that conversions have different levels of difficulty, depending on the type of register used and the combinatorial situation treated. Thus, a close relationship between representation registers and combinatorial situations is demonstrated.

Thus, it can be concluded that it is possible to develop and expand the combinatorial reasoning of 5th, 7th and 9th grade students. This can be achieved by the use of both intermediate representations used in this study, giving a better performance in the presentation of the numerical expressions corresponding to the resolution of the situations, especially with the use of trees of possibilities, since, in view of the above, they seem to have greater congruence with the numerical expression. Thus, in different grades of schooling it is possible to enable the learning of the Combinatorics, since it is possible for all to progress in their performance, guaranteeing a gradual expansion of their combinatorial reasoning.

Translated by Rute Borba and proofread by Ananyr Porto Fajardo

Received on October 26, 2018
Approved on September 21, 2019

Notes

1 Fundamental Counting Principle (FCP), also known as multiplicative principle, is a way of solving combinatorial situations and is the basis of formulas used in the study of Combinatorics, since it expresses the multiplicative nature of the different types of combinatorial problems (Lima, 2015, p. 22). PFC can also be announced as the Fundamental Enumeration Principle or Multiplication Principle.

2 The generalisation of possibilities results from the multiplication of the number of possibilities listed for a fixed element in the first choice by the total number of elements, characterizing as a relational calculation different from that ex-

pressed in the FCP. The numerical expression of the FCP is characterized by multiplying the number of possibilities for each choice.

3 In this study, 'Combinatorial Analysis' and 'Combinatorics' are considered synonyms.

4 In this study, significance level was considered p <0.05.

5 Pre x post-test: 5th grade: survey of possibilities (t (38) = -4.766; p <0.001); numerical expression (t (38) = -4.361, p <0.001). 7th grade: survey of possibilities (t (46) = -8,878; p <0.001); numerical expression (t (46) = -6.156, p <0.001). 9th grade: survey of possibilities (t (34) = -3.710, p = 0.001); numerical expression (t (34) = - 7.824, p <0.001).

6 G1 x G2: 5th grade: survey of possibilities (t (37) = 0.576, p = 0.568); numerical expression (t (37) = 0923, p = 0.362). 7th grade: survey of possibilities (t (45) = 0.440, p = 0.662); numerical expression (t (45) = 0.166, p = 0.300). 9th grade: survey of possibilities (t (33) = 0.650, p = 0.520); numerical expression (t (33) = 1.341, p = 0.189).

7 G1x G2 situations with high number of possibilities: t (119) = 3.162; p = 0.002.

References

AZERÊDO, Maria Alves. **As Representações Semióticas de Multiplicação**: um instrumento de mediação pedagógica. 2013. 279 f. Tese (Doutorado em Educação) – Programa de Pós-graduação em Educação, Universidade Federal da Paraíba, João Pessoa, 2013.

AZEVEDO, Juliana. **Alunos de Anos Iniciais Construindo Árvores de Possibilidades**: é melhor no papel ou no computador? 2013. 127 f. Dissertação (Mestrado em Educação Matemática e Tecnológica) – Programa de Pós-graduação em Educação Matemática e Tecnológica, Universidade Federal de Pernambuco, Recife, 2013.

AZEVEDO, Juliana; BORBA, Rute. Combinatória: a construção de árvores de possibilidades por alunos dos anos iniciais com e sem uso de software. **Alexandria: Revista de Educação em Ciência e Tecnologia**, Florianópolis, v. 6, n. 2, p. 113-140, jun. 2013.

BATANERO, Carmen; NAVARRO-PELAYO, Virginia; GODINO, Juan.Díaz. Effect of the Implicit Combinatorial Model on Combinatorial Reasoning in Secondary School Pupils. **Educational Studies in Mathematics**, v. 32, n. 2, p. 181-199, Feb. 1997. Disponível em: <http://www.ugr.es/~batanero/pages/ARTICULOS/Implicitmodel.pdf>. Acesso em: 6 jul. 2016.

BRASIL. Ministério da Educação, Secretaria de Educação Básica. **Parâmetros Curriculares Nacionais**. Matemática. 1º e 2º ciclos. Brasília, DF, 1997.

BORBA, Rute. O Raciocínio Combinatório na Educação Básica. In: ENCONTRO NACIONAL DE EDUCAÇÃO MATEMÁTICA, 10, 2010, Salvador. **Anais...** Salvador: Universidade Católica do Salvador (UCSal), 2010. P. 1-16.

COLOMBO, Janecler; FLORES, Cláudia; MORETTI, Méricles. Reflexões em Torno da Representação Semiótica na Produção do Conhecimento: compreendendo o papel da referência na aprendizagem da matemática. **Educação Matemática Pesquisa**, São Paulo, v. 9, n. 2, p. 181-203, 2007.

DURO, Mariana Lima; BECKER, Fernando. Análise Combinatória: do método aleatório à combinatória sistemática. **Educação & Realidade**, Porto Alegre, v. 40, n. 3, p. 859-882, 2015.

DUVAL, Raymond. **Semiósis e Pensamento Humano**: registros semióticos e aprendizagens intelectuais. Tradução: Lênio Fernandes Levy e Marisa Rosâni Abreu da Silveira. São Paulo: Livraria da Física, 2009.

DUVAL, Raymond. **Ver e Ensinar Matemática de Outra Forma – Entrar no Modo Matemático de Pensar**: os registros de representações semióticas. São Paulo: PROEM, 2011.

DUVAL, Raymond. Registros de Representação Semiótica e Funcionamento Cognitivo do Pensamento. **Revemat**: revista eletrônica de educação matemática. Florianópolis, v. 7, n. 2, p. 266-297, 2012.

FISCHBEIN, Efraim. **The Intuitive Sources of Probabilistic Thinking in Children**. Dordrecht: Reidel, 1975.

FISCHBEIN, Efraim; PAMPU, Ileana; MINZAT, Ion. Effects of Age and Instruction on Combinatory Ability in Children. **The British Journal of Educational Psychology**, Nova Jersey, n. 40, 1970.

GITIRANA, Verônica; CAMPOS, Tânia Maria Mendonça; MAGINA, Sandra; SPINILLO, Alina. **Repensando Multiplicação e Divisão**: contribuições da teoria dos campos conceituais. São Paulo: PROEM, 2014.

GUIRADO, João Cesar; CARDOSO, Evelyn. Análise Combinatória: da manipulação à formalização de conceitos. In: ENCONTRO PARANAENSE DE EDUCAÇÃO MATEMÁTICA, 9, 2007, Assis Chateaubriand. **Anais...** Assis Chateaubriand: UNIMEO, 2007.

LIMA, Ana Paula. **Princípio Fundamental da Contagem**: conhecimentos de professores de matemática sobre seu uso na resolução de situações combinatórias. 2015. 139 f. Dissertação (Mestrado em Educação Matemática e Tecnológica) – Programa de Pós-Graduação em Educação Matemática e Tecnológica – Universidade Federal de Pernambuco, Recife, 2015.

MONTENEGRO, Juliana Azevedo. **Identificação, Conversão e Tratamento de Registros de Representações Semióticas Auxiliando a Aprendizagem de Situações Combinatórias**. 2018. Tese (Doutorado em Educação Matemática e Tecnológica) – Programa de Pós-graduação em Educação Matemática e Tecnológica, Universidade Federal de Pernambuco, Recife, 2018.

MORGADO, Augusto; PITOMBEIRA DE CARVALHO, João; PINTO DE CARVALHO, Paulo; FERNANDEZ, Pedro. **Análise Combinatória e Probabilidade**. Rio de Janeiro: Graftex, 1991.

MORO, Maria Lucia; SOARES, Maria Tereza. Níveis de Raciocínio Combinatório e Produto Cartesiano na Escola Fundamental. **Educação Matemática Pesquisa**, São Paulo, v. 8, n. 1, p. 99-124, 2006.

PESSOA, Cristiane; BORBA, Rute. A Compreensão do Raciocínio Combinatório por Alunos do 2º Ano do Ensino Fundamental ao 3º ano do Ensino Médio. In: SEMINÁRIO INTERNACIONAL DE PESQUISA EM EDUCAÇÃO MATEMÁTICA, 4., 2009, Brasília. **Anais...** Brasília: UCB, 2009.

PESSOA, Cristiane; BORBA, Rute. O Raciocínio Combinatório do Início do Ensino Fundamental ao Término do Ensino Médio. ENCONTRO NACIONAL DE EDUCAÇÃO MATEMÁTICA, 10., 2010, Salvador. **Anais...** Salvador, 2010.

VERGNAUD, Gérard. Psicologia do Desenvolvimento Cognitivo e Didática das Matemáticas um Exemplo: as estruturas aditivas. **Análise Psicológica**, Lisboa, v. 1, p. 75-90, 1986.

VERGNAUD, Gerárd. **El Niño, las Matemáticas y la Realidad** – problemas de la enseñanza de las matemáticas en la escuela primaria. Mexico: Trillas, 1991.

VERGNAUD, Gerárd. A Teoria dos Campos Conceptuais. In: BRUN, Jean (Org.). **Didáctica das Matemáticas**. Horizontes pedagógicos. Lisboa: Instituto Piaget, 1996. P. 155-191.

Juliana Azevedo Montenegro holds a PhD in Mathematics and Technological Education. Professor of the Department of Teaching Methods and Techniques – Education Center of Universidade Federal de Pernambuco.
ORCID: http://orcid.org/0000-0003-3570-9581
E-mail: azevedomontenegro.ju@gmail.com

Rute Elizabete de Souza Rosa Borba holds a PhD in Mathematics Education from the Oxford Brookes University. Professor of the Graduate Program in Mathematics and Technological Education – Education Center of Universidade Federal de Pernambuco.
ORCID: http://orcid.org/0000-0002-5098-4461
E-mail: resrborba@gmail.com

Marilena Bittar holds a PhD in Didactics of Mathematics from Université Joseph Fourier, Grenoble. Professor of the Graduate Program in Mathematics Education of Universidade Federal do Mato Grosso do Sul (UFMS).
ORCID: http://orcid.org/0000-0001-9989-7871
E-mail: marilenabittar@gmail.com