SECTION: ARTICLES

Prospective mathematics teachers' knowledge and competence analysing proportionality tasks*[1]

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Abstract

This paper describes the design and implementation of a training process about proportional reasoning with students of a Master's Degree in Secondary Education in Spain. The main objective of the experiment is to explore their initial knowledge and to evaluate how competent the participants on analysing relevant aspects of the epistemic facet of the didactic-mathematical knowledge, which are concretized in the recognition of algebrization levels through different solutions to proportionality problems, are. The participants in the experience have been 33 students, with diverse background education profiles, of the course Initiation to the Teaching Innovation and Investigation in Mathematics Education. Among the results, we highlight the students' limitations and difficulties to identify propositions and procedures and their related arguments. Likewise, the assignment of algebrization levels has also been a complex and difficult task for the participants. Besides, some students have shown deficiencies in the common knowledge of proportionality, which serves as a basis for the didactic-mathematical knowledge of the content. It is concluded that an improvement of the results requires, among other actions, to increase the time allocated to the formative intervention, which will allow extending the number and variety of situations-problems posed, their solutions and discussions.

Keywords

Proportionality - Teacher's education - Onto-semiotic approach - Algebrization levels - Epistemic analysis.

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Introduction

Studying ratios, proportions and proportionality is an important topic in the school curriculum that begins in Primary Education and continues in Secondary, being transversal to different subjects: Mathematics and Art Education in Primary Education, Mathematics, Mathematics oriented to the academic or applied education, Physics and Chemistry, Biology, Technical Drawing, Visual and Audio-visual Education, Artistic Drawing, Volume and Fundamentals of Art in Secondary Education (WILHELMI, 2017, p. 1).

Proportionality can be approached from different points of view or meanings, depending on the contexts of application (daily life, scientific-technical, artistic, geometric, probabilistic, statistical, etc.), which involves the participation of specific objects and processes of these fields when solving the corresponding problems. As indicated by Obando, Vasco and Arboleda (2014, p. 60),

> Since the sixties with Piaget's work on the adolescents' formal reasoning to the present day, with a great diversity of research lines of cognitive, didactic, curricular, epistemological, etc., concern for the difficulties related to teaching and learning of these objects of knowledge remains.

Consequently, teachers' education should take into account developing the mathematical and didactical knowledge and competence regarding this topic, through specific formative interventions. However, the research conducted on the problem of proportional reasoning in teacher education is scarce, as Rivas (2013) points out. This author highlights the works of: Ben-Chaim, Keret, and Ilany (2012); Berk et al. (2009); Rivas and Godino (2010); Rivas, Godino and Castro (2012); Simon and Blume (1994); Sowder et al. (1998); Thompson and Thompson (1994); Thompson and Thompson (1996). Several investigations indicate that both teachers in initial and in service education have difficulties in teaching concepts related to proportionality. Teachers tend to rely on the cross multiplication algorithm (rule of three) in situations of proportionality, without reasoning their appropriateness (RILEY, 2010). Frequently, teachers focus the attention on providing their students with an operational understanding (application of rules and algorithms) sacrificing the development of a conceptual understanding, that is, applying proportional reasoning (LAMON, 2007).

In this article, we report on the design, implementation and results of a formative intervention with prospective secondary school mathematics teachers on the subject of proportionality, whose objective is to explore their knowledge on the subject and to develop some relevant aspects of didactic-mathematical knowledge on this content. Among the results obtained we highlight the recognition of the dialectical relationships between the mathematical knowledge in itself, which enables us to solve proportionality problems of secondary education, and relevant aspects of didactic-mathematical knowledge, such as forecasting different resolution methods for the tasks, the recognition of different algebrization levels put at stake in the solutions and the statement of related problems.

This article is organized in the following sections. In *Theoretical framework and research problem* we introduce elements of the Onto-semiotic Approach (OSA), theoretical framework born within Didactics of Mathematics, and the specific research problem. In *Method* we describe the method used, context, participants, data collection, and analysis instruments; this method can be considered as a design research methodology. *A priori analysis of an evaluation task* shows the *a priori* analysis of one of the tasks actually implemented; this informs of the type of epistemic analysis that students are expected to perform. In *Results* we present the results of the task analysis, in terms of the research questions. In *Discussion*, the results obtained are discussed, thus identifying the level of epistemic analysis competence developed by the prospective teachers. Finally, the last section includes some didactical implications.

Theoretical framework and research problem

A research on mathematics teacher education needs to make explicit the model of knowledge and professional development that is adopted, as well as the methodological approach that guides and bases the research.

The model of teacher's didactic-mathematical knowledge and competence

The aim of the formative experience is to study some specific mathematical knowledge of prospective teachers and the level of epistemic competence analysis for the recognition of such knowledge. In this respect, we have adopted the mathematics teacher's Didactic-Mathematical Knowledge and Competence model, hereinafter DMKC, proposed by Godino, Giacomone et al. (2017). This model develops the Didactic-Mathematical Knowledge model described by Godino (2009) and Pino-Fan and Godino (2015); these theoretical works extend and complement the MKT model (Mathematical Knowledge for Teaching) elaborated by Ball et al. (BALL; LUBIENSKI; MEWBORN, 2001).

In the DMKC model it is considered that the teacher should have *common mathematical knowledge* regarding a certain educational level where he/she teaches (primary, secondary, university level), as well as having an *expanded mathematical content knowledge* that allows him/her to articulate the content with higher education levels; this type of knowledge (common and expanded) is called mathematical knowledge *per se*. On the other hand, as some mathematical content is put at stake, it is clear that the teacher should have a didactic-mathematical or specialized knowledge of the different facets involved in the educational process. The OSA framework proposes taking into account the following facets: epistemic, ecological, cognitive, affective, interactional and mediational facets. Thus, both mathematical knowledge *per se*, and specialized knowledge are closely related. Given the complexity of all the factors involved in a teaching-learning process, in this article we focus on the epistemic facet, which includes as components:

- Recognizing the different meanings of the corresponding content and their interconnection.
- Recognizing the diversity of objects and processes involved (that is, the *onto--semiotic configuration*) for the different meanings.

According to the DMKC model the prospective teacher should have the knowledge mentioned, but should also be *competent* to address the basic didactic problems that are present in the teaching. Godino, Giacomone et al. (2017) define the *competence of epistemic* analysis as that which allows the teacher to identify the objects and processes involved in the mathematical practices necessary to solve the problem-situations required. This recognition allows the teacher

> [...] to anticipate potential and effective learning conflicts, to evaluate the students' mathematical competences and to identify objects (languages, concepts, propositions, procedures, arguments) that should be remembered and institutionalized at the appropriate moments of the study processes. (p. 94).

For this, other specific theoretical and methodological tools are needed, as detailed below.

Pragmatic meaning and epistemic configuration

Two key theoretical notions of the Onto-semiotic Approach are those of *pragmatic meaning* (understood as the system of practices associated with the field of problems from which the object emerges at a given moment) and *onto-semiotic configuration*, defined as the network of objects (concepts, languages, propositions, procedures and arguments) that intervene and emerge from the systems of practices. Both tools together allow us to describe the mathematical activity, from the institutional (epistemic) and personal (cognitive) point of view.

The characteristics of the practices carried out to solve mathematical tasks allow us to define different *algebrization levels*; these levels are defined taking into account the intervention of certain objects and algebraic processes in the resolution of mathematical problems. To define these levels, the degree of generality of the objects, the treatment (calculation) that is applied to these objects, as well as the types of languages used (natural, numerical, diagrammatic, symbolic-literal) are considered (GODINO et al., 2014a).

In the case of proportionality, Godino, Beltrán-Pellicer, and collaborators (2017) have identified three specific pragmatic meanings of proportionality, linked to the algebrization levels involved in solving direct proportionality tasks:

- *arithmetic*, characterized by the application of arithmetic calculation procedures (multiplication, division);
- *proto-algebraic*, focused on the application of the proportion notion;
- *algebraic-functional*, characterized by the application of linear function notion and resolution techniques based on the properties of these functions.

These authors also exemplify the use of the epistemic configuration notion (FONT; GODINO; GALLARDO, 2013; GIACOMONE; GODINO et al., 2017) to identify the network of objects and meanings put at stake in the proto-algebraic and algebraic-functional pragmatic meanings.

Algebrization levels

To achieve a detailed analysis of the network of objects and processes involved in solving proportionality tasks, we rely on the distinction of different algebrization levels of mathematical practice, as they have been developed in various publications (for example, CASTRO; PINO-FAN; MARTÍNEZ-ESCOBAR, 2017; GODINO et al., 2015; GODINO; BELTRÁN-PELLICER et al., 2017).

- Level 0 (arithmetic meaning): indicates absence of algebraic reasoning, that is, concepts or properties of structural or functional nature do not intervene in the practices.
- Level 1 (proto-algebraic meaning, incipient level of algebrization): it begins to recognize operation properties; the relational meaning of the equal sign is also used, so that the concept of equivalence intervenes. In the functional aspect, a general rule is expressed.
- Level 2 (proto-algebraic meaning, intermediate level of algebrization): in the structural aspect, properties of the operations and the relational meaning of the equal sign are used, so the notion of equivalence intervenes. In the functional aspect, a general rule is expressed.
- Level 3 (algebraic-functional meaning): indicates consolidated forms of algebraic reasoning.

The different levels are exemplified and justified in section 4, where the *a priori* analysis of the proportional sharing task proposed to the students is included. Thus, solution 1 (see the *a priori* analysis) corresponds to level 0 of algebrization (arithmetic solution), solution 2 is associated with level 1 (based on the part-whole relationship), solution 3 has a proto-algebraic character of level 2 (missing value) and finally, solution 4 have a level 3 given its more formal nature and that it involves the resolution of an equation of type $Ax + B = Cx$.

Teachers' recognition of the different levels of algebrization in solving mathematical tasks, in particular, tasks that put at stake the notion of proportionality, is considered a key aspect of the DMKC model on this specific content.

Critical semiotic function

To analyze the knowledge put at stake in proportionality tasks, we will use the notion of *semiotic function* of the OSA. In this theoretical framework, a semiotic function is defined as the correspondence between an antecedent object (expression, significant) and another consequent object (content, meaning) established by a subject (person or institution) according to certain criterion or correspondence rule (GODINO, 2017). Contreras et al. (2017) have introduced the notion of *critical semiotic function* (CSF) to identify the key knowledge required to respond to a problem or task.

From the four theoretical tools indicated in this section, we formulate the research problem in the following terms:

- Do prospective teachers have the appropriate common knowledge about proportionality to perform the required epistemic analysis?

• What objects and algebraic processes are most difficult to recognize for the prospective teachers?
• What are the critical semiotic functions in the process of solving the proposed tasks?
• To what extent, has the formative intervention implemented developed the epistemic analysis competence of proportionality tasks, in particular, the recognition of algebrization levels involved in different solutions of the task?

In the following section, we describe the formative experience, which helps to answer the questions posed.

Method

Methodological approach

Given that the research problem is to design, implement and evaluate a training intervention to develop the competences and didactic-mathematical knowledge of prospective secondary education teachers on a specific topic, the methodological approach will be didactic engineering, in our case understood in a generalized sense, as proposed by Godino and cols. (2014b). This approach extends the traditional conception of didactical engineering (ARTIGUE, 1989) in the direction of research based design (COOB et al., 2003), proposing four phases in the research cycle: 1) preliminary study; 2) design of the experiment; 3) implementation; 4) retrospective analysis.

Moreover, for the analysis of the training process the notion of *significant didactic fact* (SDF) is used: “A didactic fact is considered as significant if the actions or didactical practices that compose the fact play a role, or they admit an interpretation, in terms of the intended instructional objective” (GODINO et al., 2014b, p. 174). The SDF identified in section 6 are based on the analysis of the responses of 10 students to a task used as a final evaluation of the learning process.

Research context, participants and data collection

The training experience was carried out within the framework of the Master Degree in Teaching Secondary Education (specialty of Mathematics), during the academic year of 2016-2017, in Spain, as a part of the subject Teaching Innovation and Initiation to Educational Research in Mathematics. This one-year master degree, which includes a period of teaching practices in schools, constitutes the initial training that every university graduate must overcome to teach as a secondary teacher in Spain.

33 students (prospective teachers) with varied academic profile participated in the study; 12 (33.3%) had a mathematics degree; 15 were civil engineers or architects (44.1%), 3 were physicists, and 3 had other engineering specializations. 19 students declare that they had some mathematics teaching experience in private classes; the remaining cases had no teaching experience.

The training intervention was carried out in 4 sessions lasting two and a half hours. Two of these sessions deal with the topic of visualization, in which the analysis of objects and processes is introduced; another session on algebra in which the levels of elementary algebraic reasoning is introduced and a final session in which the competence of onto-semiotic analysis achieved with a proportionality task is evaluated, followed by the discussion of the solutions. Therefore, the fourth session is part of the instructional process and has not a merely evaluative purpose.

The third session (a two-hour workshop) is focused on developing knowledge and competence for the recognition of algebrization levels, considering three moments:

1. Presentation of the characteristics of the elementary algebraic reasoning, and the model of algebrization levels of mathematical activity, based on Godino and collaborators' works (GODINO et al., 2014a; GODINO et al., 2015).

2. Working in teams, it is proposed to carry out the following activities:
    2.1 Solve mathematical tasks (8 were proposed), typical of primary and secondary education, if possible, in several ways.
    2.2. Assign levels of algebraic reasoning to the different solutions given in the previous point to the tasks, taking into account the previously identified objects and algebraic processes.
    2.3 Enunciate related tasks whose solution involves changes in the algebrization levels put at stake.

3. Presentation, discussion of results and drawing the conclusions.

As an optional, complementary assignment to increase the final mark of the course, the solution of 5 tasks was proposed. The protocols corresponding to the solution of one of these tasks are analysed in this article with the purpose of identifying SDF in the cognitive facet of the training process implemented.

Data collection instruments

In each of the course sessions, the responses given in writing to specific tasks, solved through teamwork (of 2 or 3 students) and delivered through the Moodle platform used in the management of the course, were collected. The optional complementary work, carried out by 10 students individually after the end of the course, therefore reflects the learning achieved by these students.

A priori analysis of an evaluation task

In this section, we perform the analysis of one of the tasks proposed in the final evaluation, which will serve as a reference to interpret the students' answers. This is a problem of proportional sharing, taken from Ben-Chaim, Keret and Ilany:

Statement: *Divide 40 nuts between two children in a ratio of 3:5. How many does each child receive?* (2012, p. 134).

Although this problem can be solved by means of an arithmetical reasoning, it is possible to apply other procedures that involve the proto-algebraic levels 1 and 2, as well as the level 3 of algebrization. This type of task (part-part-whole ratio category) involves a relationship between two disjoint quantities (Juan's nuts and Saul's nuts) within a whole (nuts to be distributed), so that the sum of the parts is the whole. The issues posed to the prospective teacher on this problem were:

a) Solve the problem by at least two methods.
b) Identify the knowledge that is put at stake in the solutions.

For each solution, list the sequence of practices that are carried out to solve and justify the solution and complete the table included below, adding the necessary rows.

Chart 1.

Sequence of elementary practices to solve the task	Use and intentionality of the practices	Objects referred in the practices (concepts, propositions, procedures, arguments)
...	...	...

Source: Developed by a student.

c) Taking into account the knowledge put at stake in each solution, recognize the level of algebrization involved in each case.
d) Enunciate and solve related tasks whose solution implies changes in the levels of algebrization, justifying the assignment of the levels.

Solution 1. Arithmetic (0 algebrization level)

Sequence of mathematical practices solving the task:

- We are going to distribute 40 nuts between Juan and Saúl, so that for every 3 that Juan receives, Saúl receives 5.
- Of every 8 nuts that they receive jointly, Juan receives 3. The 40 nuts to be distributed can be grouped into 5 groups of 8, 8×5 = 40.
- Therefore, Juan will receive 3×5 = 15 and Saúl, 5×5 = 25 (nuts)

In this solution, which in the terminology of Ben-Chaim et al. (2012) would be of the type "division by the ratio", particular numerical values intervene and arithmetic operations are applied on such values. Equality has the meaning of result of an operation. Therefore, according to Godino et al. (2014a), the mathematical activity carried out is considered of 0 algebrization level.

This sequence of operative and discursive practices requires that the students be aware of the ratio given and recognize the multiplicative relationship that exists between the quantities included in the statement. The student should understand that the 3:5 ratio describes a situation in which each group would contain 8 elements (3 nuts for Juan and 5 nuts for Saul). In addition, it is necessary to recognize that this ratio 3:5 is maintained both for the total amount to be distributed (the 40 nuts) and for each group within the total. In this way, we will calculate how many groups there are in the total, arriving at the conclusion that these are 5 (40:8).

We identify, therefore, the following critical semiotic functions:

- CSF 1.1. Interpret the 3:5 notation as a multiplicative relation between the amounts of nuts of Juan and Saúl ("for every 3 that Juan receives, Saúl receives 5").
- CSF 1.2. Recognize in the ratio 3:5 the new partial unitary whole, $3 + 5 = 8$.
- CSF 1.3. Divide the total 40 nuts into 5 groups of 8, $40 = 8 \times 5$.

CSF 1.4. Recognize that the 3:5 ratio is maintained for each of the 5 groups of 8 nuts.

- CSF 1.5. Apply a procedure, such as multiplication ($5 \times 3 = 15$; $5 \times 5 = 25$) or repeated addition to arrive at the solution.

Solution 2. Part-whole (proto-algebraic, algebrization level 1)

Sequence of mathematical practices solving the task:

- Since the sharing ratio of the nuts between Juan and Saúl is 3:5, Juan will receive 3/8 of the nuts to be shared.
- That is, Juan receives nuts.
- To know how many nuts Saúl will receive, we only have to subtract from the total number of nuts, the nuts that Juan receives, that is, $40-15 = 25$.

A general relation is established between the ratio of nuts that each child receives and the total nuts to be shared, although this rule is stated with arithmetic and natural language. The mathematical activity carried out involves, therefore, a level 1 of algebraic reasoning.

We can distinguish the following critical semiotic functions, in addition to CSF 1.1 and 1.2:

- CSF 2.1 Establish the correspondence between the sharing ratio and the fraction of the unit that corresponds to each child.
- CSF 2.2. Recognize the use of the fraction as operator (that applied on the initial number of nuts allows us to find the final number of nuts that corresponds to one of the children)
- CSF 2.3. Identify that the number of nuts of the other child is the difference with the total.

Solution 3. Missing value (proto-algebraic, algebrization level 2)

Sequence of mathematical practices solving the task:

- It is intended to distribute 40 nuts between Juan and Saúl, so that for every 3 nuts that Juan receives, Saúl receives 5.
- Of every 8 nuts that they receive jointly, Juan receives 3, that is, 3/8.
- The relation between the number of nuts that Juan receives and the total nuts distributed is of direct proportionality.
- In direct proportionality, the ratios of the corresponding quantities are the same: $3/8 = x/40$; where x is the number of nuts that Juan receives.
- Therefore $x = (3\times40)/8=15$.
- That is, Juan receives 15 nuts and Saúl, $40-15=25$.

Although the solution of a missing value problem, based on the use of ratios and proportions, involves an unknown value and using an equation, the algebrization activity that is carried out is level 2, according to the model of Godino et al. (2014a), since the unknown value appears in a member of the equation established by the proportion.

With this technique, first of all, it is necessary to identify the quantities involved and to recognize the direct proportionality relationship between the magnitudes. The equality of ratios of the corresponding amounts and the equality of cross-products in a proportion to get the unknown value must be evoked.

In addition to CSF 1.1 and CSF 1.2 the following critical semiotic functions are involved:

- CSF 3.1. Recognize that the correspondence between the discrete magnitudes that intervene is of direct proportionality.
- CSF 3.2. Represent the unknown quantity as such unknown and write the equality of ratios.
- CSF 3.3. Solve the first-degree equation posed.

Solution 4. Formal/algebraic (level 3 of algebrization)

Sequence of mathematical practices solving the task:

- We represent by x the number of nuts that Juan receives and by y the number of nuts Saúl receives.
- In the distribution of nuts the proportion must be respected, $5/5 = x/y$
- Also, $x + y = 40$, that is, $y = 40x$. Therefore, $3/(5\ ) = x/(40-x)$
- We proceed to get the unknown: $3(40-x) = 5x$, so that:
$120-3x=5x$; $120=8x$; $x=120/8=15$
- In this way, Juan receives 15 nuts and Saúl $40-15 = 25$.

Assigning an appropriate algebraic level (level 3) to a practice requires the use of symbolic-literal language and to operate analytically/syntactically with this language (GODINO et al., 2014a). In the previous practice, the equations have been presented in a symbolic way and a substitution technique is applied to solve the required equation.

In addition to CSF 1.1, we distinguish the following critical semiotic functions:

- CSF 4.1. Symbolically represent the unknown quantities, x and y.
- CSF 4.2. Establish the proportion based on the sharing ratio (the sharing ratio must be respected for any pair of corresponding amounts).
- CSF 4.3. Express one unknown depending on the other.
- CSF 4.4. Apply a procedure to solve the first-degree equation.

Results

To complete the class sessions, students were proposed to solve 5 tasks, in a non-face-to-face manner, being a complementary work to increase the final grade of the course. The tasks were presented through the Moodle platform. In this section, we include specific examples of students' responses, obtained from the assessment instrument, which will allow us to determine the content common knowledge and the degree of epistemic--cognitive analysis competence achieved with the implementation of the training process.

Solution methods and assignment of algebrization levels

Of the 10 students who performed the optional complementary task, 7 proposed at least one correct solution with 0 algebrization level. The proposed solutions correspond to the categories of "pre-formal" strategies of Ben-Chaim et al. (2012).

In an additive, pre-formal strategy, such as the one shown in Chart 2, students take 8 nuts and distribute them individually giving 3 nuts to one child and 5 nuts to another. Then they take another 8 nuts and distribute them similarly until the 40 nuts to be distributed are exhausted.

Chart 2- Example of an arithmetic solution following an additive strategy

The ratio can be interpreted like this: for every 3 nuts Juan receives, Saúl receives 5. Therefore, the problem solving can be addressed generating successive groups of 3 + 5 nuts until reaching the total number of 40. Therefore, Juan should receive 3 × 5 = 15 nuts and Saúl the remaining: 5 × 5 = 25 nuts.

Source: Developed by a student.

Chart 3 shows another prototypical solution of 0 algebrization level (pre-formal strategy iii) by BEN-CHAIM et al., 2012). In this solution, the student divides the whole, that is, the 40 nuts, into 5 groups of 8 nuts. In each group, for every 3 nuts Juan receives, Saúl receives 5.

Chart 3- Example of arithmetic solution obtained by dividing by the ratio.

If you want to distribute the nuts between the two friends, when I give 5 to one I give 3 to the other, that is, I distribute the nuts from 8 to 8. If I divide 40 from 8 I will get the number of groups I will make, which will be 5. In each cast, I give Juan 3 nuts, therefore, Juan receives 5 × 3 = 15 nuts, while Saúl receives 5 × 5 = 25 nuts.

Source: Developed by a student.

All the students who justified the level of algebrization in the arithmetic or additive strategy solutions did it correctly.

Five students propose proto-algebraic solutions of level 1. These respond to two categories: a) Tabular, b) Part-whole. Two of them elaborate a similar table to the one shown in Table 1:

Table 1- Example of tabular solution

Nuts to share	8	16	24	32	40
Nuts for Juan	3	6	9	12	15
Nuts for Saúl	5	10	15	20	25

Source: Developed by a student.

The use of the table introduces some potential generality in the procedure. The sequence of rounds can be prolonged, which indicates an intensive object of second degree of generality. We therefore consider this activity as level 1 of algebrization; however, both students assign to this solution level 0 of algebrization — one of them does not justify it and the other states that,

The task involves a level 0 of algebrization for the following reasons:

- Extensive (particular) objects are being used, to which only arithmetic operations are applied.
- The sign of equality is purely operational.
- No symbols or variables are used as unknown.

The other three students offer a solution based on the part-whole relationship (BEN-CHAIM et al., 2012, p.138) that agrees with solution 2 of the a priori analysis. Two of the students correctly assign the algebrization level, although one of them does not explain it and the other does it in a confusing way. We present the solution and the justification of the algebrization level assigned to this task by this student in Chart 4.

Chart 4- Proto-algebraic solution of level 1

The sharing ratio is 3:5. That is, of 8 equal parts, 3 correspond to Juan and 5 to Saúl. $\frac{3}{8}$ of 40 to Juan and $\frac{5}{8}$ of 40 to Saúl. $\frac{3}{8}$ of 40 = 15; $\frac{5}{8}$ of 40 = 25 The solution is: 15 nuts for Juan and 25 nuts for Saúl.	To solve the task by this method it has been necessary using basic algebraic knowledge, symbols such as percentage (%) are used, operations with first degree of operationality are performed and equality is used as equivalence. It is therefore a level 1 of algebrization.

Source: Developed by a student.

For the third student, this resolution is typical of the level 0 of algebrization, since it is a resolution without unknowns, which uses the operational meaning of equality.

Two students propose different solutions using the "missing value" strategy (BEN-CHAIM et al., 2012, p. 138). A diagrammatic variant of this solution strategy is the well-known 'rule of three'. This technique somewhat conceals the intervention of ratios and proportion, which may involve a degenerate meaning of the arithmetical proportionality. For example, we present in the left part of Chart 5 the solution given by one of the two students who use the rule of three; in the right part the argument used to affirm level 1 of algebrization of the proposed solution is shown, which is not correct. The other student who solves by the same method assigns the level of algebrization appropriately; however, he does not justify it.

Chart 5- Solution by the rule of three (proto-algebraic level 2)

Since we have a ratio of 3:5, this tells us that of 8 nuts Juan will receive 3 and Saúl 5. We can arrange the following proportionality 8 nuts → 3 Juan 40 nuts → x Juan So, Juan will receive x = (3 × 40) / 8 = 15 nuts. We do the same in the case of Saúl 8 nuts → 5 Saúl 40 nuts → *y* Saúl Saúl will receive x = (5/8) × 40 = 25 nuts	Corresponds to level 1 of algebrization, since there are unknown, but no operations are performed with them nor equations of the form Ax = B are solved.

Source: Developed by a student.

Finally, 5 students elaborate solutions with a proper algebraic level, similar to the formal-algebraic solution number 4 that we included in the a priori analysis of the tasks. Of these 5 students, 2 adequately recognize the level 3 of algebrization in the developed activity although the explanation may not be sufficiently precise (*relational meaning of equality, finding variables also as unknowns and the use of symbolic-literal language, equations appear and it is operated with these unknowns*). One of the students assures that such a solution *follows a level 2 of algebrization, since a system of equations appears, but it is not operated with the unknowns but with numbers.*

Identifying knowledge put at stake in tasks

When a mathematical practice is carried out, a network (configuration) of mathematical objects linked to each other intervenes. On the other hand, the meaning of a mathematical object is determined by the system of mathematical practices in which it appears involved.

These objects intervene as antecedent and/or consequent of semiotic functions. By identifying the critical semiotic functions that connect the different objects present in the configurations, helps to show the complexity of the meanings that the teacher should construct and recognize when solving a problem.

Prospective teachers' answers given to the proportional sharing task show certain difficulties to perform the sequencing of elementary practices, as well as to distinguish the objects (concepts, propositions, procedures and arguments) referred to in them. Half of the students did not distinguish elementary practice within the sequence of resolution, or the configurations they made were very scarce (they only partially recognized some concepts as mathematical objects involved). We have been able to identify that although some students do not differentiate the meanings of fraction as part-whole relationship or as operator, they propose different solutions that involve the two uses. Some confuse the meaning of ratio or interpret its terms incorrectly.

In general, the prospective teachers recognize appropriately the concepts considered in the *a priori* analysis. For example, regarding this practice: "*Since the sharing ratio of the nuts between Juan and Saúl is 3:5, Juan will receive 3/8 of the nuts to be shared*"; (where he critical semiotic functions CSF 1.2 and CSF 2.1 intervene), these concepts should be identified: proportional sharing, ratio, fraction (part-whole).

However, in some students we observe confusion with the meaning of a concept as primary object. For example, the answers given by two students to different practices indicate the *relation of unknown* and the *resolution of equations* as concepts. It is also common considering the rule of three as a concept in the configuration.

Prospective teachers are often imprecise with the notion of proposition. Sometimes it is interpreted as a premise or argument instead of a statement about concepts that needs justification or proof.

The student who made the additive solution shown in Chart 1 (without listing the sequence of practices) pointed out as an associated proposition: *the sharing ratio can be expressed as a fraction of the total.*

In general, we can observe that the students recognize the procedures considered in the a priori analysis. However, they show difficulties by explaining or justifying the intended use of the involved textualized practice. For example, in a solution of additive pre-formal strategy provided by a student, regarding the elementary practice *if they receive nuts at a ratio of 3:5, start by distributing 3 nuts to Juan and 5 nuts to Saúl*, the student includes as a procedure: "*express a proportion as a sum in which the summands meet that proportion*".

The student who developed the solution shown in Chart 3, includes as an elementary practice: *establish that the ratio, can be expressed as a fraction on the whole: 3/8 and 5/8*; referring to this he includes the procedure: *creation of two fractions that represent the parts of a whole.*

The object *argument* is the least identified, and when it is, it does not usually allude to justification of a proposition or procedure, but to the description of the practice.

A student, who solves the problem following a formal/algebraic strategy, includes as arguments: *deduction from the equations or resolution of the system of equations explained in the previous steps.*

The student, who solves the problem using the rule of three, as it is included in Chart 4, indicates that: *the argument is based on the rule of three used.* The same student includes as a concept, the rule of three in the same elementary practice.

Statement of new tasks

Research related to didactic experiences developed with prospective teachers on creating problems for teaching purposes reveals the close link between these tasks and teaching competencies. We emphasize the statement of Malaspina, Mallart and Font (2015, p. 2861-2862):

> A teacher must not only be good at solving problems, but also needs to know how to choose, modify and create them with a didactical purpose. A teacher also needs to be able to critically evaluate the quality of the mathematical activity required to solve the proposed problem and, if necessary, to be able to modify the problem in order to facilitate a richer mathematical activity.

To answer the request posed to the prospective teachers “*enunciate and solve related tasks whose solution involves changes in levels of algebrization, justifying the assignment of these levels*” it is important that they have previously identified the mathematical objects involved in the problem solution and to establish the interrelations between them (in terms of semiotic functions).

For the most part, the prospective teachers have difficulties in elaborating correctly problems that suppose a variation with respect to the initial statement. The proposed statements are too far from the original problem, they are not significant or the context is not proportionality. They interpret that when new variables, coefficients, etc., are introduced; this increases the level of algebrization, and often sustains the belief that greater complexity in solving the problem is associated with a higher level of algebrization.

The types of problems proposed in a pertinent way by the students include the use of parameters to elaborate variants, using mostly the parameter: *number of nuts to be distributed* (Chart 6).

Chart 6- Prototypical example of a problem that refers to a quantity parameter

Juan and Saúl want to share their nuts at a ratio 3:5. Indicate how many nuts correspond to each one based on the total number of nuts.	If you want to share *k* nuts between Juan and Saúl according to ratio 3:5. Determine the value of *k* based on Juan’s number of nuts.

Source: Developed by a student.

The use of parameters, as numerical register and variable coefficients, implies the ability to discriminate the domain and the range of the corresponding parametric function and is indicative of a fourth level of algebrization, according to Godino et al. (2015). The students justify level 4 of algebrization arguing that “*parameters appear, but operations are not carried out with them*”.

Discussion. Application of the CSF to the results analysis

In the OSA theoretical framework, on which this research is based, students’ errors are interpreted in terms of discordance between the institutional meanings of the objects involved in mathematical practices and the personal meanings. A more detailed analysis of such disagreements is made by identifying the semiotic functions established between the objects involved in the corresponding practices.

In this study, the semiotic functions linked to the practices with the highest frequency of errors in solving the problem are CSF 1.1., and 1.2. Next, in Chart 7 we show part of the sequence of practices developed by a student in which we can identify a semiotic conflict related to CSF 1.1. The student does not adequately relate antecedent and consequent in the 3:5 ratio.

Chart 7- Prototypical example 1 of semiotic conflict

We denote x the number of Juan's nuts and we name y the number of Saul's nuts. We get the equations	Argumentation: - First equation: The sum of the nuts of both Juan and Saúl is 40. - Second equation: Three times the number of Juan's nuts is 5 times the number of Saul's nuts for the ratio 3:5.

Source: Developed by a student.

On the other hand, in the solution presented in Chart 8 the student confuses the consequent of the ratio with the number of parts of the whole in the sharing of nuts. He interprets then that the fraction of nuts of the total that Juan receives is 3/5 and, therefore, the ones that Saúl receives are 2/5.

Chart 8- Prototypical example 2 of semiotic conflict

Juan would have 3/5 of the nuts and Saúl 2/5, therefore Juan $=3/5 \times 40 = 24$ Saúl $= 2/5 \times 40 =16$

Source: Developed by a student.

We can conclude that the student has not correctly established CSF 1.1 (he does not recognize the multiplicative relationship between the numbers of nuts of Juan and Saúl) and CSF 1.2. (he does not identify in the ratio 3:5 the new partial unitary whole, 3 + 5 = 8).

A third incorrect solution is shown in Chart 9 where the sharing of nuts is established according to the proportion 3/(2)=x/(40-x). In the latter case, the critical semiotic function CSF 1.1 and CSF 4.2 (to establish the proportion based on the sharing ratio) explain the erroneous response.

Chart 9- Prototypical example 3 of semiotic conflict

Juan: x nuts and Saúl 40–x (x/40) / (40–x/40) = 3/2   x/40–x =3/2 2x=120–3x 5x= 120   x=24; Juan 24 nuts, Saúl: 40 – 24 = 16

Source: Developed by a student.

The data analysis has allowed us to identify some *significant didactical facts* (SDF) in the cognitive facet of the training process implemented. These SDF have a certain incidence in the subjects' sample and, therefore, can be indicative of the manifestation of *didactical phenomena*:

- Despite having completed the mathematics degree, some students show deficiencies in the common content knowledge of proportionality.

- The following facets of the epistemic analysis competence of the tasks have hardly been developed after the formative process:

• Splitting the problem resolution in elementary practices.
• Identification of propositions and procedures, and consequently, the argumentation of these objects.
• Recognition of the proto-algebraic levels of mathematical practices.
• Elaborate new problems by variation of a given statement.

The students of our sample have revealed important deficiencies in their didactic-mathematical knowledge, possibly due to its intrinsic complexity and the limited time dedicated to its development. These limitations lead us to recognize a poor and biased conception of the nature of elementary algebraic reasoning. However, these results were expected given the difficulties highlighted by previous research regarding this type of task (VAN DOOREN; VERSCHAFFEL; ONGHENA, 2003).

Final reflections

Sowder et al. (1998) propose a set of recommendations for teachers' education in the field of proportional reasoning. In particular, they state that "Teachers of compulsory secondary education (middle-grade) should have a deep understanding of the conceptual components of proportional reasoning and its centrality in all mathematical thinking" (p.144). This is so because an important part of mathematics teacher's role is to engage students in experiences that involve critical concepts, while challenging them in their previous ideas with which they arrive at instruction.

In this article, we have described the design and implementation of a formative intervention to develop knowledge and competence for the epistemic analysis of prospective mathematics teachers. On the one hand, it is about innovation based on reflective practice, being this reflection a key aspect of the teacher's professional development (POCHULU; FONT; RODRÍGUEZ, 2016; PONTE et al., 2017). On the other hand, the complexity of the proposed objectives has been highlighted; as pointed out by Giacomone, Godino, and Beltrán-Pellicer (2017), developing this type of competence is a challenge for teacher education and even more so when it involves the content of proportionality and the associated algebraic knowledge, as our results show.

The activity of epistemic reflection that we have implemented in our formative intervention is aimed at achieving a deep understanding, not only of the conceptual components of proportional reasoning, but also the propositional and argumentative components. The recognition of different levels of algebrization in the resolution of proportionality tasks constitutes another important aspect of the epistemic facet of the didactic-mathematical knowledge required for a suitable teaching of this content.

Having an adequate professional relationship with the nature of algebraic reasoning, and mathematical argumentation is essential to manage mathematical learning processes with high epistemic suitability that is, with a high degree of representativeness of institutional meanings implemented with respect to the reference meanings.

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