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A New Jacobi-based Iterative Method for the Classical Analysis of Structures

Abstract

Traditionally, classical methods of structural analysis such as slope-deflection and moment distribution methods (Cross method) are used for primary analysis of structures and also controlling the results of computer programs. The main objective of this paper is to introduce a new method for classical computing and extending it to a matrix formulation. The proposed approach, named the "Slope Distribution Method (SDM)", is based on a Jacobi iterative procedure, in which without forming the system of linear equations, structural displacement values are obtained. Also, to make the method applicable and to use it in computer softwares, the matrix formulation of the approach is developed, where there is no need for iterative procedures and the nodal rotations are obtained through solving only one matrix equation. The SDM is able to analyze frames with non-vertical columns and those with nodal vertical displacement. Whereas current analysis softwares have some elimination for the analysis of non-prismatic members, the proposed method can be applied to analyze structures with any non-prismatic member. The SDM process is also developed for the analysis of dual lateral load resisting systems (moment resisting frames with other lateral load resisting elements such as bracings and shear walls). The advantages of the method over previous ones and also, its accuracy and reliability are presented through the article.

Keywords

Slope-Deflection method, Jacobi iterative procedure, Non-prismatic elements, Non-vertical columns, Matrix formulation, Nodal displacements

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1 INTRODUCTION

Multi-storey building frames may be considered the most widely used kind of structures, especially in urban and residential areas. Population growth and land scarcity increase the need of these types of structures. Substantial and rapid expansion of this necessity in the early decades of the twentieth century, led to the creation of different methods of frames analysis. On the other hand, because of the high indetermination degree of frames, their analysis with traditional force methods was very time consuming. (However, some researchers have conducted a number of studies on the analysis time reduction in the force method, recently (Kaveh, 1992, 2006)). Based on these two reasons, engineers tried to use more applicable and less time consuming methods such as slope-deflection, cross and Kani methods(Cross, 1930; Kani, 1957). Slope-deflection is a manual method to analyze the beams and bending frames and was introduced in 1915 by George A. Maney (Maney, 1915). In this method, with formation of equations and applying nodal and shear force equilibrium conditions, rotate angle of nodes and members are calculated which are placed in corresponding relations to determine end moments of members. Slope-deflection was a revolutionary approach in comparison to the previous structural analysis methods. Instead of calculating static redundancies and using them to find structural deformations, nodal displacements are computed firstly and used to achieve support reactions.

The fundamental problem of the Slope-deflection method appears in structures with a high degree of indetermination, which leads to the formation and solving of linear equation systems; so a lot of efforts were done to find methods that do not have this problem. This method was used for one decade in many cases, until the moment distribution method was developed. In 1930, Cross proposed the new method and could eliminate this problem for non-sway structures (Cross, 1930). In the first step of MDM, the rigid connections of bending frames were assumed to be fixed, and the moments created by external loads on these connections were obtained. These moments are unbalanced at the joints of the original non-restrained structure, and in order to equilibrate the joints, the moments are distributed proportionally to corresponding members' stiffness. The procedure repeats until the unbalanced moments become negligible. The final moments at the joints of members are the sum of all distributed incremental moments (Volokh, 2002). The Cross method in the sway structures needs forming and solving algebraic equations with fewer numbers of unknowns.
The Cross approach has easy interpretation and has been taught in different universities. This method could be used in simple programming of structural analysis, in which end moment of members is considered as the unknown. By using an iterative procedure, (Consecutive moment distribution and transmission among the members connected to rigid nodes), the system of linear equations created by the slope – deflection method is solved. Consequently, beams and frames are analyzed without solving any system of equation, directly. End shear force for each member is also obtained through static equilibrium.

In addition, the Gaspar Kani's method (Kani, 1957) which is used in structural analysis, uses iterative procedures to solve the slope deflection equations system (Behravesh and Kaveh, 1990). In the moment distribution method, the unknowns (end moments of structural members) are obtained through performing iteration on their changes (increments), whereas in the Kani method, the iteration procedure is applied on the unknowns themselves.

Behravesh and Kaveh(Behravesh and Kaveh, 1990) explained the relationship between moment distribution and Kani methods and a numerical iterative procedure, and showed that the calculation trends in these two methods are similar to the Jacobi iteration procedure that has been used to solve the equations of classical displacement. The Jacobi iteration approach in both methods is converged, if the stiffness matrix is diagonally dominant. A study of Volokh (Volokh, 2002)also shows the correlation of MDM to the Jacobi iteration approach.

In this paper, a new analysis approach is proposed which is named "Slope Distribution Method" (SDM). The method is explained in manual formulation and is extended in matrix form. In this approach, there is no need to form and solve the system of algebraic equations which is its advantage compared to the slope–deflection approach. In comparison to the moment distribution method(Cross method), in the proposed method, the distribution and carry-over procedures are merged, and unlike the Cross and Kani approaches, instead of distributing and transmitting the moments at several members' ends attached to each structural rigid node, only the nodal rotations (slopes) are distributed. These properties decrease the analysis parameters and lead to analysis time reduction of the proposed method. As the numbers of unknowns depends on the number of nodes and not the number of members connected to each node, the unknowns are limited in comparison to the Cross and Kani methods. According to the above-mentioned advantages, it will be shown that the proposed approach is less time-consuming than well-known methods of slope-deflection, Cross and Kani. The SDM process is also a proper and low-cost procedure to analyze the structures with non-prismatic members. It should be noted that the current analysis softwares are not able to model every kind of non-prismatic member; whereas by defining the corresponding coefficients, the SDM is capable of analyzing these structures properly.

The paper is organized as follows:

The following section describes the Jacobi scheme. Afterwards, a brief review of the slope – deflection method is explained. Thereafter, the proposed slope distribution method (SDM) will be illustrated. The relation between the new method and the Jacobi scheme is shown, and the relations of the proposed method for non-prismatic members are expanded. In the next part, the matrix formulation of the method is extended. The approach is followed with some numerical examples and is compared with other traditional methods to show its advantages. The ability of the approach in analysis of dual systems, frames with non-vertical columns and also frames with vertical nodal displacements are shown. The accuracy and reliability of the approach are shown through the paper, and finally, the paper ends with some conclusions that are useful for researchers.

2 THE JACOBI SCHEME

The Jacobi scheme is a iterative method for solving the diagonally-dominant systems of equations (Golub and Van Loan, 1996; Young, 2013). In mathematics, a system of equations can be presented by a matrix format. If the magnitude of the diagonal entry in each row exceeds the sum of the magnitudes of the non-diagonal entries in that row, the matrix is called diagonally-dominant. In fact, matrix A is strictly diagonally dominant if (Behravesh and Kaveh, 1990; Datta, 2010).

$$\sum_{j\neq i}|a_{ij}| < |a_{ii}| \tag{1}$$

Where, a_{ij} denotes the entry in the ith row and jth column. For solving a set of equations in the matrix form Ax=B, the Jacobi relation is illustrated as (Golub and Van Loan, 1996; McCormac and Nelson, 1997; McGuire et al., 2000):

$$x_i^{(k)} = \frac{b_i - \sum_{j\neq i} a_{ij}\ x_j^{(k-1)}}{a_{ii}} \tag{2}$$

In which, a_{ij} and b_i denote the elements of coefficient matrix.

3 A BRIEF REVIEW OF SLOPE – DEFLECTION METHOD

As mentioned previously, in the suggested method, nodal displacements are considered the main unknowns. As nodal displacements are also the unknowns of the equations in the slope deflection method, this method will be reviewed briefly in this part. It should be noticed that unlike the slope-deflection method, the proposed method uses iterative procedures to solve the equations, and in general, it will not be necessary to solve the equations directly. The Slope-deflection method, as applied nowadays in analyzing the structures with rigid joints, was introduced by G.A.Maney in 1915. This method is based on evaluating the nodal rotations and displacements.

After finding nodal rotations, end moment and shear of members can be calculated using the slope-deflection relations. These relations express end moment of a member based on rotations of nodes and elements, and also the external loads on the members. For element ij with constant bending rigidity, and length L (Figure 1), the relation is as follows (McCormac and Nelson, 1997; Norris et al., 1976).

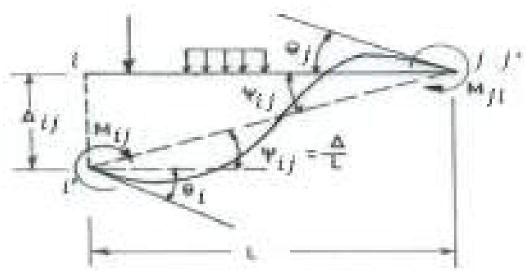


Figure 1: Deformation curve of the elastic beam.

$$M_{ij} = \frac{2EI}{L}\{2\Theta_i + \Theta_j - 3\varphi_{ij}\} + FEM_{ij} \tag{3}$$

$$M_{ji} = \frac{2EI}{L}\{2\Theta_j + \Theta_i - 3\varphi_{ij}\} + FEM_{ji} \tag{4}$$

In these equations, the effects of axial and shear deformation are neglected, because the effect of these two parameters on the bending deformation is small. Using equations (3) and (4) and writing the equations of static equilibrium for the member ij, the end shear will be found as follows(Megson, 2005):

$$V_{ij} = -\frac{6EI}{L^2}\{\Theta_i + \Theta_j - 2\varphi_{ij}\} + FEV_{ij} \tag{5}$$

$$V_{ji} = \frac{6EI}{L^2}\{\Theta_i + \Theta_j - 2\varphi_{ij}\} + FEV_{ji} \tag{6}$$

Θ_i and Θ_j are the rotation of node i and j respectively, and φ_{ij} shows the sway rotation of member ij that is calculated through the below equation. FEM and FEV are the fixed end moment and shear.

$$\varphi_{ij} = \frac{\Delta_{ij}}{L} = \frac{\Delta_i - \Delta_j}{L} \tag{7}$$

Δ:Relative displacement of the member ends.
In Table 1, values of fixed end shear, moment and sign convention for different loading cases are shown(Mau, 2003):

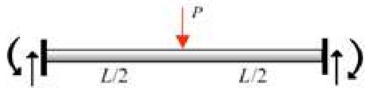
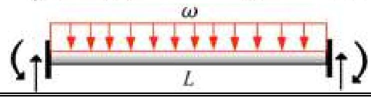
FEM	FEV	Loads	FEV	FEM
$-\frac{PL}{8}$	$\frac{P}{2}$		$\frac{P}{2}$	$\frac{PL}{8}$
$-\frac{\omega L^2}{12}$	$\frac{\omega L}{2}$		$\frac{\omega L}{2}$	$\frac{\omega L^2}{12}$

Table 1: Values of fixed end shear and moment (Mau, 2003).

Also, Equations (3) to (6) could be written in matrix formulation (Megson, 2005):

$$\begin{Bmatrix} M_{ij} \\ M_{ji} \\ V_{ij} \\ V_{ji} \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & \frac{-6EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} & \frac{-6EI}{L} \\ \frac{-6EI}{L^2} & \frac{-6EI}{L^2} & \frac{12EI}{L^2} \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{-12EI}{L^2} \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \\ \varphi_{ij} \end{Bmatrix} + \begin{Bmatrix} FEM_{ij} \\ FEM_{ji} \\ FEV_{ij} \\ FEV_{ji} \end{Bmatrix} \tag{8}$$

By replacing the values of nodal displacements in the above matrix formulation, end shear force and moment of members are calculated. This matrix formation can be used in programming of the SDM matrix procedure.

If a static equilibrium equation is written for a non-prismatic member, the shear force at the end of each member can be calculated. Given that the values of bending moments are defined according to nodal rotations, lateral rotation of members and also the external loads, the shears of members' ends are also obtained based on these three parameters:

$$V_{ij} = -\left\{T_i\theta_i + T_j\theta_j - (T_i + T_j)\varphi_{ij}\right\} + FEV_{ij} \quad (9)$$

$$V_{ji} = \left\{T_i\theta_i + T_j\theta_j - (T_i + T_j)\,\varphi_{ij}\right\} + FEV_{ji} \quad (10)$$

where T_i and T_j are defined as follows:

$$T_i = \frac{S_{ij} + C_{ji}S_{ji}}{L} = \frac{S_{ij} + C_{ij}S_{ij}}{L} = \frac{S_{ij}\,(1 + C_{ij})}{L} \quad (11)$$

$$T_j = \frac{S_{ji} + C_{ij}S_{ij}}{L} = \frac{S_{ji} + C_{ji}S_{ji}}{L} = \frac{S_{ji}\,(1 + C_{ji})}{L} \quad (12)$$

In the above equations, C_{ij} and C_{ji} are moment carry-over factors. Also, S_{ij} and S_{ji} are bending stiffness values of the two ends of the non-prismatic member and could be calculated by integration methods or using related handbooks(Association, 1958).

4 THE PROPOSED APPROACH: SLOPE DISTRIBUTION METHOD (SDM)

4.1 Introduction

SDM is an iterative method which is based on a series of computational cycles that are repeated until the results converge to a final value in each stage. By this repetition, simultaneous solving of algebraic equations is not necessary. This is a unique structural analysis method, in which the solution is obtained through an iterative process, without solving any equations to find problem parameters. SDM uses a Jacobi iterative scheme to find displacements in the system of equations produced by the slope-deflection method.

4.2 SDM equations

From static equilibrium, if external moment (M^{ex}) is applied at the node B, the rotation at this node will take place until moment equilibrium is achieved, $\sum M_B = M^{ex}$ (Figure 2).

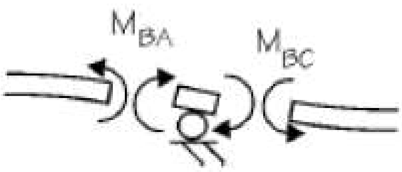


Figure 2: Free-body diagram for moment at node B (Lopes et al., 2011).

$$M_{BA} + M_{BC} = M^{ex} \quad (13)$$

Using the equations from the Slope-Deflection Method given by Laursen (Laursen, 1988) and Kassimali (Kassimali, 1995), we have:

$$M_{BA} = \left[\frac{2EI}{L}\right]_{BA} \{2\theta_B + \theta_A - 3\varphi_{BA}\} + FEM_{BA} = \left[\frac{4EI}{L}\right]_{BA} (\theta_B) - \left[\frac{6EI}{L}\right]_{BA} \varphi_{BA} + FEM_{BA} \quad (14)$$

$$M_{BC} = \left[\frac{2EI}{L}\right]_{BC} \{2\theta_B + \theta_C - 3\varphi_{BC}\} + FEM_{BC} = \left[\frac{4EI}{L}\right]_{BC} (\theta_B) - \left[\frac{6EI}{L}\right]_{BC} \varphi_{BC} + FEM_{BC} \quad (15)$$

L is the length of the member, EI is the cross-section rigidity, and θ_B is the rotation at B (Kassimali, 1995; Laursen, 1988).

Placing Equations (14) and (15) into equation (13) and solving for θ_B, we have:

$$\theta_B = \frac{M_B^{ex} - FEM_{BA} - FEM_{BC}}{\left[\frac{4\,EI}{L}\right]_{BA} + \left[\frac{4\,EI}{L}\right]_{BC}} + \frac{\left[\frac{6\,EI}{L}\right]_{BA}}{\left[\frac{4\,EI}{L}\right]_{BA} + \left[\frac{4\,EI}{L}\right]_{BC}} \varphi_{BA} + \frac{\left[\frac{6\,EI}{L}\right]_{BC}}{\left[\frac{4\,EI}{L}\right]_{BA} + \left[\frac{4\,EI}{L}\right]_{BC}} \varphi_{BC} \quad (16)$$

In general, initial rotation of rigid node "i" has a direct relationship with the external load and the lateral rotation angle of members connected to the node. This dependency is shown in equation (17):

$$\theta_i^{(0)} = \frac{M_i^{ex} - \sum_{j=1}^{j=N} FEM_{ij}}{4\left(\sum_{j=1}^{j=N} K_{ij}\right)} + \sum_{j=1}^{j=N} 1.5\, R_{ij}\, \varphi_{ij}^{(0)} \quad (17)$$

In the mentioned equation, N is the number of elements connected to node i, $R_{ij} = \frac{K_{ij}}{\sum_{j=1}^{j=N} K_{ij}}$ is the relative bending stiffness of members connected to node i and $K_{ij} = \left(\frac{EI}{L}\right)_{ij}$ is bending stiffness of each of these members.

In SDM, to calculate the slope of node i, the recursive series is defined as in equation (18):

$$\theta_i^{(n)} = \theta_i^{(n-1)} + \Delta\theta_i^{(n-1)} = \theta_i^{(0)} + \sum_{\varepsilon=0}^{\varepsilon=n-1} \Delta\theta_i^{(\varepsilon)} \qquad n \geq 1 \quad (18)$$

Where n is the number of analysis stage and $\theta_i^{(0)}$ is the rotation value of node i under primary loads that are obtained through Equation (17). $\Delta\theta_i^{(\varepsilon)}$ is the difference between the connection slope at node i, in two successive stages, due to incremental unbalanced moments. The value of $\Delta\theta_i^{(\varepsilon)}$ will decrease during the analysis procedure and finally tends toward zero. In this condition, the connection rotation converges to its actual value, and the system will be balanced.

As discussed before, in the SDM method, incremental unbalanced moments at members' ends decrease to achieve nodes' equilibrium. In each step, for moment equilibrium condition at each node, it is necessary that: $\Delta M_i^{(n)} = M_i^{ex} - \sum_{j=1}^{j=N} M_{ij}^{(n)} \rightarrow 0$.

Repeating the slope distribution approach, equilibrium is established in connection when $n \rightarrow \infty$. In this condition, $\Delta M_i^{(n)} = M_i^{ex} - \sum_{j=1}^{j=N} M_{ij}^{(n)} = 0$. Also, incremental unbalanced moments at node i, lead to change in rotation of this node in each cycle of the iterative procedure ($\Delta\theta_i^{(n)}$). For nth Cycle, $\Delta\theta_i^{(n)}$ is calculated through Equation (19).

$$\Delta\boldsymbol{\theta}_i^{(n)} = \frac{\Delta \boldsymbol{M}_i^{(n)}}{4\sum_{j=1}^{j=N} \boldsymbol{K}_{ij}} = \frac{\boldsymbol{M}_i^{ex} - \sum_{j=1}^{j=N} \boldsymbol{M}_{ij}^{(n)}}{4\sum_{j=1}^{j=N} \boldsymbol{K}_{ij}} \tag{19}$$

Where, M_i^{ex} is external moment and $\sum_{j=1}^{j=N} M_{ij}^{(n)}$ is the sum of incrementally unbalanced moments at node i and $\sum_{j=1}^{j=N} K_{ij}$ is the sum of the flexural stiffness of members connected to the ith node. n is the number of the analysis cycle and N is the number of members connected to node i. Equation (14) is obtained through a simple static equilibrium on node i. The equation is the same as the Jacobi iterative procedure for solving the system of linear equations which has been described in previous parts. Using equation (19), the incremental unbalanced rotations caused by unbalanced moments, and consequently, the nodal rotations are calculated.

Inserting equations (3), (4) and (18) in equation (19), the fundamental equations of the SDM are consequently achieved:

$$\Delta\theta_i^{(0)} = \sum_{j=1}^{j=N} -\frac{1}{2} R_{ij}\theta_j^{(0)} \tag{20}$$

$$\Delta\theta_i^{(n)} = \sum_{j=1}^{j=N} -\frac{1}{2} R_{ij}\Delta\theta_j^{(n-1)} + \sum_{j=1}^{j=N} 1.5\, R_{ij} \left(\Delta\varphi_{ij}^{(n-1)} = \varphi_{ij}^{(n)} - \varphi_{ij}^{(n-1)}\right) \tag{21}$$

To compute the values of lateral rotation of the member ij,($\varphi_{ij}^{(n)}$), a shear equilibrium equation should be used in each storey of the structure (Figure 3).

Through writing static equilibrium in each storey of the structure (i.e. Sth storey), the auxiliary equation is obtained to analyze structure without forming any equation system:

$$V_S = \sum_{j=1}^{j=m} V_j \tag{22}$$

In which, V_S is the shear of the Sth storey resulted from lateral loads such as earthquake or wind and V_j is the shear of the jth column of that storey and obtained through relations (5) and (6). The $\varphi_s^{(n)}$ is calculated as follows:

$$\varphi_s^{(n)} = \varphi_s^{(0)} + \sum_{1}^{m} 0.5\, D_{ij} \left(\theta_i^{(n)} + \theta_j^{(n)}\right) \tag{23}$$

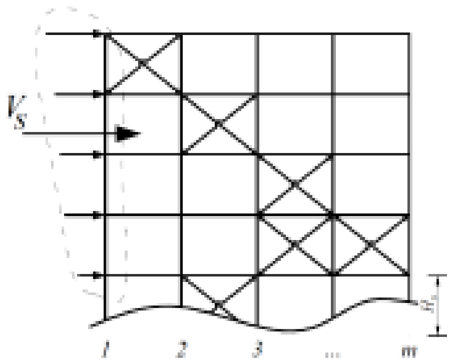


Figure 3: The horizontal Shear of the frame in the Sth Storey (Rezaee Pajand and Aftabi Suny, 2010).

m is the number of columns in the Sth storey and i and j are the two ends of column ij; $\varphi_S^{(0)}$ is the initial rotation of member and D_{ij} is the relative shear stiffness of the storey columns which is defined through equations (24) and (25):

$$\varphi_S^{(0)} = \frac{V_S - \sum_1^m FEV}{12\sum_1^m \mu_{ij}} \tag{24}$$

$$D_{ij} = \left(\frac{\mu_{ij}}{\sum_1^m \mu_{ij}}\right)_S \qquad \mu_{ij} = \left(\frac{EI}{L^2}\right)_{ij} \tag{25}$$

FEV is the fixed end shear of columns in the Sth storey; μ_{ij} is shear stiffness of column ij, and D_{ij} is the shear stiffness ratio of columns in the storey of interest.

The SDM method can be extended for the condition where the beams jointed to a node have a lateral displacement, and also when the φ values of structure members in a storey are not the same. As the lateral displacements of structural members (Δ) in each storey are related by geometric relationships, the equations can be expressed based on the lateral rotation of a storey member (φ_S) that is considered as the base and independent one. So, the above equations will be corrected as follows:

$$\theta_i^{(0)} = \{\frac{M_i^{ex} - \sum_{j=1}^{j=N} FEM_{ij}}{4\sum_{j=1}^{j=N} K_{ij}}\} + \sum_{j=1}^{j=N} 1.5\, R_{ij}\, \left(\frac{\varphi_{ij}}{\varphi_S}\right)\varphi_S^{(0)} \tag{26}$$

$$\Delta\theta_i^{(n)} = \sum_{j=1}^{j=N} -\frac{1}{2} R_{ij}\Delta\theta_j^{(n-1)} + \sum_{j=1}^{j=N} 1.5\, R_{ij}\left(\frac{\varphi_{ij}}{\varphi_S}\right)\left(\Delta\varphi_S^{(n-1)} = \varphi_S^{(n)} - \varphi_S^{(n-1)}\right) \tag{27}$$

$$\varphi_S^{(0)} = \frac{V_S - \sum_1^m FEV}{12\sum_1^m \left(\frac{\varphi_{ij}}{\varphi_S}\right)\mu_{ij}} \tag{28}$$

$$D_{ij} = \left(\frac{\mu_{ij}}{\sum_{1}^{m}\left(\frac{\varphi_{ij}}{\varphi_S}\right)\mu_{ij}}\right)_S \tag{29}$$

In the above equations, N is the number of elements connected to node i, m is the number of columns in the storey "S" and n is the step number of the recursive process.

As the lateral independent rotation of each storey of structure ($\varphi_s^{(n)}$) is related to nodal rotations ($\theta_i^{(n)}$), we can place equation (23) into equation (27) and simplify the analysis stages:

$$\Delta\theta_i^{(1)} = \sum_{j=1}^{j=N} \omega_{ij}\,\Delta\theta_j^{(0)} + \sum_{j=1}^{j=N} \widetilde{\omega}_{ij}\left\{\sum\nolimits_{1}^{m} \delta_{ij}\left(\theta_i^{(1)} + \theta_j^{(1)}\right)\right\} \tag{30}$$

For $2 \leq n$ we have:

$$\Delta\theta_i^{(n)} = \sum_{j=1}^{j=N} \omega_{ij}\,\Delta\theta_j^{(n-1)} + \sum_{j=1}^{j=N} \widetilde{\omega}_{ij}\left\{\sum\nolimits_{1}^{m} \delta_{ij}\left(\Delta\theta_i^{(n-1)} + \Delta\theta_j^{(n-1)}\right)\right\} \tag{31}$$

Where, $\omega_{ij} = -\frac{1}{2}R_{ij}$ and $\widetilde{\omega}_{ij} = \left(\frac{\phi_{ij}}{\phi_s}\right)\omega_{ij}$ and R_{ij} is the relative bending stiffness of members connected to node i which was introduced before. Also, the value of the carry-over factor (δ_{ij}) is defined as: $\delta_{ij} = -1.5\,D_{ij}$.

The current equation performs as a recurrence relation. The rotation value of each node will be updated in each stage and finally converges to its accurate values. Closing the values of nodal rotations in two consecutive steps by desired accuracy is considered as the end of the SDM process.
If a structure has non-prismatic members, then the above equations will be changed as follows:

$$\theta_i^{(0)} = \left(\frac{M_i^{ex} - \sum_{j=1}^{j=N} FEM_{ij}}{\sum_{j=1}^{j=N} S_{ij}}\right) + \sum_{j=1}^{j=N} \widetilde{\omega}_{ij}\,\varphi_S^{(0)} \tag{32}$$

$$\Delta\theta_i^{(0)} = \sum_{j=1}^{j=N} \omega_{ij}\;\theta_j^{(0)} \tag{33}$$

$$\varphi_S^{(n)} = \varphi_S^{(0)} + \sum_{1}^{m} D_r\,\theta_r^{(n)} \tag{34}$$

$$\Delta\theta_i^{(1)} = \sum_{j=1}^{j=N} \omega_{ij}\;\Delta\theta_j^{(0)} + \sum_{j=1}^{j=N} \widetilde{\omega}_{ij}\left(\sum_{1}^{m} D_r\,\theta_r^{(1)}\right) \tag{35}$$

For n≥ 2 we have:

$$\Delta\theta_i^{(n)} = \sum_{j=1}^{j=N} \omega_{ij}\ \Delta\theta_j^{(n-1)} + \sum_{j=1}^{j=N} \widetilde{\omega}_{ij} \left(\sum_{1}^{m} D_r \Delta\theta_r^{(n-1)} \right) \tag{36}$$

$$\phi_s^{(0)} = \frac{V_S - \sum_1^m FEV_{ij}}{\sum_1^m \left(\frac{\varphi_{ij}}{\varphi_s}\right) T_r} \tag{37}$$

As the bending stiffness coefficients and carry-over factor are different at two ends of storey columns in non-prismatic structures, the shear stiffness ratio (D_r) in node "r" of the column element should be calculated separately as follows:

$$D_r = \frac{T_r}{\sum_1^m \left(\frac{\varphi_{ij}}{\varphi_s}\right) T_r} \tag{38}$$

T_r is the shear stiffness of node "r" of the non-prismatic column which is obtained through equations (11) and (13), and $\sum_1^m T_r$ is the total shear stiffness of all nodes of columns (nodes of the two ends of the columns) in the storey of interest (i.e. storey "s") of the structure. Also, θ_r is the rotation of node "r" of the non-prismatic column.

$$R_{ij} = \frac{S_{ij}}{\sum_{j=1}^{j=N} S_{ij}} \tag{39}$$

$$\omega_{ij} = -\ C_{ij}\, R_{ij} \tag{40}$$

$$\widetilde{\omega}_{ij} = \left(1 +\ C_{ij}\right)\left(\frac{\varphi_{ij}}{\varphi_s}\right) R_{ij} \tag{41}$$

When the non-prismatic structure has no lateral displacement in the storey "S"($\varphi_s = 0$), by simplification, the below relations are conducted:

$$\Delta\theta_i^{(0)} = \sum_{j=1}^{j=N} \omega_{ij}\ \theta_j^{(0)} \tag{42}$$

$$\Delta\theta_i^{(n)} = \sum_{j=1}^{j=N} \omega_{ij}\ \Delta\theta_j^{(n-1)} \tag{43}$$

For a special situation, prismatic structure without any lateral displacement in the storey "S" ($\varphi_s = 0$), $C_{ij} = \frac{1}{2}$ is replaced in the above equations.

4.3 Sign Convention

Sign convention in the proposed SDM method is similar to other manual analysis methods in which clockwise direction for θ,φ, FEM and M is considered as positive. According to this convention, the forces and linear displacements are taken as positive when the element rotates clockwise. It is in accordance with the positive directions used by some other researchers(e.g.(Timoshenko and Goodier, 1987). Also, the positive direction for shear force is the direction in which the force rotates the element clockwise (Figure 4).

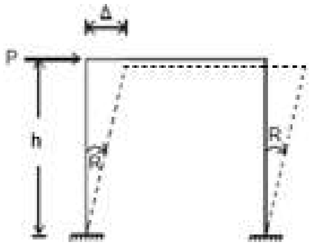


Figure 4: Positive sign convention for force and displacement.

5 MATRIX FORMULATION OF SDM

Manual approach of SDM should be repeated in a consecutive process until the final values of structure node displacements are obtained; this process is time-consuming and boring in structures with high degrees of indeterminacy. In fact, in matrix formulation of SDM, the initial information such as($\varphi^{(0)}, \theta^{(0)}$, δ and ω) will be calculated manually by the user and a computer will be used to perform the analysis process. Therefore, introducing an accurate solution without any repetition procedure and forming and solving only one matrix equation to give the unknowns, is the main purpose of the SDM matrix procedure.

If the equations (23), (25) and (26) are defined by matrix formulation, then we will have:

$$\{\Delta\theta^{(0)}\} = [\omega]^{\wedge 1}\{\theta^{(0)}\} \tag{44}$$

$$\{\Delta\theta^{(1)}\} = [\omega]\{\Delta\theta^{(0)}\} + [\tilde{\omega}][\delta]\{\theta^{(1)}\} \tag{45}$$

If we have:

$$[\tau] = [\tilde{\omega}][\delta] \tag{46}$$

And by inserting equation (46) in (47) we have:

$$\{\Delta\theta^{(1)}\} = [\omega]^{\wedge 2}\{\theta^{(0)}\} + [\tau]\{\theta^{(1)}\} \tag{47}$$

$\{\theta\}$ is a vector including the last rotations in structure nodes and has a rank of (p×1) and $\{\theta^{(0)}\}$ is also a column vector including the initial rotations of structure nodes due to external loads with a rank of (p×1); $[\omega]$ is a square matrix with a rank of (p×p) which includes the slope distribution factors of the members, and p is the number of rigid nodes and structure supports, and [I] is a unique matrix with an equal rank with [ω]. The matrices of $[\tilde{\omega}]$ and [δ] have the same rank with [ω] and their entries are calculated just in two end nodes of the columns of each structure storey. They are replaced in corresponding matrixes to estimate the lateral displacement of structure (φ). In non-prismatic structures, due to the difference between bending stiffness coefficients and carry-over factors in two ends of storey columns, we have $\delta_{ij} \neq \delta_{ji}$ and they should be calculated separately for each column node. In a multi-storey structure, $[\tilde{\omega}]$and [δ] matrices should be formed separately for each storey of the structure and the matrix [τ] for all of the structure is calculated through the superposition principle by using the partial matrix of [τ] in each storey of the structure. Therefore:

$$[\tau] = \sum_{r=1}^{r=N_s} [\tau]_r \tag{48}$$

In which, Ns is the number of stories in the structure.

Each row in matrix $[\tilde{\omega}]$ is corresponded to the rigid node of structure storey, and the value of $[\tilde{\omega}]_i$ in node ί is equal to a total of $(1 + C_{ij})R_{ij}(\frac{\varphi_{ij}}{\varphi_s})$ for the members connected to node ί on the storey "S"; which is placed in each entry of the ith row of matrix $[\tilde{\omega}]$. Therefore:

$$[\tilde{\omega}]_i^S = \sum_{j=1}^{j=N} (1 + C_{ij}) \left(\frac{\varphi_{ij}}{\varphi_s}\right) R_{ij} \tag{49}$$

Indeed, the values of relative shear stiffness (D_r), calculated through equation (38) are placed in matrix D, and matrix [δ] is obtained through equation (50):

$$[\delta_{ij}] = [D_{ji}] \Rightarrow [\delta] = [D]^{\text{Transpose}} \tag{50}$$

It should be noted that during the process of placing the entries in the global matrix of the structure, the zero value is considered for entries corresponding to nodes which do not exist in the given storey. If the structure includes prismatic members, equations (49) and (50) are changed as follows:

$$[\tilde{\omega}]_i^S = \sum_{j=1}^{j=N} \left(\frac{\varphi_{ij}}{\varphi_s}\right) \omega_{ij} \tag{51}$$

$$[\delta_{ij}] = [-1.5\, D_{ij}] \tag{52}$$

By calculating the above parameters, the rotations in structural nodes can be calculated. To this end, equation (27) can be changed to a matrix format. Equations (53) and (54) show the 2nd step of calculating ΔΘ:

$$\{\Delta\Theta^{(2)}\} = [\omega]\{\Delta\Theta^{(1)}\} + [\tau]\{\Delta\Theta^{(1)}\} \tag{53}$$

$$\{\Delta\Theta^{(2)}\} = ([\omega] + [\tau])^{\wedge 1}\{\Delta\Theta^{(1)}\} \quad (54)$$

In this way, through expanding relationship (27), other values of $\{\Delta\theta^{(n)}\}$ can be obtained for n≥ 2:

$$\{\Delta\theta^{(n)}\} = ([\omega] + [\tau])^{\wedge n-1}\{\Delta\theta^{(1)}\} \quad (55)$$

Then, through placing equations (44), (45) and (55) in (18), the values of $\{\theta^{(n)}\}$ will be calculated in the last cycle. Therefore, it can be concluded that:

$$\{\theta\} = \left[([I] + [\omega]) + [\,[I] - ([\omega] + [\tau])\,]^{-1} \times ([\omega]^{\wedge 2} + [\tau]([I] + [\omega]))\right]\{\theta^{(0)}\} \quad (56)$$

If we define: $[Z] = \text{Inverse}\{[I] - [V]\}$, $[V] = [\omega] + [\tau]$ and $[U] = [I] + [\omega]$, then we will have:

$$\{\theta\} = \left[[U] + [Z]\left([\omega]^{\wedge 2} + [\tau][U]\right)\right]\{\theta^{(0)}\} \quad (57)$$

Final values of nodal rotations can be calculated through equation (56) by forming and solving only one matrix equation. End moments and shears of members can be evaluated through placing the nodal rotations in slope-deflection equations. The above equation can be used in computer programming of SDM. It should be noted that the slope distribution factors in matrix formulations of SDM are calculated for all members connected to the structure nodes, whereas the supports will also be considered as nodes. At first, to compute the bending stiffness, carry-over factors of members and fixed-end moments for non-prismatic members, the "hand book of frame constants"((Association, 1958)) could be used. Then, the matrix of slope distribution factor, [ω], will be formed for a structure. This matrix is a square one with a rank of (p×p), where p is the number of structure nodes. The entry ω_{ij} in matrix [ω] is equal to the slope distribution factor of member ij, which connects the nodes i and j. For those nodes which are not connected to each other by any member, $\omega_{ij}= 0$; for original diameter entries, $\omega_{ii}= 0$ and for fixed and two-roller supports in which $R_{ij} = 0$, ω_{ij} is also considered as zero. For hinge and roller supports, $\omega_{ij}= -C_{ij}$, because the value of R_{ij} is equal to unity (see equation (40)). The value of initial rotation for a rigid node of structure under external loading is obtained by equations (26) and (32). For fixed and two-roller supports, $\theta_i^{(0)} = 0$ and for hinged support and roller support, it is computed through equation (26). It is worth noting that the initial rotation value in matrix formulation of SDM is computed for a hinged support; So, the bending stiffness, carry-over factor and the fixed-end moments obtained through "handbooks of frame constants" ((Association, 1958)) will be used in the equations without any changes. If the structure has no lateral displacement ($\varphi_{ij} = 0$), equation (56) will be simplified as follows:

$$\{\theta\} = \left[[I] - [\omega]\right]^{-1}\{\theta^{(0)}\} \quad (58)$$

6 NUMERICAL EXAMPLES

6.1 In the first example, the proposed procedure of SDM is applied on the continuous beam of Figure 5 from reference(Lopes et al., 2011). This beam has 4 supports in A, B, C and D. The supports

A and D are fixed. Other specifications are shown in Figure 5. The cross section of beam is U-shape (UPN 80).EI is assumed to be constant through the length of the beam.

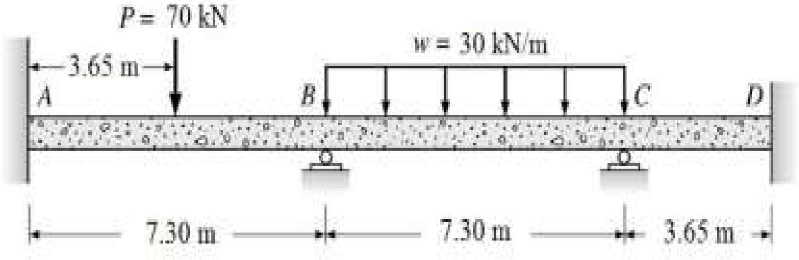


Figure 5: Continuous beam with asymmetric loading (Lopes et al., 2011).

Fixed end moment of members and initial rotation of the structure nodes and slope distribution factors are calculated in Table 2:

Node	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
FEM	-63.875	63.875	-133.225	133.225	0	0
$\theta^{(0)}$	0	$63.282/EI$		$-81.0452/EI$		0
ω	0	-0.25	-0.25	-0.1666	-0.3333	0

Table 2: Fixed end moment, initial rotations and slope distribution factors for the example 6.1.

Table 3 shows the steps of Jacobi-based SDM for analysis of the continuous beam of example 6.1:

Node	*B*	*C*
Member	*BC*	*CB*
ω	-0.25	-0.1666
$\theta^{(0)}$	63.282	-81.0452
$\Delta\theta^{(0)}$	20.2613	-10.5427812
$\Delta\theta^{(1)}$	2.6356953	- 3.37553258
$\Delta\theta^{(2)}$	0.843883145	-0.439106837
$\Delta\theta^{(3)}$	0.109776709	-0.140590932
$\Delta\theta^{(4)}$	0.035147732	-0.0108288799
$\Delta\theta^{(5)}$	0.004572199	-0.005855612
$\theta^{(6)}$	+87.17237509	-95.55989604

Table 3 The steps of Jacobi-based SDM for analysis of the continuous beam of example 6.1.

Below, the parametric calculation of some steps of the SDM is shown:

$$\Delta\theta_B^0 = \omega_{BC}\,\theta_C^0\,,\Delta\theta_C^0 = \omega_{CB}\,\theta_B^0,\quad \Delta\theta_B^1 = \omega_{BC}\,\Delta\theta_C^0\,,\Delta\theta_C^1 = \omega_{CB}\,\Delta\theta_B^0,\quad \Delta\theta_B^2 = \omega_{BC}\,\Delta\theta_C^1\,,\Delta\theta_C^2 = \omega_{CB}\,\Delta\theta_B^1$$

Using slope deflection relations, the values of end moments and shears for the beam are calculated as below:

M_{AB}= - 39.9921, M_{BA}=111.6406, M_{BC}= -111.6401, M_{CB}=104.7462, M_{CD}= -104.7231, M_{DC}= - 52.3615, V_{AB}=25.1851, V_{BA}=44.8148, V_{BC}=110.4443, V_{CB}=108.5556, V_{CD}=43.0369, V_{DC}= - 43.0369.

To apply the matrix formulation on the above example, [ω] and $\{\vartheta^0\}$ are formed for the beam of Figure 5, by using the parameter values in Table 2:

$$[\omega] = \begin{bmatrix} & A & B & C & D \\ A & 0 & 0 & 0 & 0 \\ B & -0.25 & 0 & -0.25 & 0 \\ C & 0 & -\frac{1}{6} & 0 & -\frac{1}{3} \\ D & 0 & 0 & 0 & 0 \end{bmatrix}$$

By applying equation (58) and using the calculated [ω] and$\{\theta^0\}$, nodal rotation matrix is obtained as follow:

$$\{\theta^{(0)}\} = \begin{Bmatrix} A = 0 \\ B = \dfrac{63.282}{EI} \\ C = \dfrac{-81.0452}{EI} \\ D = 0 \end{Bmatrix} \qquad \Rightarrow \qquad \{\theta\} = \begin{Bmatrix} A = 0 \\ B = \dfrac{87.1756}{EI} \\ C = \dfrac{-95.5744}{EI} \\ D = 0 \end{Bmatrix}$$

As could be seen, the results of the manual and matrix formulation are in good agreement.
The above beam is also analyzed by Sap2000 software. The results are shown in Figures. 6 and 7.

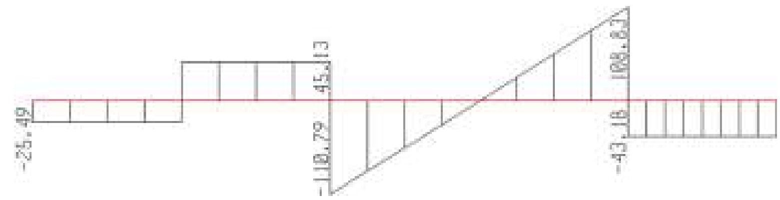


Figure 6: Shear diagram (KN) for the beam of the example 6.1.

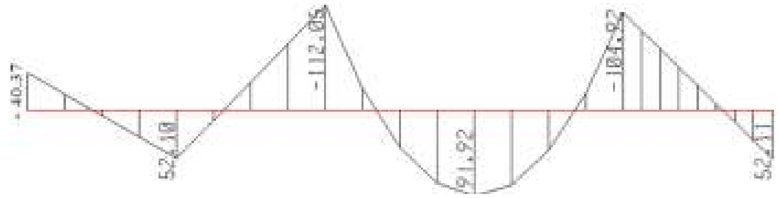


Figure 7: Bending moment diagram (KN.m) for the beam of the example 6.1.

As can be seen, the difference between the results of the SDM with those of these diagrams is negligible and is less than 1% .It seems this small difference is because of considering shear deformation in software calculations.

6.2 In this example, bending frame with lateral displacement and different columns heights, will analyze by classic and matrix procedure of SDM and results will be compared with those of Sap2000 software. Characteristics, loading and support conditions are shown in Figure 8 and the flexural rigidity of frame members is constant.

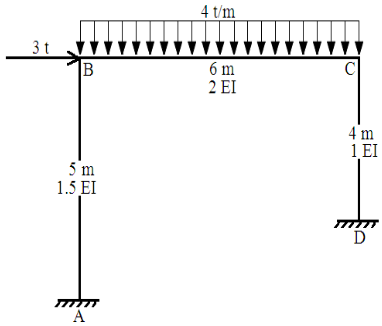


Figure 8: The Frame with different column heights (example 6.2).

If it is supposed that φ_{AB} is considered as the independent φ of the storey, and we have: $\varphi_s = \varphi_{AB}$, then regarding geometric relationships between the members of the frame, it can be obtained that:

$$\varphi_{BC} = 0 \qquad \text{and} \qquad \varphi_{CD} = +1.25\phi_s$$

At first, primarily needed parameters are calculated. These values are presented in Table 4.

Node	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
FEM	0	0	-12	12	0	0
$\theta^{(0)}$	0	$6.023/EI +$		$-3.6884/EI$		0
R	0	0.4737	0.5263	0.5714	0.4286	0
ω	0	-0.2368	- 0.2631	- 0.2857	-0.21428	0
$\tilde{\omega}$	0	-0.2368		-0.26785		0
δ	-0.6516	-0.6516	− − − −	− − − −	-0.6787	-0.6787

Table 4: Fixed-end moment of members, slope distribution and carry-over factors for the frame of Figure 8.

$$D_{AB}=\frac{\left(\frac{1.5\text{ EI}}{5^2}\right)}{\left(\frac{1.5\text{ EI}}{5^2}\right)+(1.25)\left(\frac{1\text{ EI}}{4^2}\right)}=0.43439 \qquad D_{CD}=\frac{\left(\frac{1\text{ EI}}{4^2}\right)}{\left(\frac{1.5\text{ EI}}{5^2}\right)+(1.25)\left(\frac{1\text{ EI}}{4^2}\right)}=0.4524887$$

$$\delta_{AB}=\text{-}1.5D_{AB}=\text{-}0.6516 \quad , \quad \delta_{CD}=\text{-}1.5D_{CD}=\text{-}0.678733 \qquad \varphi_S^{(0)}=\frac{3-(0)}{12\left\{\left(\frac{1.5\text{ EI}}{5^2}\right)+(1.25)\left(\frac{1\text{ EI}}{4^2}\right)\right\}}=\frac{1.81}{\text{EI}}$$

$$\theta_B^{(0)}=\frac{0-(-12)}{4*0.6333\text{ EI}}+1.5\,(0.4737)\left(\frac{1.81}{\text{EI}}\right)=\frac{6.023}{\text{EI}} \quad \theta_C^{(0)}=\frac{0-(12)}{4*0.58333\text{ EI}}+1.5\,(0.4286)\,(1.25)\left(\frac{1.81}{\text{EI}}\right)=-\frac{3.6884}{\text{EI}}$$

$$[\widetilde{\omega}]_{\text{c}}=\left(\frac{\phi_{CD}}{\phi_S}\right)\omega_{CD}=-0.26785 \qquad [\widetilde{\omega}]_{\text{B}}=\left(\frac{\phi_{BA}}{\phi_S}\right)\omega_{BA}=-0.2368$$

Now, slope distribution relationships between rigid nodes are simply produced through computing needed parameters. The parametric form of the slope distribution procedure for calculating nodal rotations of Figure 8 is shown in Table 5 (for the first three steps). Similarly, by repeating this procedure, other step values of nodal rotations' changes are calculated. The results of all stages are shown in Figure 9.

Node	B	C
$\Delta\theta^{(0)}$	$\omega_{BC}\theta_C^{(0)}$	$\omega_{CB}\theta_B^{(0)}$
$\Delta\theta^{(1)}$	$\omega_{BC}\Delta\theta_C^{(0)}+\widetilde{\omega}_{\text{B}}\left[\delta_{AB}\theta_B^{(1)}+\delta_{CD}\theta_C^{(1)}\right]$	$\omega_{CB}\Delta\theta_B^{(0)}+\widetilde{\omega}_{\text{C}}\left[\delta_{AB}\theta_B^{(1)}+\delta_{CD}\theta_C^{(1)}\right]$
$\Delta\theta^{(2)}$	$\omega_{BC}\Delta\theta_C^{(1)}+\widetilde{\omega}_{\text{B}}\left[\delta_{AB}\Delta\theta_B^{(1)}+\delta_{CD}\Delta\theta_C^{(1)}\right]$	$\omega_{CB}\Delta\theta_B^{(1)}+\widetilde{\omega}_{\text{C}}\left[\delta_{AB}\Delta\theta_B^{(1)}+\delta_{CD}\Delta\theta_C^{(1)}\right]$

Table 5: Parametric form of SDM procedure for calculating nodal rotation of example 6.2.

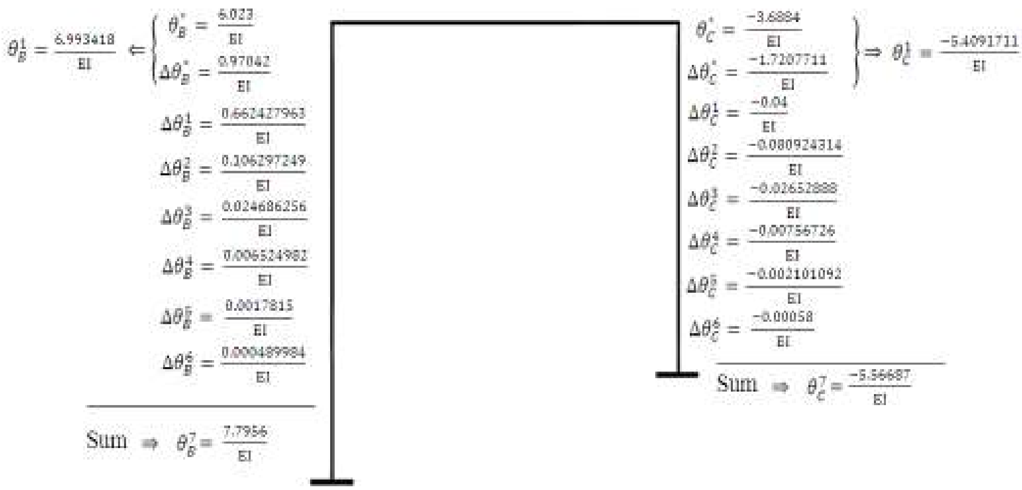


Figure 9: SDM procedure for analysis of the moment frame of example 6.2.

Now, for the matrix procedure of SDM, by calculating the required parameters, the corresponding matrices can be formed. Then, through SDM matrix procedure, frame node rotations will be defined:

$$[\omega]=\begin{bmatrix} & A & B & C & D \\ A & 0 & 0 & 0 & 0 \\ B & -0.236842 & 0 & -0.263158 & 0 \\ C & 0 & -0.285714 & 0 & -0.214286 \\ D & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\widetilde{\omega}]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.236842 & -0.236842 & -0.236842 & -0.236842 \\ -0.267857 & -0.267857 & -0.267857 & -0.267857 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\delta]=\begin{bmatrix} 0 & -0.6516 & 0 & 0 \\ -0.6516 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.678733 \\ 0 & 0 & -0.678733 & 0 \end{bmatrix}$$

$$\{\theta^{(0)}\}=\begin{Bmatrix} A=0 \\ B=\dfrac{6.023}{EI} \\ C=-\dfrac{3.6884}{EI} \\ D=0 \end{Bmatrix} \qquad \Rightarrow \qquad \{\theta\}=\begin{Bmatrix} A=0 \\ B=\dfrac{7.7963}{EI} \\ C=-\dfrac{5.56734}{EI} \\ D=0 \end{Bmatrix}$$

As can be observed, the calculated nodes rotation values through manual procedure of SDM after the sixth stages are equal to the results from their matrix procedure, till two decimals value; this shows the proper accuracy and convergence of the proposed method. Now, the lateral rotation of members, end moment and end shear values of members can be computed using equation (23) and slope deflection relations:

$$\varphi_S=+\frac{2.244}{\mathrm{EI}}$$

M_{AB}= 0.64, M_{BA}= 5.32, M_{BC}= -5.32, M_{CB}=9.77, M_{CD}= -9.77, M_{DC}= -7

V_{AB}=1.2→, V_{BA}=1.2←, V_{BC}= 11.257↑, V_{CB}=12.743↑, V_{CD}= 4.2→, V_{DC}=4.2←

To compare the results obtained through the proposed method with those of computer programs; the above frame is modeled and analyzed through Sap2000 software. In Figure 10, the shear and moment diagrams of frame members are shown as the two outputs of software:

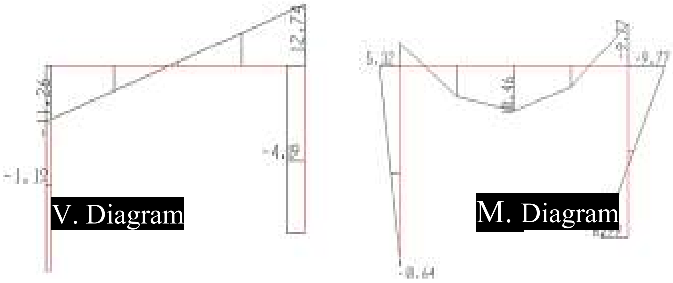


Figure 10: The shear diagram (in Ton) and moment diagram (in Ton-m) of the frame of example 6.2

6.3 In this example, one single-bay moment resisting frame with a single-storey is considered which has non-prismatic columns and a set of cross bracing (x-bracing)(Rezaee Pajand and Aftabi Suny, 2010). As is clear in Figure 11, the frame has a bay of 4b and a height of 3b. Columns of frame are non-prismatic with two different end sections as shown in the figure and its beam is prismatic with a section like the column in connecting joint. The beams and columns of the frame are made from materials with an elastic modulus of 10E. On the other hand, braces have an elastic modulus of E and a section area of 0.01b2.

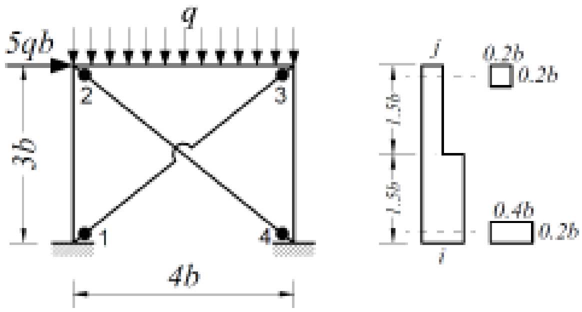


Figure 11: The one bay moment resisting frame with non-prismatic columns and x-bracing system(Rezaee Pajand and Aftabi Suny, 2010)

In such a problem, the stiffness of braces in the storey should be included in calculations. What analyzing relations reveals is, while designing of the two given bracing members is performed based on allowable compression force, both members will participate in the lateral resisting system; so, the total lateral stiffness of the braces is equal to total lateral stiffness of the two members (Rezaee Pajand and Aftabi Suny, 2010; Zalka, 2002):

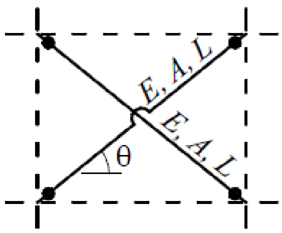


Figure 12: A X-bracing frame(Rezaee Pajand and Aftabi Suny, 2010).

$$K_v = 2K_{Brace} = 2\,(\frac{EA}{L})\,cos^2\theta \tag{59}$$

Where, E is elasticity modulus, A is cross-section area, L, is the length of bracing member and Θ is the angle between the bracing member axis and horizontal direction. Regarding that the above system is a dual bending–bracing frame system, it is needed to consider the stiffness of these two systems during the computations processes, simultaneously. According to Figure 3, the storey shear (V_s) is equal to the sum of the column's shear forces and the horizontal components of the bracing members forces in the desired storey:

$$V_v + \sum_{1}^{m} V_{ij} = V_s \tag{60}$$

In which, m is the number of column members in the storey and V_v is obtained by the below equation:

$$V_v = K_v \Delta_s \tag{61}$$

In this equation, the sum of lateral stiffness of the bracing system in the Sth storey is defined by K_v. According to Figure 4 and based on the main assumption of the SDM method, i.e. neglecting axial deformation of frame members, relative displacements of all Sth storey columns are the same and equal to Δ_s.

All bending frame equations are used in dual frames, but member shear stiffness will be changed. Therefore equations (28) and (29) are modified as follows:

$$D_r = \frac{T_r}{(K_v H_s) + \sum_1^m (\frac{\varphi_{ij}}{\varphi_s})\,T_r} \tag{62}$$

$$\varphi_s^{(0)} = \frac{V_S - \sum_1^m FEV_{ij}}{(K_v H_s) + \sum_1^m (\frac{\varphi_{ij}}{\varphi_s})\,T_r} \tag{63}$$

If the structural members are prismatic, the above equations will be simplified as below:

$$D_{ij} = \frac{\mu_{ij}}{(\frac{K_v H_s}{12}) + \sum_1^m (\frac{\varphi_{ij}}{\varphi_s})\mu_{ij}} \tag{64}$$

$$\varphi_s^{(0)} = \frac{V_S - \sum_1^m FEV_{ij}}{(K_v H_s) + 12\sum_1^m (\frac{\varphi_{ij}}{\varphi_s})\mu_{ij}} \tag{65}$$

Now, to solve the above problem, stiffness and carry-over coefficients of non-prismatic columns are computed at first, using the tables of "handbook of frame constants" (Association, 1958), as follows:

$$S_{12} = 20.61\left(\frac{(10E)(\frac{(0.2\,b)^4}{12})}{3b}\right) = 0.00916\,Eb^3 = S_{43}$$

$$S_{21} = 5.42 \left(\frac{(10E)(\frac{(0.2\ b)^4}{12})}{3b} \right) = 0.00241\ Eb^3 = S_{34}$$

The frame beam is prismatic and has stiffness coefficient and carry-over factors as below:

$$S_{23} = 4 \left(\frac{(10E)(\frac{(0.2\ b)^4}{12})}{4b} \right) = 0.0013333\ Eb^3 = S_{32}$$

$$K_v = 2\ \frac{E(0.01\ b^2)}{5b} (\frac{4}{5})^2 = 0.00256\ \mathrm{Eb} \qquad K_v\ H_s = 0.00768\ Eb^2$$

Now, by defining the required data, the rotation coefficients are calculated:

$$R_{21} = \frac{S_{21}}{\sum S_2} = \frac{0.00241\ Eb^3}{0.00241\ Eb^3 + 0.0013333\ Eb^3} = 0.64381122 = R_{34}$$

$$\omega_{21} = -C_{21}\ R_{21} = -\ 0.77257 = \omega_{34}$$

$$R_{23} = \frac{S_{23}}{\sum S_2} = \frac{0.0013333\ Eb^3}{0.00241\ Eb^3 + 0.0013333\ Eb^3} = 0.3562 = R_{32} \qquad \omega_{23} = -C_{23}\ R_{23} = -\ 0.1781 = \omega_{32}$$

$$T_1 = \frac{S_{12} + C_{21}S_{21}}{L_{12}} = 0.00402\ Eb^2 = T_4 \qquad T_2 = \frac{S_{21} + C_{12}S_{12}}{L_{12}} = 0.00177\ Eb^2 = T_3$$

$$(K_v H_s) + \sum_{1}^{m} \{ (\frac{\phi_{ij}}{\phi_s}) (T_r) \} = 0.01926\ Eb^2$$

$$D_1 = 0.208723 = D_4 \qquad D_2 = 0.0919 = D_3$$

$$[\tilde{\omega}]_2 = (1 + C_{21})R_{21} \left(\frac{\varphi_{21}}{\varphi_S} \right) = 1.4164 = [\tilde{\omega}]_3$$

$$\varphi_S^{(0)} = \frac{5qb - 0}{0.01926\ Eb^2} = 259.6054 \frac{q}{Eb}$$

$$\theta_2^{(0)} = \frac{0 - (-\left(\frac{4}{3}\right)\ qb^2)}{0.0037433\ Eb^3} + (1 + 1.2)(0.64381122)(259.6054 \frac{q}{Eb}) = 723.89387 \frac{q}{Eb}$$

$$\theta_3^{(0)} = \frac{0 - (\left(\frac{4}{3}\right)\ qb^2)}{0.0037433\ Eb^3} + (1 + 1.2)(0.64381122) \left(259.6054 \frac{q}{Eb}\right) = 11.51632 \frac{q}{Eb}$$

Node	2	3
$\Delta\theta^{(0)}$	$\omega_{23}\theta_3^{(0)}$	$\omega_{32}\theta_2^{(0)}$
$\Delta\theta^{(1)}$	$\omega_{23}\Delta\theta_3^{(0)} + \widetilde{\omega}_2\left[D_{21}\theta_2^{(1)} + D_{34}\theta_3^{(1)}\right]$	$\omega_{32}\Delta\theta_2^{(0)} + \widetilde{\omega}_3\left[D_{21}\theta_2^{(1)} + D_{34}\theta_3^{(1)}\right]$
$\Delta\theta^{(2)}$	$\omega_{23}\Delta\theta_3^{(1)} + \widetilde{\omega}_2\left[D_{21}\Delta\theta_2^{(1)} + D_{34}\Delta\theta_3^{(1)}\right]$	$\omega_{32}\Delta\theta_2^{(1)} + \widetilde{\omega}_3\left[D_{21}\Delta\theta_2^{(1)} + D_{34}\Delta\theta_3^{(1)}\right]$

Table 6: Parametric form of SDM procedure for nodal rotation of example 6.3.

Node	2	3
$\theta^{(0)}$	$723.89387(\frac{q}{Eb})$	$11.51632\frac{q}{Eb}$
$\Delta\theta^{(0)}$	$-2.051056592(\frac{q}{Eb})$	$-128.9254982\frac{q}{Eb}$
$\theta^{(1)}$	$721.8428134(\frac{q}{Eb})$	$-117.4091782(\frac{q}{Eb})$
$\Delta\theta^{(1)}$	$101.6390409(\frac{q}{Eb})$	$79.04270288(\frac{q}{Eb})$
$\Delta\theta^{(2)}$	$9.441324069(\frac{q}{Eb})$	$5.416916267(\frac{q}{Eb})$
$\Delta\theta^{(3)}$	$0.96930216(\frac{q}{Eb})$	$0.25255513(\frac{q}{Eb})$
$\Delta\theta^{(4)}$	$0.114065624(\frac{q}{Eb})$	$-0.013587021(\frac{q}{Eb})$
$\Delta\theta^{(5)}$	$0.015498862(\frac{q}{Eb})$	$-0.007236073(\frac{q}{Eb})$
$\Delta\theta^{(6)}$	$0.002364(\frac{q}{Eb})$	$-0.00276(\frac{q}{Eb})$
$\theta^{(7)}$	$834.0244093(\frac{q}{Eb})$	$-32.71951182(\frac{q}{Eb})$

Table 7: The results of SDM procedure for nodal rotation of example 6.3.

By calculating the required parameters, the corresponding matrices for the structure are formed:

$$[\omega] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.77257 & 0 & -0.1781 & 0 \\ 0 & -0.1781 & 0 & -0.77257 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\widetilde{\omega}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.4164 & 1.4164 & 1.4164 & 1.4164 \\ 1.4164 & 1.4164 & 1.4164 & 1.4164 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\delta] = [D]^{Transpose} = \begin{bmatrix} 0 & 0.0919 & 0 & 0 \\ 0.208723 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.208723 \\ 0 & 0 & 0.0919 & 0 \end{bmatrix}$$

$$\{\theta^{(0)}\} = \begin{Bmatrix} A = 0 \\ B = 723.894\dfrac{q}{Eb} \\ C = 11.5163\dfrac{q}{Eb} \\ D = 0 \end{Bmatrix} \quad \Rightarrow \quad \{\theta\} = \begin{Bmatrix} A = 0 \\ B = 834.025\dfrac{q}{Eb} \\ C = -32.7199\dfrac{q}{Eb} \\ D = 0 \end{Bmatrix}$$

Through comparing manual and matrix procedures of SDM, it can be shown that the value of frame nodal rotation after the sixth step using a manual procedure is equal to its matrix results till two decimal values which indicate the accuracy and good convergence speed of the proposed method.

After calculating the nodal displacements, members' lateral rotations and bending moments are resulted by equation (34) and slope deflection equation, respectively:

$$\varphi_S = 333.24533\left(\frac{q}{Eb}\right)$$

$M_{12} = -1.603\, qb^2$, $M_{21} = 0.2431\, qb^2$, $M_{23} = -0.2431 qb^2$, $M_{32} = 1.846\, qb^2$, $M_{34} = -1.846\, qb^2$, $M_{43} = -4.112\, qb^2$

In the above example, if the bracing system is eccentric, the only difference is the value of the stiffness of bracing members. Regarding studies (Rezaee Pajand and Aftabi Suny, 2010; Zalka, 2002), it is just enough that parameter "A" in equation (59) is replaced by "A_e" which is defined as follows:

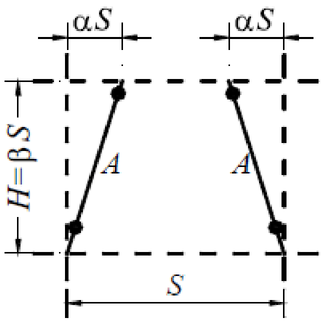


Figure 13: An eccentric bracing frame(Rezaee Pajand and Aftabi Suny, 2010).

$$A_e = \alpha^2 \left(\frac{1+\beta^2}{\alpha^2+\beta^2}\right)^{\frac{3}{2}} A \tag{66}$$

In the above equation, β is the proportion of height to length of braced span and α is the ratio of distance between the brace connection place to its neighbor column to the length of braced span.

These two factors can be seen in Figure 13. In a situation where shear walls are used in the structure, it is just enough to define wall lateral stiffness through the following equation to analyze the structure by the SDM method:

$$K_w = \frac{3EI}{\gamma H^3} \tag{67}$$

Where, E is elasticity modulus for concrete shear wall and I is the moment inertia of shear wall and are calculated by following equation:

$$E = 5000\sqrt[2]{f_c} \tag{68}$$

$$I = \frac{bL_w^3}{12} \tag{69}$$

f_c is compressive strength of concrete based on N/mm2 . b and L_w are the thickness and the length of the wall, respectively. Coefficient γ in equation (67) shows the effect of shear deformation on the wall stiffness and obtained by the following equation:

$$\gamma = 1 + 0.75\left(\frac{L_w}{H}\right) \tag{70}$$

where, H is the height of the wall.

7 BENDING FRAME WITH NON-VERTICAL COLUMNS.

7.1

The SDM method can be used for frames at which the beams at storey level have lateral rotation or the columns are non-vertical. Through writing horizontal equilibrium equations for shear force, the horizontal components of the axial forces of non-vertical columns are participated in the equilibrium. Therefore, to ease the process, moment equilibrium equation around a virtual point resulted from the intersection of members or the continuation (point I in Figure 14), are used to remove the effect of column axial force in equations. If the concentrated load of P is applied on the connection “B” in the frame of Figure 14, the parameters of the SDM approach can be defined for the frame through writing moment equilibrium for point “I”.

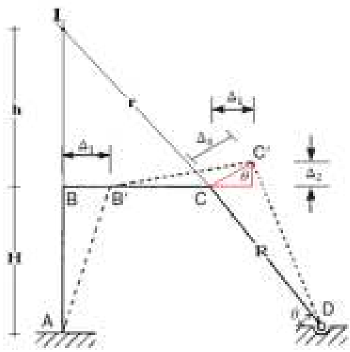


Figure 14: An example of a frame with non-vertical columns and Lateral displacements of its members (Kaveh, 2012; Megson, 2005).

As mentioned in (Kaveh, 2012), lateral displacements of members can be simply described based on frame independent displacement through geometric relationships. This dependency is defined for the frame of Figure 14 by following equations:

$$\frac{\Delta_1}{\sin\theta} = \frac{\Delta_2}{\sin(90^\circ - \theta)} = \frac{\Delta_3}{\sin 90^\circ} \tag{71}$$

$$\Delta_1 = \Delta_2 \tan\theta = \Delta_3 \sin\theta \tag{72}$$

Lateral rotation of member "AB" is supposed as the basis ($\varphi_s = \varphi_{AB}$) and the length of h and r is equal to $r = \frac{L_{BC}}{\cos\theta}$ and $h = L_{BC}\tan\theta$, therefore:

$$\varphi_S^{(0)} = \frac{P\left(\frac{h}{r}\right) - \left(\frac{FEM_{AB} + FEM_{DC}}{r}\right) - \left(\frac{FEV_{AB}(H+h) + FEV_{DC}(R+r)}{r}\right)}{12\left\{\left(\frac{EI}{H^2}\right)\left[\left(\frac{h}{r}\right) + \left(\frac{H}{2r}\right)\right] + \left(\frac{EI}{R^2}\right)\left(\frac{\varphi_{DC}}{\varphi_S}\right)\left[1 + \left(\frac{R}{2r}\right)\right]\right\}} \tag{73}$$

Also, the relative shear stiffness and lateral rotation of members are defined as follows:

$$D_{AB} = \frac{\left(\frac{EI}{H^2}\right)\left[\left(\frac{h}{r}\right) + \left(\frac{2H}{3r}\right)\right]}{\left(\frac{EI}{H^2}\right)\left[\left(\frac{h}{r}\right) + \left(\frac{H}{2r}\right)\right] + \left(\frac{EI}{R^2}\right)\left(\frac{\varphi_{DC}}{\varphi_S}\right)\left[1 + \left(\frac{R}{2r}\right)\right]} \tag{74}$$

$$D_{CD} = \frac{\left(\frac{EI}{R^2}\right)\left[1 + \left(\frac{2R}{3r}\right)\right]}{\left(\frac{EI}{H^2}\right)\left[\left(\frac{h}{r}\right) + \left(\frac{H}{2r}\right)\right] + \left(\frac{EI}{R^2}\right)\left(\frac{\varphi_{DC}}{\varphi_S}\right)\left[1 + \left(\frac{R}{2r}\right)\right]} \tag{75}$$

$$\varphi_S^{(n)} = \varphi_S^{(0)} + 0.5\, D_{AB}\left(\Theta_A^{(n)} + \Theta_B^{(n)}\right) + 0.5\, D_{CD}\left(\Theta_C^{(n)} + \Theta_D^{(n)}\right) \tag{76}$$

It can be seen that if $\theta \to \frac{\pi}{2}$, then r&h$\to \infty$, $\frac{h}{r} \to 1$ and $\frac{1}{r} \to 0$ and the equations will be simplified.

7.2 In this part, an example from (Kaveh, 2012) is presented. In the frame of Figure 15, A and D are fixed supports and the column"CD" makes an angle of 53 degrees with horizontal direction. Structural loading and the relative value of $\left(\frac{EI}{L}\right)$ for each member are shown in Figure 15.

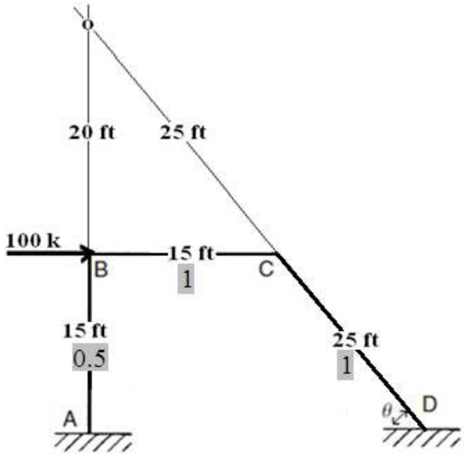


Figure 15: Frame of example 7.2 and the structure characteristics(Kaveh, 2012).

If $\varphi_s = \varphi_{AB}$, then from equation (72), we have the following relations:

In this example, the required parameters are calculated firstly and the structure is analyzed by the two manual and matrix forms of the proposed method.

Node	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
FEM	0	0	0	0	0	0
$\theta^{(0)}$	0	-20.4082		0		0
R	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0
ω	0	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0
D	0.4898	0.4898	----	----	0.81632	0.81632
δ	-0.7347	-0.7347	----	----	-1.22449	-1.22449
$\widetilde{\omega}$	0	0.08333		0		0

Table 8: Fixed-end moment of the members, slope distribution and carry-over factors of the frame of Figure 15.

$$\varphi_S^{(0)} = \frac{100\left(\frac{20}{25}\right) - (0) - (0)}{12\left\{\left(\frac{0.5}{15}\right)\left[\left(\frac{20}{25}\right) + \left(\frac{15}{2*25}\right)\right] + \left(\frac{1}{25}\right)(0.75)\left[1 + \left(\frac{25}{2*25}\right)\right]\right\}} = 81.6326$$

$$D_{AB} = \frac{\left(\frac{0.50}{15}\right)\left[\left(\frac{20}{25}\right)+\left(\frac{2*15}{3*25}\right)\right]}{\left(\frac{0.5}{15}\right)\left[\left(\frac{20}{25}\right)+\left(\frac{15}{2*25}\right)\right]+\left(\frac{1}{25}\right)(0.75)\left[1+\left(\frac{25}{2*25}\right)\right]} = 0.4898 \qquad D_{CD} = \frac{\left(\frac{1}{25}\right)\left[1+\left(\frac{2*25}{3*25}\right)\right]}{\left(\frac{0.5}{15}\right)\left[\left(\frac{20}{25}\right)+\left(\frac{15}{2*25}\right)\right]+\left(\frac{1}{25}\right)(0.75)\left[1+\left(\frac{25}{2*25}\right)\right]} = 0.81632$$

$$\delta_{CD} = -1.5D_{CD} = -1.22449 \quad \delta_{AB} = -1.5D_{AB} = -0.7347$$

$$\theta_C^{(0)} = \{(1.5)(-0.75)\left(\tfrac{1}{2}\right) + (1.5)(0.75)\left(\tfrac{1}{2}\right)\}(81.6326) = 0 \;\;, \;\; \theta_B^{(0)} = \{1.5\left(\tfrac{1}{3}\right) + (1.5)(-0.75)\left(\tfrac{2}{3}\right)\}(81.6326) = -20.4082$$

$$\widetilde{\omega}_B = \left(\frac{\phi_{BA}}{\phi_S}\right)\omega_{BA} + \left(\frac{\phi_{BC}}{\phi_S}\right)\omega_{BC} = 0.08333 \;\;, \;\; \widetilde{\omega}_C = \left(\frac{\varphi_{CB}}{\varphi_S}\right)\omega_{CB} + \left(\frac{\phi_{CD}}{\phi_S}\right)\omega_{CD} = 0$$

Now, through calculating required parameters, the slope distribution process will be repeated between rigid connections. Parametric form of slope distribution procedure for calculating nodal rotation of the frame of Figure 15 is similar to Table 5. In Table 9, the calculation process of the rotations for nodes B and C are presented:

Node	B	C
$\theta^{(0)}$	−20.4082	0
$\Delta\theta^{(0)}$	0	5.10505
$\theta^{(1)}$	−20.4082	5.10505
$\Delta\theta^{(1)}$	−0.971813	0
$\Delta\theta^{(2)}$	0.0595	0.243
$\Delta\theta^{(3)}$	−0.10944	−0.014875
$\Delta\theta^{(4)}$	0.013176	0.02736
$\Delta\theta^{(5)}$	−0.01272	−0.00323
$\theta^{(6)}$	−21.4295	5.3542

Table 9: Calculation process of the rotation values for nodes B and C of the frame of Figure 15.

For analysis of the frame by matrix formulation of the presented approach, the corresponding matrices should be formed at first:

$$[\omega] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & -\frac{1}{3} & 0 \\ 0 & -0.25 & 0 & -0.25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\widetilde{\omega}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.08333 & 0.08333 & 0.08333 & 0.08333 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\delta] = \begin{bmatrix} 0 & -0.7347 & 0 & 0 \\ -0.7347 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.22449 \\ 0 & 0 & -1.22449 & 0 \end{bmatrix}$$

By forming the initial rotation matrix and using equation (56), the final rotation value of the nodes are calculated:

$$\{\theta^{(0)}\} = \begin{Bmatrix} A = 0 \\ B = -20.4082 \\ C = 0 \\ D = 0 \end{Bmatrix} \qquad \Rightarrow \qquad \{\theta\} = \begin{Bmatrix} A = 0 \\ B = -21.4286 \\ C = 5.35715 \\ D = 0 \end{Bmatrix}$$

By using equation (23) and applying slope-deflection equations, the end-moments and shears of the members are calculated.

M_{AB}= -257.8923, M_{BA}= -279.321, M_{BC}= 279.695,
M_{CB}=333.267, M_{CD}= -333.267, M_{DC}= -344

V_{AB}=35.8142←, V_{BA}=35.8142→, V_{BC}= 40.864↓, V_{CB}=40.864↑, V_{CD}= 27→, V_{DC}=27←

The comparison of the results with those of (Kaveh, 2012), shows the accuracy of the proposed method.

8 BENDING FRAME WITH VERTICAL DISPLACEMENTS OF THE NODES

8.1

If a frame column does not continue to the foundation level, due to vertical displacements, the frame beams will have lateral rotation. This is a limitation of the Kani method which is not able to analyze a frame with vertical displacement, and structure columns should be continued to the foundation level. But through the SDM method, these kinds of structures can be analyzed. An example of this structure is shown in Figure 16.

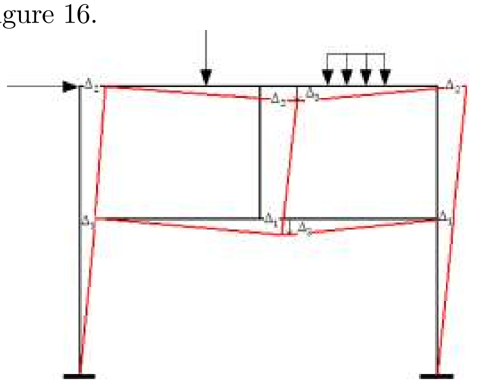


Figure 16: A moment resisting frame with vertical displacements of the nodes.

In this condition, through writing the shear equilibrium equation in vertical direction (Figure 17) and replacing the end shear of members, an auxiliary equation will be obtained. This equation considers the vertical displacement, Δ_3, and consequently the lateral rotation of the beam (φ_3):

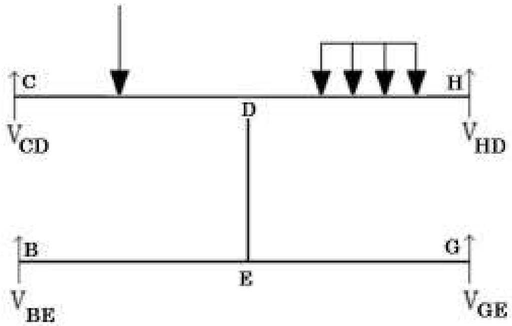


Figure 17: The free diagram of a part of the frame of Figure 16.

$$\mathrm{V_{CD}} + \mathrm{V_{BE}} + \mathrm{V_{HD}} + \mathrm{V_{GE}} = \mathrm{F}_{ex} \tag{77}$$

$\mathrm{F_{ex}}$, is the resultant of external forces in vertical direction which are applied on the frame part that is shown in Figure 17. Through simplifying, the below auxiliary equation will be obtained:

$$\varphi_3^{(n)} = \varphi_3^{(0)} + 0.5\, D_{BE}\left(\Theta_B^{(n)} + \Theta_E^{(n)}\right) + 0.5\, D_{CD}\left(\Theta_C^{(n)} + \Theta_D^{(n)}\right) - 0.5\, D_{HD}\left(\Theta_H^{(n)} + \Theta_D^{(n)}\right) - 0.5\, D_{GE}\left(\Theta_G^{(n))} + \Theta_E^{(n)}\right) \tag{78}$$

$$\varphi_3^{(0)} = \frac{\mathrm{F_{ex}} - \sum FEV}{12\left\{\left(\frac{\varphi_{BE}}{\varphi_3}\right)\mu_{BE} + \left(\frac{\varphi_{CD}}{\varphi_3}\right)\mu_{CD} - \left(\frac{\varphi_{GE}}{\varphi_3}\right)\mu_{GE} - \left(\frac{\varphi_{HD}}{\varphi_3}\right)\mu_{HD}\right\}} \tag{79}$$

$$D_{ij} = \frac{\mu_{ij}}{\left(\frac{\varphi_{BE}}{\varphi_3}\right)\mu_{BE} + \left(\frac{\varphi_{CD}}{\varphi_3}\right)\mu_{CD} - \left(\frac{\varphi_{GE}}{\varphi_3}\right)\mu_{GE} - \left(\frac{\varphi_{HD}}{\varphi_3}\right)\mu_{HD}}, \qquad \mu_{ij} = \left(\frac{EI}{L^2}\right) \tag{80}$$

It should be noted that the geometrical relations between structural members are as follows:

$$\varphi_{\mathrm{BE}} = +\frac{\Delta_3}{\mathrm{L_{BE}}}, \qquad \varphi_{\mathrm{CD}} = +\frac{\Delta_3}{\mathrm{L_{CD}}}, \qquad \varphi_{\mathrm{GE}} = -\frac{\Delta_3}{\mathrm{L_{GE}}}, \qquad \varphi_{\mathrm{HD}} = -\frac{\Delta_3}{\mathrm{L_{HD}}}$$

In the following example, the classical and matrix computing stages of SDM in a frame with vertical displacement are explained more:

8.2. In this example, a two-storey bending frame containing horizontal and vertical displacement is evaluated through classical and matrix approach of SDM. This frame is influenced by uniform dead load and horizontal force resulted from an earthquake. The frame geometrical characteristics and loading are represented in Figure 18. Finally, the obtained results will be compared with those of SAP2000 software. Bending rigidity values of all frame members are constant and equal.

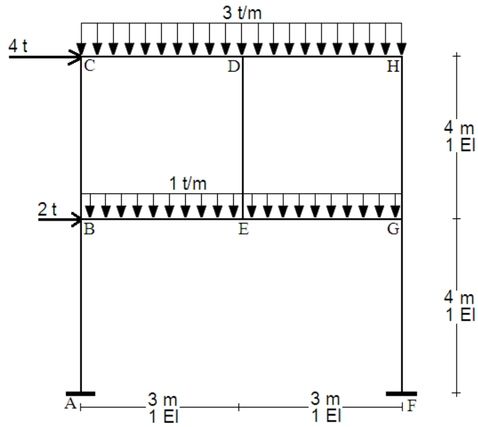


Figure 18: The bending frame of Figure 17.

Considering the geometrical relation between structural members, we have:

$$\varphi_{AB} = \varphi_{FG} = +\varphi_1, \quad \varphi_{BC} = \varphi_{ED} = \varphi_{GH} = +\varphi_2, \quad \varphi_{BE} = \varphi_{CD} = +\varphi_3, \quad \varphi_{EG} = \varphi_{DH} = -\varphi_3$$

The required coefficients and factors are calculated through simple relations of structural analysis (Tables 10 and 11).

Node	A , F	C		D			H	
Member	AB , FG	CB	CD	DC	DE	DH	HD	HG
FEM	0	0	-2.25	2.25	0	-2.25	2.25	0
$\varphi^{(0)}$	4/EI	1.7777/EI	2.25/EI	2.25/EI	1.7777/EI	2.25/EI	2.25/EI	1.7777/EI
$\theta^{(0)}$	0	4.035694 /EI			0.727273 /EI		-1.74983/EI	
R	0	0.4286	0.5714	0.3636	0.2727	0.3636	0.5714	0.4286
ω	0	-0.2143	-0.2857	-0.1818	-0.1364	-0.1818	-0.2857	-0.2143
D	0.5	0.3333	0.25	0.25	0.3333	0.25	0.25	0.3333
δ	-0.75	-0.5	-0.375	-0.375	-0.5	+0.375	+0.375	-0.5

Table 10: Fixed-end moment of members, slop distribution and carry-over factors for example 8.2.

Node		B			E			G	
Member	BA	BE	BC	EB	ED	EG	GE	GH	GF
FEM	s0	-0.75	0	0.75	0	-0.75	0.75	0	0
$\varphi^{(0)}$	4/EI	2.25/EI	1.7777/EI	2.25/EI	1.7777/EI	2.25/EI	2.25/EI	1.7777/EI	4/EI
$\theta^{(0)}$		4.175/EI			0.727273 /EI			1.025 /EI	
R	0.3	0.4	0.3	0.3636	0.2727	0.3636	0.4	0.3	0.3
ω	-0.15	-0.2	-0.15	-0.1818	-0.1364	-0.1818	-0.2	-0.15	-0.15
D	0.5	0.25	0.3333	0.25	0.3333	0.25	0.25	0.3333	0.5
δ	-0.75	-0.375	-0.5	-0.375	-0.5	+0.375	+0.375	-0.5	-0.75

Table 10 (cont.): Fixed-end moment of members, slop distribution and carry-over factors for example 8.2.

	Node	A , F	B	C	D	E	G	H
$\tilde{\omega}$ values for each node	Due to φ_1	0	-0.15	----	----	----	-0.15	----
	Due to φ_2	----	-0.15	-0.2143	-0.1364	-0.1364	-0.15	-0.2143
	Due to φ_3	----	-0.2	-0.2857	0	`0	+0.2	+0.2857

Table 11: $\tilde{\omega}$ values for nodes in the structure of Figure 18.

Through computing required parameters, the classical iterative procedure of SDM is repeated through structure rigid nodes till changes in nodal rotations values become negligible in two successive steps. The results are shown on Figure 19.

As EI is constant and equal for all frame members, it is not shown in Figure 19. The final values of rotations should be divided by this parameter. Through the sum of partial rotations calculated in each node, the values of final node rotations and also, the member lateral rotations are obtained:

$$\theta_A^{(7)} = \frac{0}{EI}, \qquad \theta_B^{(7)} = \frac{6.16457}{EI}, \qquad \theta_C^{(7)} = \frac{4.86486}{EI}, \qquad \theta_D^{(7)} = \frac{1.12757}{EI}$$

$$\theta_E^{(7)} = \frac{-0.171009}{EI}, \qquad \theta_F^{(7)} = \frac{0}{EI}, \qquad \theta_G^{(7)} = \frac{2.45871}{EI}, \qquad \theta_H^{(7)} = \frac{-2.4981}{EI}$$

$$\varphi_3^{(7)} = \frac{3.6336025}{EI}, \quad \varphi_2^{(7)} = \frac{3.768878}{EI}, \quad \varphi_1^{(7)} = \frac{6.15582}{EI}$$

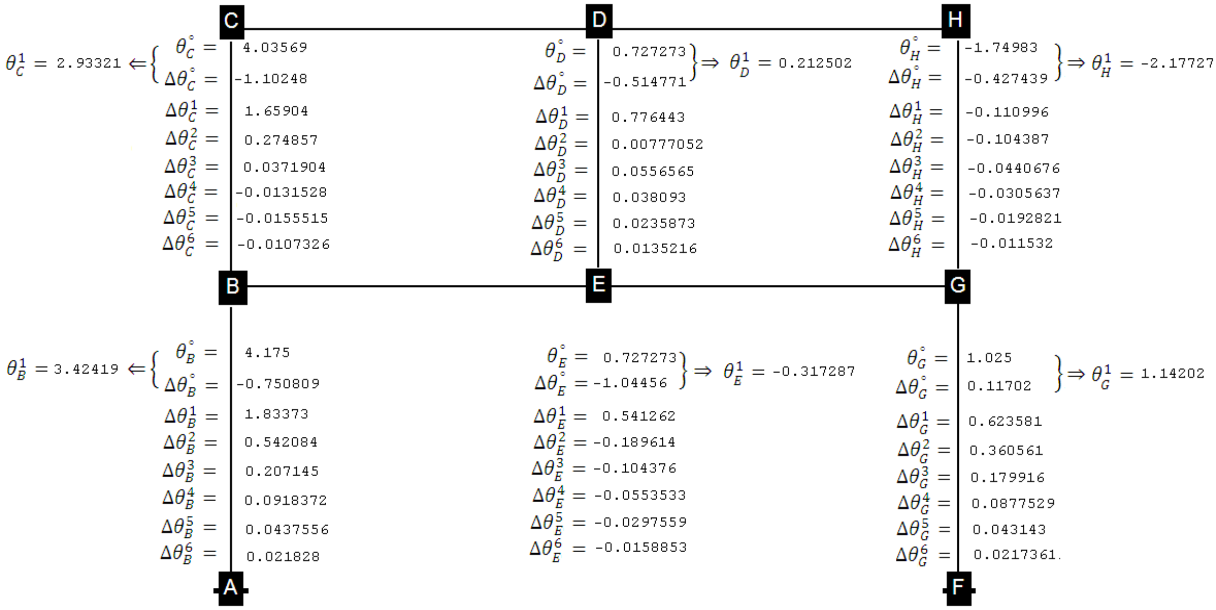


Figure 19: Classical procedure of SDM for analysis of the frame of example 2.8.

For analysis of the frame by matrix procedure, the corresponding matrices are formed firstly based on Tables 10 and 11.

$$[\omega] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.15 & 0 & -0.15 & 0 & -0.2 & 0 & 0 & 0 \\ 0 & -0.2143 & 0 & -0.2857 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1818 & 0 & -0.1364 & 0 & 0 & -0.1818 \\ 0 & -0.1818 & 0 & -0.1364 & 0 & 0 & -0.1818 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & -0.15 & 0 & -0.15 \\ 0 & 0 & 0 & -0.2857 & 0 & 0 & -0.2143 & 0 \end{bmatrix}$$

$$[\tilde{\omega}]_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.15 & -0.15 & 0 & 0 & -0.15 & -0.15 & -0.15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.15 & -0.15 & 0 & 0 & -0.15 & -0.15 & -0.15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\delta]_1 = \begin{bmatrix} 0 & -0.75 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.75 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\widetilde{\omega}]_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.15 & -0.15 & -0.15 & -0.15 & 0 & -0.15 & -0.15 \\ 0 & -0.2143 & -0.2143 & -0.2143 & -0.2143 & 0 & -0.2143 & -0.2143 \\ 0 & -0.1364 & -0.1364 & -0.1364 & -0.1364 & 0 & -0.1364 & -0.1364 \\ 0 & -0.1364 & -0.1364 & -0.1364 & -0.1364 & 0 & -0.1364 & -0.1364 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.15 & -0.15 & -0.15 & -0.15 & 0 & -0.15 & -0.15 \\ 0 & -0.2143 & -0.2143 & -0.2143 & -0.2143 & 0 & -0.2143 & -0.2143 \end{bmatrix}$$

$$[\delta]_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 \end{bmatrix}$$

$$[\widetilde{\omega}]_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & -0.2 & -0.2 & -0.2 & 0 & -0.2 & -0.2 \\ 0 & -0.2857 & -0.2857 & -0.2857 & -0.2857 & 0 & -0.2857 & -0.2857 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 \\ 0 & 0.2857 & 0.2857 & 0.2857 & 0.2857 & 0 & 0.2857 & 0.2857 \end{bmatrix}$$

$$[\delta]_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.375 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.375 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.375 & 0 & 0 & 0 & 0 & 0.375 \\ 0 & -0.375 & 0 & 0 & 0 & 0 & 0.375 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.375 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.375 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.375 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, by forming the primary rotation matrix and using equation (56), the final values of rotations at structural nodes are computed.

$$\{\theta^{(0)}\} = \begin{Bmatrix} A = 0 \\ B = 4.175 \\ C = 4.03569 \\ D = 0.727273 \\ E = 0.727273 \\ F = 0 \\ G = 1.025 \\ H = -1.74983 \end{Bmatrix} \quad \Rightarrow \quad \{\theta\} = \begin{Bmatrix} A = 0 \\ B = 6.18804 \\ C = 4.85051 \\ D = 1.14392 \\ E = -0.189134 \\ F = 0 \\ G = 2.48216 \\ H = -2.51266 \end{Bmatrix}$$

Using equation (23), the lateral independent rotation of the stories is calculated:

$$\varphi_3 = \frac{3.63363125}{EI} \varphi_2 = \frac{3.771583777}{EI} \varphi_1 = \frac{6.16755}{EI}$$

To evaluate the results, the values of end moments of the members, resulting from manual and matrix procedures of SDM are compared with those of SAP2000. On Figure 20, three values are shown on each end of the member; the top ones are the moments resulted from the software and the second and third ones are the results of matrix and manual procedures of the SDM, respectively. The values show good agreement between these results.

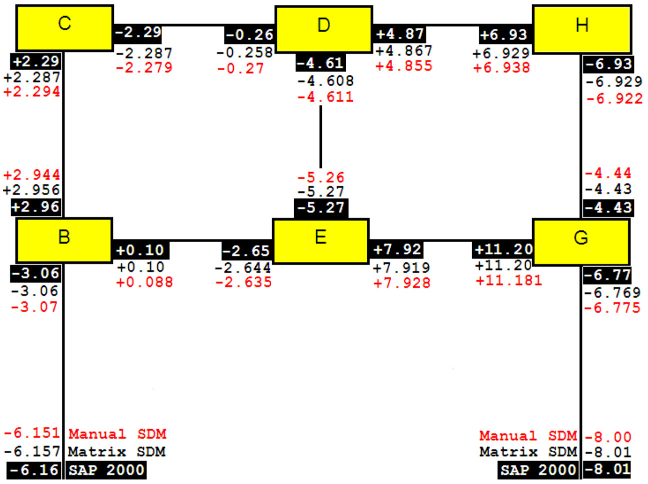


Figure 20: Comparison of the results of Sap 2000, manual and matrix formulation of SDM.

9 CONCLUSION

In this paper, two manual and matrix formulations of SDM for structural analysis were introduced. Based on Jacobi approach, the manual SDM uses an iterative process to reach the answers without solving any linear equation. The efficiency and reliability of the manual SDM are shown through some examples. In comparison to Cross method, in the new method, the distribution and carry-over process are merged together. This characteristic and considering the fact that the SDM is repeated only on rigid nodes and not on the moment values of all the members connected to rigid nodes, analysis time consuming and steps are reduced, comparatively. Regarding the later characteristics, analysis time consuming and the number of stages is also less than the Kani method; and by obtaining nodal rotation and lateral rotation of the storey, end moment and shear values of members can be calculated, simultaneously. In another work, the steps of the manual formulation were shown by a geometric progression, which led to matrix formulation. The main advantage of the matrix formulation is using only a matrix equation for the unknowns that could be used in computer programming. To show the accuracy of the Matrix formulation, some examples were solved. The results also show the accuracy and effectiveness of the proposed matrix formulation of SDM.

Another advantage of the proposed method is that the approach does not contain limitations as those in the Kani method and is capable of analyzing frames with non-vertical columns and also, those with vertical displacement. In the proposed method, with some modification, lateral stiffness of bracing members could be applied in SDM equations, so the proposed method is able to analyze bracing and dual systems. The SDM process is also capable of analyzing structures containing non-prismatic members. It should be noted that the current analysis software are not usually able to model every kind of non-prismatic member; while by specifying some basic parameters, the SDM approach is able to simply analyze structures with these kinds of members. The results of this study could be used in structural engineering calculations and the method could be expanded for other specific structures.

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