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An Analytical Solution for Free Vibration of Elastically Restrained Timoshenko Beam on an Arbitrary Variable Winkler Foundation and Under Axial Load

Abstract

Natural frequencies are important dynamic characteristics of a structure where they are required for the forced vibration analysis and solution of resonant response. Therefore, the exact solution to free vibration of elastically restrained Timoshenko beam on an arbitrary variable elastic foundation using Green Function is presented in this paper. An accurate and direct modeling technique is introduced for modeling uniform Timoshenko beam with arbitrary boundary conditions. The applied method is based on the Green Function. Thus, the effect of the translational along with rotational support flexibilities, as well as, the elastic coefficient of Winkler foundation and other parameters are assessed. Finally, some numerical examples are shown to present the efficiency and simplicity of the Green Function in the new formulation.

Keywords

Timoshenko beam, free vibration, general boundary conditions, Winkler foundation and Green Function.

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1 INTRODUCTION

Free vibration analysis has an important role in the structural design of buildings. In fact, the free vibration behavior of structures influences their response to earthquake and wind. Numerous studies are devoted to obtaining the free vibration analysis of civil engineering constructions both in the past and recent years (Carrera and Pagani, 2014). On the other hand, structures resting on foundation are an important class of problems in civil engineering. Therefore, numerous researches are presented pertaining to reports involving the calculation and analysis approach for beams and plates on foundation. These various types of foundation models include such as Winkler, Pasternak, Hetenyi, Kerr, Vlasov and Viscoelastic that are applied in the analysis of structures on elastic foundations (Mahrenholtz, 2010; Wang et al., 2005). The Winkler foundation model is frequently used in the analysis of structures on elastic foundation problems.

The natural vibrations of a Timoshenko beam on Pasternak foundation is studied by Wang and Stephens (Wang and Stephens, 1977). Moreover, the appropriate frequency equations are derived for different end restraints. Wang and Gagnon present the dynamic analysis of the continuous Timoshenko beams on Winkler-Pasternak foundations (Wang and Gagnon, 1978). The free and forced vibrations of a three span continuous beam resting on a Winkler-Pasternak foundation are studied by means of the general dynamic slope-deflection equations. In addition, the natural response of an Euler-Bernoulli beam supported by an elastic foundation is investigated by Doyle and Pavlovic (Doyle and Pavlovic, 1982). Ultimately, this paper considers the vibration problem of Euler-Bernoulli beam partially supported by a Winkler foundation. Abbas utilized the free vibration of the Timoshenko beam using the unique finite element model (Abbas, 1984). All the geometric and natural boundary conditions of Timoshenko beam with elastically supported ends can satisfy by the proposed method. Natural frequencies and normal modes of a spinning Timoshenko beam for the six classical boundary conditions are analytically solved by Zu and Han (Zu and Han, 1992). The backward and forward precession normal modes have become identical for beam with simply-supported boundary conditions. The vibration of uniform Euler-Bernoulli beam on a two-parameter elastic foundation with initial stress is investigated by Naidu and Rao (Naidu and Rao, 1995). Furthermore, the finite element formulation is applied to obtain the vibration parameter of simply supported and clamped beams.

Thambiratnam and Zhuge presented the free vibration analysis of beams supported on elastic foundations by a simple finite element method (Thambiratnam and Zhuge, 1996). An accurate solution of Timoshenko beam resting on two-parameter elastic foundation is exhibited by Wang et al. (Wang et al., 1998). In this study, the Green function is presented for bending, buckling, and vibration problems of Euler–Bernoulli and Timoshenko beams. Li applies a simple approach for the free vibration analysis of Euler-Bernoulli beam with general boundary conditions (Li, 2000). The displacement of the beam is determined as the linear combination of a Fourier series and an auxiliary polynomial function. Ying et al. investigated the precise solutions for free vibration and bending of functionally graded beams on a Winkler–Pasternak elastic foundation (Ying et al., 2008). The beam is considered as orthotropic at any point, while material properties varying exponentially along the thickness direction. In addition, the differential transform method is applied to the vibration of an Euler–Bernoulli and Timoshenko beam on an elastic soil by Balkaya et al. (Balkaya et al., 2009). In this method, precise solutions are obtained without the requirement for serious calculations.

Motaghian et al. studied the free vibration of Euler-Bernoulli beam on Winkler foundation (Motaghian et al., 2011). A mathematical approach is used to find the precise analytical solution of the free vibration of Euler-Bernoulli beam with mixed boundary conditions. The double Fourier transform is employed for the free vibration analysis of the semi-rigid connected Reddy–Bickford beam with variable cross-section on elastic soil and under axial load by Yesilce and Catal (Yesilce and Catal, 2011). Bayat et al. presented the analytical study on the vibration frequencies of tapered beams (Bayat et al., 2011). The Max-Min Approach and Homotopy Perturbation Method are employed in order to solve the governing equations of tapered beams. Thus the nonlinear vibration of the clamped-clamped Euler-Bernoulli beam subjected to the axial loads is investigated by Barari et al (Barari et al., 2011). Xing and Wang explained a general model for the free vibration of the Euler-Bernoulli beam restrained with two rotational and two transversal elastic springs under a constant axially load (Xing and Wang, 2013). In this paper, an analytical approach is used to find the frequency equations and the shape functions. Ratazzi et al. considers free vibrations of Euler-Bernoulli beam system structures with elastic boundary conditions and an internal elastic hinge (Ratazzi et al., 2013). The beam system is clamped at one end and elastically restrained at the other. Furthermore, the free vibration of the Euler-Bernoulli beam with variable cross-section on elastic foundation and under axial load is considered by Mirzabeigy (Mirzabeigy, 2014). Bazehhour et al. utilized a new analytical solution for the free vibration of the rotating Timoshenko shaft with various boundary conditions (Bazehhour et al., 2014). The effect of the axial load on the natural frequencies is investigated as the rotational speed increases. At the same time, the numerical method for solution of the free vibration of Timoshenko beams with arbitrary boundary conditions is presented by Prokić et al. (Prokić et al., 2014). Basically, the numerical method is based on numerical integration rather than the numerical differentiation. Yayli et al. (Yayli et al., 2014) investigated the analytical method for free vibration of the elastically restrained Euler-Bernoulli beam on elastic foundation. The Fourier sine series with the Stoke's transformation is used to obtain the free vibration response of the beam on elastic foundation.

In previous studies regarding free vibration of the beam rested on a foundation, the Euler–Bernoulli and Timoshenko beams on uniform foundation are analyzed. On the other hand, only the solution taken from few previous researchers can be generalized to general boundary conditions for Euler–Bernoulli beam on uniform foundation. In this study, an accurate solution in closed forms is presented for free vibration behavior of elastically restrained Timoshenko beam on an arbitrary variable Winkler foundation and under axial load. The Green Function method is utilized to evaluate the free vibration of the Timoshenko beam. Furthermore, the free vibration expression for the Timoshenko beam is written in a general form. Hence, the computation becomes more efficient. Also, through the application of the Green function method, the boundary conditions are embedded in the Green functions of the corresponding beams. Therefore, the objective of this paper is:

- To present a very simple and practical analytical–numerical technique for determining the free vibration of Timoshenko beams, with elastically restrained boundary conditions, rested on a partial Winkler foundation and under axial load.
- To state precise solutions in closed forms using the Green function for free vibration of the Timoshenko beam with and without the partial Winkler foundation along with the axial load.

This article is organized as follows. Section 2 outlines the basic equations of the Timoshenko beam resting on the uniform elastic foundation. Then, in section 3, the Green function and the natural frequency equation of the elastically restrained Timoshenko beam on an arbitrary variable Winkler foundation and under axial load are explained. Section 4 presents some numerical examples to illustrate the efficiency of the Green Function in the new formulation. Finally, in section 5, the conclusions are drawn, briefly.

2 MODELLING OF TIMOSHENKO BEAM ON WINKLER FOUNDATION

In this paper, it is supposed that a Timoshenko beam on elastic foundation where it is partially restrained against translation and rotation at its ends. The model of elastic foundation is assumed as Winkler foundation, as shown in Figure 1. K_{TL}, K_{TR}, K_{RL} and K_{RR} are the transverse and rotational elastic coefficients at the supports at the left and right boundary ends, respectively. Thus, the coupled system of differential equations for the vibration of the uniform Timoshenko beam can be given by:

$$\kappa GA\left(\theta_{,x} - w_{,xx}\right) + \rho A\, w_{,tt} - N w_{,xx} + K_W w = q(x,t) \tag{1}$$

$$EI\, \theta_{,xx} + \kappa GA\left(w_{,x} - \theta\right) - \rho I\, \theta_{,tt} = 0 \tag{2}$$

where w(x,t) is the transverse deflection of the mid-surface of the beam, $\theta(x,t)$ represents the anti-clockwise angle of rotation of the normal to the mid-surface, $q(x,t)$ is the external load force on the beam. In addition, I, A, E, G, N, κ and ρ are, the second moment of area, the cross-sectional area of the beam, the Young's modulus of elasticity, the shear modulus, the axial load, the sectional shear coefficient, and the beam material density, respectively. It is assumed that each function $w(x,t)$, $\theta(x,t)$ and $q(x,t)$ can be presented as a product of a function dependent on the coordinate x and a function dependent on the time t (with the same time function):

$$w(x,t) = W(x)\ exp(i\omega t) \tag{3}$$

$$\theta(x,t) = \Theta(x)\ exp(i\omega t) \tag{4}$$

$$q(x,t) = Q(x)\, exp(i\omega t) \tag{5}$$

where $W(x)$, $\Theta(x)$ and $Q(x)$ are the beam deflection amplitude, the amplitude angle of rotation of the normal to the mid-surface in point x of the Timoshenko beam and the external load on the beam, respectively. In addition, ω is the circular frequency of the Timoshenko beam. Substituting Eqs. (3), (4) and (5) into Eqs. (1) and (2), result in:

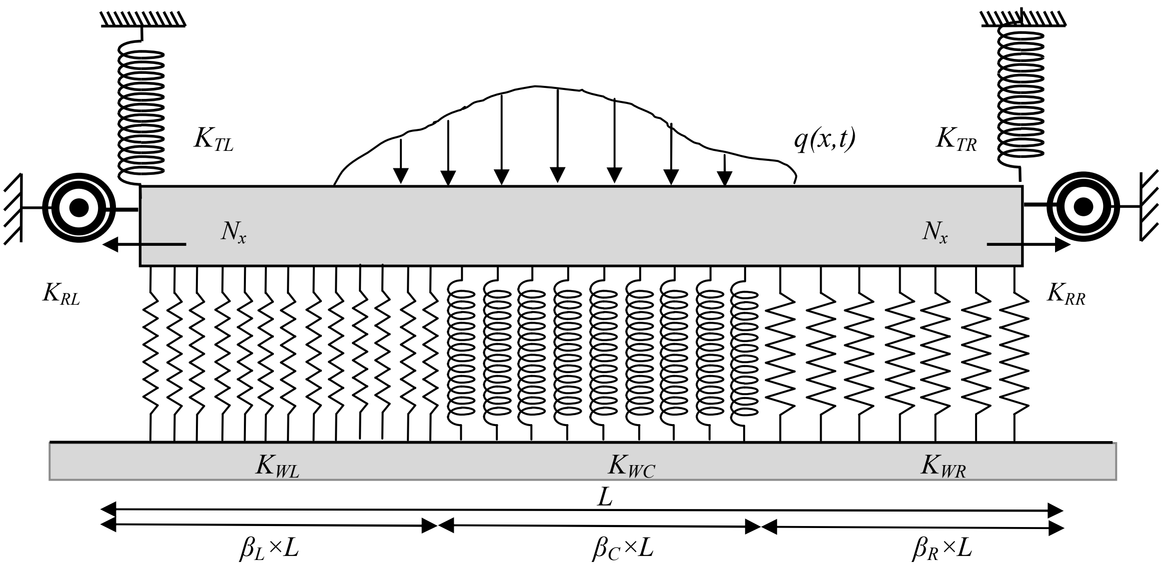


Figure 1: Timoshenko beam with general boundary conditions resting on Winkler foundation.

$$(-\kappa GA - N_x)W_{,xx} + (K_W - \rho A\omega^2)W + \kappa GA\Theta_{,x} = Q(x) \tag{6}$$

$$EI\,\Theta_{,xx} + (\rho I\omega^2 - \kappa GA)\Theta + \kappa GAW_{,x} = 0 \tag{7}$$

For a linear elastic, isotropic, homogeneous and uniform Timoshenko beam, these two equations can be combined after several transformations. The vibration equations for Timoshenko beam can be expressed in a form dependent only on the functions of the displacement w(x, t):

$$\begin{gathered} EI\left(1+\frac{N_x}{\kappa GA}\right)W_{,xxxx} + \left(\rho I\omega^2 - \frac{EI}{\kappa GA}(K_W - \rho A\omega^2) + N_x\left(\frac{\rho I\omega^2}{\kappa GA} - 1\right)\right)W_{,xx} \\ -(K_W - \rho A\omega^2)\left(\frac{\rho I\omega^2}{\kappa GA} - 1\right)W = -\frac{EI}{\kappa GA}Q_{,xx}(x) - \left(\frac{\rho I\omega^2}{\kappa GA} - 1\right)Q(x) \end{gathered} \tag{8}$$

$$\begin{gathered} EI\left(1+\frac{N_x}{\kappa GA}\right)\Theta_{,xxxx} + \left(\rho I\omega^2 - \frac{EI}{\kappa GA}(K_W - \rho A\omega^2) + N_x\left(\frac{\rho I\omega^2}{\kappa GA} - 1\right)\right)\Theta_{,xx} \\ -(K_W - \rho A\omega^2)\left(\frac{\rho I\omega^2}{\kappa GA} - 1\right)\Theta = Q_{,x}(x) \end{gathered} \tag{9}$$

For tension $N_x > 0$, as well as, for compression, one is required to apply $N_x < 0$. It is to be noticed that when N_x and K_W are equal to zero, the expression given by Eqs. (8) and (9) does reduce to the differential equations of the motion which are obtained by Ghannadiasl and Mofid (Ghannadiasl and Mofid, 2014). In this paper, the initial conditions and the general boundary conditions associated with the Timoshenko beam theory are given below:

$\forall t \;@\; x = 0 \quad : \; M(0,t) = K_{RL}\,\theta(0,t) \qquad Q(0,t) = -K_{TL}\,w(0,t)$

$\forall t \;@\; x = L \quad : \; M(L,t) = -K_{RR}\,\theta(L,t) \qquad Q(L,t) = K_{TR}\,w(L,t)$

$\forall x : \qquad \rho I\,\theta_{,t}\,\delta\theta|_{t_0}^{t} = 0 \qquad \rho A\,w_{,t}\,\delta w|_{t_0}^{t} = 0$

where M and Q are the bending moment ($M = EI\,\theta_{,x}$) and the shear force ($Q = -\kappa AG(w_{,x} - \theta)$), respectively (Wang, 1995).

3 GREEN FUNCTION FOR TIMOSHENKO BEAM

The Green function is utilized to find the solution for Eqs. (8) and (9). Therefore in this case, if G(x,u) was the Green function for the submitted problem, the solution of Eq. (8) can be exhibited in the form of:

$$W(x) = \frac{\kappa GA - \rho I\omega^2}{EI(\kappa GA + N_x)}\int_0^L Q(u)\,G(x,u)\,du \tag{10}$$

where $G(x,u)$, the Green function for the Timoshenko beam must satisfy the boundary conditions. Hence, the Green function, G(x,u), is the solution of the differential equation:

$$G_{,xxxx}+\frac{\rho I\omega^2-\frac{EI}{\kappa GA}(K_W-\rho A\omega^2)+N_x\left(\frac{\rho I\omega^2}{\kappa GA}-1\right)}{EI\left(1+\frac{N_x}{\kappa GA}\right)}G_{,xx}+\frac{\left(1-\frac{\rho I\omega^2}{\kappa GA}\right)(K_W-\rho A\omega^2)}{EI\left(1+\frac{N_x}{\kappa GA}\right)}G=\delta(x-u) \tag{11}$$

where $\delta(x-u)$ is the Dirac delta function which is defined as:

$$\delta(x-u)=\begin{cases}+\infty & if\ x=u\\ 0 & if\ x\neq u\end{cases}$$

By applying the relationships between the individual physical quantities, Eqs. (10) and (11) can be written as the following:

$$W(x)=\frac{1-\phi^2r^2\alpha^2}{EI(1+\gamma\alpha^2)}\int_0^L Q(u)\,G(x,u)du \tag{12}$$

$$G_{,xxxx}+\left(\frac{\alpha^2(\phi^2-\eta)-\gamma}{1+\gamma\alpha^2}+\phi^2r^2\right)G_{,xx}+\frac{(1-\phi^2r^2\alpha^2)(\eta-\phi^2)}{1+\gamma\alpha^2}G=\delta(x-u) \tag{13}$$

where ϕ is the parameter proportional to the natural frequency ($\phi^2=\frac{\omega^2\rho A}{EI}$), α, the parameter proportional to the rigidity of the beam ($\alpha^2=\frac{EI}{\kappa AG}$), r, the radius of gyration of the beam cross section ($r^2=\frac{I}{A}$), γ, the parameter proportional to the axial load $\left(\gamma=\frac{N_x}{EI}\right)$, and η is the parameter proportional to the elastic coefficient of Winkler foundation $\left(\eta=\frac{K_W}{EI}\right)$. The free vibration equation of uniform Timoshenko beam on Winkler foundation and under axial load can be obtained in the form of:

$$G_{,xxxx}+2\,\phi^2p_1G_{,xx}+\phi^2p_2G=0 \tag{14}$$

where:

$$p_1=\frac{1}{2}\left(r^2-\frac{1}{\phi^2}\frac{\alpha^2(\eta-\phi^2)+\gamma}{1+\gamma\alpha^2}\right)$$

$$p_2=\frac{1}{\phi^2}\frac{(1-\phi^2r^2\alpha^2)(\eta-\phi^2)}{1+\gamma\alpha^2}$$

The general solution of Eq. (14) can be stated as:

$$G(x)=C_1\sin(\phi\lambda_1x)+C_2\cos(\phi\lambda_1x)+C_3\sinh(\phi\lambda_2x)+C_4\cosh(\phi\lambda_2x) \tag{15}$$

where $x\in[0,L]$, λ_1 and λ_2 are calculated as:

$$\lambda_1=\sqrt{p_1+\sqrt{{p_1}^2-\frac{p_2}{\phi^2}}} \tag{16}$$

$$\lambda_2 = \sqrt{-p_1 + \sqrt{{p_1}^2 - \frac{p_2}{\phi^2}}} \tag{17}$$

$C_1, \dots, C_4$ are the integration constants that are evaluated such that the Green function satisfies two boundary conditions at each end of the beam depending on the type of end support and the continuity conditions of displacement, slope and moment along with the shear force.

In this paper, the Timoshenko beam divides into three segments with the different elastic coefficient of Winkler foundation. It can be possible to write the differential equation of the free vibration of each segment. Therefore, the general solution for the first segment can be stated as:

$$\mathrm{G_L}(x) = C_{1L}\sin(\phi\lambda_{1L}x) + C_{2L}\cos(\phi\lambda_{1L}x) + C_{3L}\sinh(\phi\lambda_{2L}x) + C_{4L}\cosh(\phi\lambda_{2L}x) \tag{18}$$

where $x \in [0, \beta_L L]$, λ_{1L} and λ_{2L} are calculated as:

$$\lambda_{1L} = \sqrt{p_{1L} + \sqrt{{p_{1L}}^2 - \frac{p_{2L}}{\phi^2}}} \tag{19}$$

$$\lambda_{2L} = \sqrt{-p_{1L} + \sqrt{{p_{1L}}^2 - \frac{p_{2L}}{\phi^2}}} \tag{20}$$

where

$$p_{1L} = \frac{1}{2}\left(r^2 - \frac{1}{\phi^2}\frac{\alpha^2(\eta_L - \phi^2) + \gamma}{1 + \gamma\alpha^2}\right)$$

$$p_{2L} = \frac{1}{\phi^2}\frac{(1 - \phi^2 r^2 \alpha^2)(\eta_L - \phi^2)}{1 + \gamma\alpha^2}$$

where η_L is equal to $\frac{K_{WL}}{EI}$. For the middle section, the general solution of free vibration takes the form of the following equation:

$$\mathrm{G_C}(x) = C_{1C}\sin(\phi\lambda_{1C}x) + C_{2C}\cos(\phi\lambda_{1C}x) + C_{3C}\sinh(\phi\lambda_{2C}x) + C_{4C}\cosh(\phi\lambda_{2C}x) \tag{21}$$

where $x \in (\beta_L L, L(\beta_L + \beta_C)]$, λ_{1C} and λ_{2C} are calculated as:

$$\lambda_{1C} = \sqrt{p_{1C} + \sqrt{{p_{1C}}^2 - \frac{p_{2C}}{\phi^2}}} \tag{22}$$

$$\lambda_{2C} = \sqrt{-p_{1C} + \sqrt{{p_{1C}}^2 - \frac{p_{2C}}{\phi^2}}} \tag{23}$$

where in p_{1C} and p_{2C}, η_C is equal to $\frac{K_{WC}}{EI}$. Similarly, it is possible to develop the general solution of free vibration for the last section of the Timoshenko beam:

$$G_R(x) = C_{1R}\sin(\phi\lambda_{1R}x) + C_{2R}\cos(\phi\lambda_{1R}x) + C_{3R}\sinh(\phi\lambda_{2R}x) + C_{4C}\cosh(\phi\lambda_{2R}x) \tag{24}$$

where $x \in (L(\beta_L + \beta_C), L]$, λ_{1R} and λ_{2R} are calculated as:

$$\lambda_{1R} = \sqrt{p_{1R} + \sqrt{{p_{1R}}^2 - \frac{p_{2R}}{\phi^2}}} \tag{25}$$

$$\lambda_{2R} = \sqrt{-p_{1R} + \sqrt{{p_{1R}}^2 - \frac{p_{2R}}{\phi^2}}} \tag{26}$$

where in p_{1R} and p_{2R}, η_R is equal to $\frac{K_{WR}}{EI}$. $C_{1L} - C_{4L}$, $C_{1C} - C_{4C}$ and $C_{1R} - C_{4R}$ are the constant unknowns of the three above-mentioned solutions. In order to find these unknowns, it is required to develop twelve equations. Moreover, in which are explicitly obtained using two boundary conditions at each end of the beam depending on the type of end support and the continuity conditions of displacement, slope and moment along with the shear force in the vicinities of the different segment connections. The boundary conditions are given below:

$$\frac{1}{1-\phi^2 r^2 \alpha^2}\left(\left(\phi^2(r^2+\alpha^2) - \alpha^2\,\eta_L\right)G_{L,x}(0) + (1+\alpha^2\,\gamma)G_{L,xxx}(0)\right) + \frac{K_{TL}}{EI}G_L(0) = 0 \tag{27a}$$

$$(1+\alpha^2\,\gamma)G_{L,xx}(0) + \alpha^2\left(\phi^2 - \eta_L\right)G_L(0) - \frac{K_{RL}}{EI}\frac{\left(\phi^2\alpha^4 - \alpha^4\,\eta_L + 1\right)G_{L,x}(0) + \alpha^2(1+\alpha^2\,\gamma)G_{L,xxx}(0)}{1-\phi^2 r^2 \alpha^2} = 0 \tag{27b}$$

$$\frac{1}{1-\phi^2 r^2 \alpha^2}\left(\left(\phi^2(r^2+\alpha^2) - \alpha^2\,\eta_R\right)G_{R,x}(L) + (1+\alpha^2\,\gamma)G_{R,xxx}(L)\right) - \frac{K_{TR}}{EI}G_R(L) = 0 \tag{27c}$$

$$(1+\alpha^2\,\gamma)G_{R,xx}(L) + \alpha^2\left(\phi^2 - \eta_R\right)G_R(L) + \frac{K_{RR}}{EI}\frac{\left(\phi^2\alpha^4 - \alpha^4\,\eta_R + 1\right)G_{R,x}(L) + \alpha^2(1+\alpha^2\,\gamma)G_{R,xxx}(L)}{1-\phi^2 r^2 \alpha^2} = 0 \tag{27d}$$

Also, the continuity conditions are defined as:

$$G_L(\beta_L L) = G_C(\beta_L L) \tag{28a}$$

$$\theta_L(\beta_L L) = \theta_C(\beta_L L) \tag{28b}$$

$$M_L(\beta_L L) = M_C(\beta_L L) \tag{28c}$$

$$Q_L\left(\beta_L L\right) = Q_C\left(\beta_L L\right) \tag{28d}$$

and

$$G_C(\beta_L L + \beta_C L) = G_R(\beta_L L + \beta_C L) \quad (29a)$$

$$\theta_C(\beta_L L + \beta_C L) = \theta_R(\beta_L L + \beta_C L) \quad (29b)$$

$$M_C(\beta_L L + \beta_C L) = M_R(\beta_L L + \beta_C L) \quad (29c)$$

$$Q_C\left(\beta_L L + \beta_C L\right) = Q_R\left(\beta_L L + \beta_C L\right) \quad (29d)$$

By applying the relationships between the individual physical quantities and the Green function, the continuity conditions can be rewritten as follows:

$$\begin{aligned}\left((\phi^2\alpha^4 - \alpha^4\,\eta_L + 1)G_{L,x}(\beta_L L) - (\phi^2\alpha^4 - \alpha^4\,\eta_C + 1)G_{C,x}(\beta_L L)\right)\\ + \alpha^2(1 + \alpha^2\,\gamma)\left(G_{L,xxx}(\beta_L L) - G_{C,xxx}(\beta_L L)\right) = 0\end{aligned} \quad (30a)$$

$$\begin{aligned}(1 + \alpha^2\,\gamma)\left(G_{L,xx}(\beta_L L) - G_{C,xx}(\beta_L L)\right) +\\ \alpha^2\left((\phi^2 - \eta_L)G_L(\beta_L L) - (\phi^2 - \eta_C)G_C(\beta_L L)\right) = 0\end{aligned} \quad (30b)$$

$$\begin{aligned}\left((\phi^2(r^2 + \alpha^2) - \alpha^2\,\eta_L)G_{L,x}(\beta_L L) - (\phi^2(r^2 + \alpha^2) - \alpha^2\,\eta_C)G_{C,x}(\beta_L L)\right)\\ + (1 + \alpha^2\,\gamma)\left(G_{L,xxx}(\beta_L L) - G_{C,xxx}(\beta_L L)\right) = 0\end{aligned} \quad (30c)$$

and

$$\begin{aligned}\left((\phi^2\alpha^4 - \alpha^4\,\eta_C + 1)G_{C,x}(\beta_L L + \beta_C L) - (\phi^2\alpha^4 - \alpha^4\,\eta_R + 1)G_{R,x}(\beta_L L + \beta_C L)\right)\\ + \alpha^2(1 + \alpha^2\,\gamma)\left(G_{C,xxx}(\beta_L L + \beta_C L) - G_{R,xxx}(\beta_L L + \beta_C L)\right) = 0\end{aligned} \quad (31a)$$

$$\begin{aligned}(1 + \alpha^2\,\gamma)\left(G_{C,xx}(\beta_L L + \beta_C L) - G_{R,xx}(\beta_L L + \beta_C L)\right)\\ + \alpha^2\left((\phi^2 - \eta_C)G_C(\beta_L L + \beta_C L) - (\phi^2 - \eta_R)G_R(\beta_L L + \beta_C L)\right) = 0\end{aligned} \quad (31b)$$

$$\begin{aligned}\left((\phi^2(r^2 + \alpha^2) - \alpha^2\,\eta_C)G_{C,x}(\beta_L L + \beta_C L) - (\phi^2(r^2 + \alpha^2) - \alpha^2\,\eta_R)G_{R,x}(\beta_L L + \beta_C L)\right)\\ + (1 + \alpha^2\,\gamma)\left(G_{C,xxx}(\beta_L L + \beta_C L) - G_{R,xxx}(\beta_L L + \beta_C L)\right) = 0\end{aligned} \quad (31c)$$

Finally, the matrix equation is given as:

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{3,9} & A_{3,10} & A_{3,11} & A_{3,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{4,9} & A_{4,10} & A_{4,11} & A_{4,12} \\ A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & A_{5,7} & A_{5,8} & 0 & 0 & 0 & 0 \\ A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} & A_{6,8} & 0 & 0 & 0 & 0 \\ A_{7,1} & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} & A_{7,8} & 0 & 0 & 0 & 0 \\ A_{8,1} & A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & A_{8,7} & A_{8,8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{9,5} & A_{8,6} & A_{8,7} & A_{8,8} & A_{4,9} & A_{4,10} & A_{4,11} & A_{4,12} \\ 0 & 0 & 0 & 0 & A_{10,5} & A_{10,6} & A_{10,7} & A_{10,8} & A_{10,9} & A_{10,10} & A_{10,11} & A_{10,12} \\ 0 & 0 & 0 & 0 & A_{11,5} & A_{11,6} & A_{11,7} & A_{11,8} & A_{11,9} & A_{11,10} & A_{11,11} & A_{11,12} \\ 0 & 0 & 0 & 0 & A_{12,5} & A_{12,6} & A_{12,7} & A_{12,8} & A_{12,9} & A_{12,10} & A_{12,11} & A_{12,12} \end{bmatrix} \begin{bmatrix} C_{1L} \\ C_{2L} \\ C_{3L} \\ C_{4L} \\ C_{1C} \\ C_{2C} \\ C_{3C} \\ C_{4C} \\ C_{1R} \\ C_{2R} \\ C_{3R} \\ C_{4R} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

where the coefficient matrix $[A_{i,j}]$ is cited in the Appendix. The nontrivial solution to Eq. (32) is obtained from the condition where the main matrix determinant is equal to zero. Furthermore, the Green function for free vibration of the Timoshenko beam that is obtained by the above procedure has a general form. By moving close to the spring constants of the rotational and translational restraint to extreme values (zero and/or infinity), the suitable Green function can be attained for the desired combinations of end boundary conditions (i.e. simply supported, clamped and free boundary conditions). For example, the natural frequency equation for a general Timoshenko beam with elastic end restraints ($K_{RR} = K_{RL} = K_R$ and $K_{TR} = K_{TL} = K_T$) resting on a uniform Winkler elastic foundation ($K_{WL} = K_{WC} = K_{WR}$) and under axial load is given by:

$$\begin{aligned} & sin(\phi\lambda_1 L)\,(2A_1\,cosh(\phi\lambda_2 L) + A_2\,sinh(\phi\lambda_2 L)) \\ & \qquad + 2\,cos(\phi\lambda_1 L)\big(B_1(A_3\,cosh(\phi\lambda_2 L) - A_4\,sinh(\phi\lambda_2 L)) \\ & \qquad + B_3(A_5\,cosh(\phi\lambda_2 L) + A_6\,sinh(\phi\lambda_2 L))\big) + A_7 = 0 \end{aligned} \tag{33}$$

where

$$A1 = \frac{K_R}{EI}B_1B_6(B_2B_3 - B_1B_5) + \frac{K_T}{EI}\left(B_2B_4(B_6 - B_4) + B_3(B_2B_3 - B_1B_5)\left(\frac{K_R}{EI}\right)^2\right) + B_5\frac{K_R}{EI}\left(\frac{K_T}{EI}\right)^2(B_6 - B_4)$$

$$\begin{aligned} A_2 = B_2^2\left(\frac{K_T}{EI}\right)^2\left((B_4 - B_6)^2 + \left(\frac{K_R}{EI}\right)^2(B_5^2 - B_3^2)\right) + 2B_2B_5K_R\left(\frac{K_R}{EI}B_1B_3 + \frac{K_T}{EI}B_4\right) + \left(B_4^2 - \left(\frac{K_R}{EI}\right)^2 B_3^2\right) \\ - B_1^2\left(B_6^2 + \left(\frac{K_R}{EI}\right)^2 B_5^2\right) - 2\frac{K_R}{EI}\frac{K_T}{EI}B_1B_3B_6 \end{aligned}$$

$$A_3 = B_5\frac{K_R}{EI}\frac{K_T}{EI}(B_4 - 2B_6) - B_2B_4B_6$$

$$A_4 = \frac{K_R}{EI} B_2 B_4 B_5 + \frac{K_T}{EI}\left(\left(\frac{K_R}{EI}\right)^2 B_5^2 + B_6^2 - B_4 B_6\right)$$

$$A_5 = \frac{K_T}{EI}\left(B_2(B_6 - 2B_4) - \frac{K_R}{EI}\frac{K_T}{EI} B_5\right)$$

$$A_6 = B_2^2 B_4 + \frac{K_R}{EI}\frac{K_T}{EI} B_2 B_5 + \left(\frac{K_T}{EI}\right)^2 (B_4 - B_6)$$

$$A_7 = 2\left(B_1 B_4 + \frac{K_R}{EI}\frac{K_T}{EI} B_3\right)\left(B_2 B_6 + \frac{K_R}{EI}\frac{K_T}{EI} B_5\right)$$

$$B_1 = \frac{\phi\lambda_1}{1 - r^2\alpha^2\phi^2}(B_4 + r^2\phi^2)$$

$$B_2 = \frac{\phi\lambda_2}{1 - r^2\alpha^2\phi^2}(B_6 + r^2\phi^2)$$

$$B_3 = \frac{\phi\lambda_1}{1 - r^2\alpha^2\phi^2}(1 + \alpha^2 B_4)$$

$$B_4 = \alpha^2(-\eta + \phi^2) - (1 + \alpha^2\gamma)\phi^2\lambda_1^2$$

$$B_5 = \frac{\phi\lambda_2}{1 - r^2\alpha^2\phi^2}(1 + \alpha^2 B_6)$$

3

$$B_6 = \alpha^2(-\eta + \phi^2) + (1 + \alpha^2\gamma)\phi^2\lambda_2^2$$

Although the frequency Eq. (33) is complicated and long, it can be simulated by all the boundary conditions that are appeared in previous studies and practical situations, which have not been possible to describe before. After finding the natural frequencies, the mode shape corresponding to each natural frequency can be generated. Three of the constant unknowns in the shape function (Eq. (15)) can be solved by three equations of boundary conditions at each end of the beam. The coefficients in mode shape function for a uniform Timoshenko beam on a uniform Winkler elastic foundation with classical end conditions are listed in Table 1. Here, C_2, C_3 and C_4 are solved by the three equations of Eq. (27). On the other hand, C_1 is assumed to be non-zero in order to demonstrate vibration amplitude.

4 NUMERICAL RESULTS

In this section, to validate the presented Green Function, the results of different examples, which are solved by the new formulations, are presented. First, the high computational efficiency of the method is shown and then it is examined for feedback with general boundary conditions.

End boundary conditions	C_1	C_2 *	C_3 *	C_4 *
Pined-pinned	1	0	$-\frac{\sin(\phi\lambda_1 L)}{\sinh(\phi\lambda_2 L)}$	0
Fixed-fixed	1	$-\frac{\sin(\phi\lambda_1 L)-\frac{D_3}{D_4}\sinh(\phi\lambda_2 L)}{\cos(\phi\lambda_1 L)-\cosh(\phi\lambda_2 L)}$	$-\frac{D_3}{D_4}$	$\frac{\sin(\phi\lambda_1 L)-\frac{D_3}{D_4}\sinh(\phi\lambda_2 L)}{\cos(\phi\lambda_1 L)-\cosh(\phi\lambda_2 L)}$
Fixed-pinned	1	$-\frac{\sin(\phi\lambda_1 L)}{\cos(\phi\lambda_1 L)}$	$-\frac{\cosh(\phi\lambda_2 L)}{\sinh(\phi\lambda_2 L)}\frac{\sin(\phi\lambda_1 L)}{\cos(\phi\lambda_2 L)}$	$\frac{\sin(\phi\lambda_1 L)}{\cos(\phi\lambda_1 L)}$
Fixed-free	1	$\frac{\frac{D_2 D_3}{D_4}\sinh(\phi\lambda_2 L)-D_1\sin(\phi\lambda_1 L)}{D_2\cosh(\phi\lambda_2 L)-D_1\cos(\phi\lambda_1 L)}$	$-\frac{D_3}{D_4}$	$\frac{\frac{D_2 D_3}{D_4}\sinh(\phi\lambda_2 L)-D_1\sin(\phi\lambda_1 L)}{D_2 \mathrm{Cosh}[L\lambda 2\phi]-D_1 \mathrm{Cos}[L\lambda 1\phi]}$
Pinned- fixed	1	0	$-\frac{D_3}{D_4}\frac{\cos(\phi\lambda_1 L)}{\cosh(\phi\lambda_2 L)}$	0
Free - fixed	1	$\frac{\cos(\phi\lambda_1 L)+\frac{\lambda_2 D_5}{\lambda_1 D_6 D_2}\cosh(\phi\lambda_2 L)}{\sin(\phi\lambda_1 L)+\frac{D_1 D_4}{D_2 D_3}\sinh(\phi\lambda_2 L)}$	$-\frac{D_5}{D_6}$	$-\frac{D_1}{D_2}\frac{\cos(\phi\lambda_1 L)+\frac{\lambda_2 D_5}{\lambda_1 D_6 D_2}\cosh(\phi\lambda_2 L)}{\sin(\phi\lambda_1 L)+\frac{D_1 D_4}{D_2 D_3}\sinh(\phi\lambda_2 L)}$

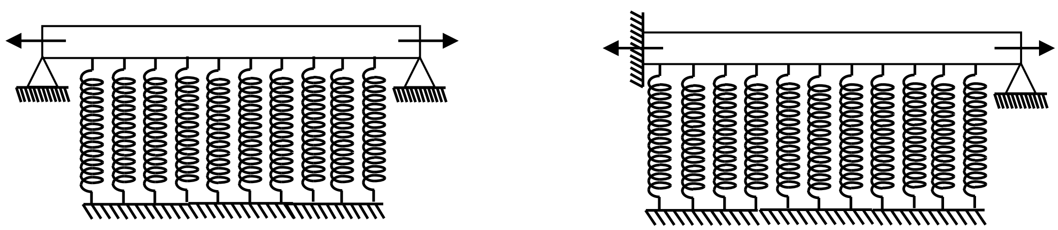
* $D_1 = \phi^2(\alpha^2 - {\lambda_1}^2) - \alpha^2\eta$ $D_2 = \phi^2(\alpha^2 + {\lambda_2}^2) - \alpha^2\eta$ $D_3 = \lambda_1(\alpha^2 D_1 + 1)$
$D_4 = \lambda_2(\alpha^2 D_2 + 1)$ $D_5 = \lambda_1(D_1 + r^2\phi^2)$ $D_6 = \lambda_2(D_2 + r^2\phi^2)$

Table 1: The coefficients in mode shape function for Timoshenko beams on a uniform Winkler elastic foundation and without axial load for classical boundary conditions.

4.1 The Timoshenko Beam on the Uniform Winkler Foundation Under Constant Axial Load

In order to illustrate the accuracy of the presented method in this paper, a uniform Timoshenko beam on the uniform Winkler elastic foundation and under constant axial load with two different boundary conditions, i.e. simply supported–simply supported (S–S) and clamped–simply supported (C–S), are considered (Figure 2). The beam is supposed with the following characteristics:

$\bar{\phi} = \omega\sqrt{\frac{\rho A L^4}{EI}}$ $\frac{r^2}{L^2} = 0.01$ $r^2 = \frac{I}{A}$



a. Simply supported beam b. Clamped – simply supported beam

Figure 2: Timoshenko beams on the uniform Winkler foundation and under constant axial load

$\upsilon = 0.25$ $\kappa = 2/3$ $G = \frac{E}{2(1+\upsilon)}$

$K_{WL} = K_{WC} = K_{WR} = K_W$ $N_x = -\,0.6\,\pi^2 \frac{EI}{L^2}$

Table 2 compares the frequency parameters of free vibration of the simply supported–simply supported and clamped–simply supported Timoshenko beam resting on the uniform Winkler foundation and under axial load using the Green Function method along with the differential quadrature element method (Malekzadeh et al., 2003). It is seen that the results are fairly close and the maximum difference is 0.057%. It is informed from Table 2 that the first mode of the Timoshenko beam on the uniform Winkler elastic foundation and under constant axial load is more sensitive to the elastic coefficient of Winkler foundation. At the same time, it can also be seen that the maximum difference of the first, the second and the third frequency parameter for the simply supported beam with and without the uniform Winkler foundation ($K_W = 0.8\,\pi^4 \frac{EI}{L^2}$) are approximately 167.44%, 9.39% and 2.94%, respectively. On the other hand, the maximum difference of the first, second and third frequency parameter for the clamped–simply supported beam with and without the uniform Winkler foundation ($K_W = 0.8\,\pi^4 \frac{EI}{L^2}$) are approximately 54.80%, 8.05% and 2.82%, respectively.

K_W	Mode	Simply supported		Clamped – simply supported	
		Present study (%Error)	Malekzadeh et al. (Malekzadeh et al., 2003)	Present study (%Error)	Malekzadeh et al. (Malekzadeh et al., 2003)
0	1	3.46648 (0.101)	3.47	7.32425 (0.058)	7.32
	2	19.2209 (0.005)	19.22	20.9311 (0.005)	20.93
	3	35.0792 (0.002)	35.08	35.7458 (0.012)	35.75
$0.2\,\pi^4 \frac{EI}{L^2}$	1	5.52398	-	8.50792	-
	2	19.6879	-	21.3650	-
	3	35.3404	-	36.0005	-
$0.4\,\pi^4 \frac{EI}{L^2}$	1	7.00019	-	9.54555	-
	2	20.1439	-	21.7900	-
	3	35.5996	-	36.2532	-
$0.6\,\pi^4 \frac{EI}{L^2}$	1	8.21469 (0.057)	8.21	10.4806 (0.008)	10.48
	2	20.5896 (0.002)	20.59	22.2068 (0.014)	22.21
	3	35.8568 (0.009)	35.86	36.5041 (0.011)	36.50
$0.8\,\pi^4 \frac{EI}{L^2}$	1	9.27091	-	11.3384	-
	2	21.0257	-	22.6157	-
	3	36.1122	-	36.7532	-

Table 2: The frequency parameters ($\bar{\phi}$) of free vibration of Timoshenko beams on the uniform Winkler elastic foundation and under constant axial load

4.2 The Influence of the Axial Load and the Elastic Foundation on the Frequency of Timoshenko Beam

As an interesting application of the present method, the influence of the elastic coefficient of Winkler foundation and the axial load on free vibration characteristics of simply supported Timoshenko beam is evaluated. The beam characteristics are as follows:

$\upsilon = 0.3$ $\kappa = 0.85$

$\frac{r}{L} = 0.05$ $\bar{\phi} = \omega\sqrt{\frac{\rho A L^4}{EI}}$

Variation of the first frequency parameter ($\bar{\phi}$) of free vibration of Timoshenko beam is shown in Table 3. In addition, it is evident from the obtained values of the frequency parameter that the natural frequencies will increase when the values of K_W and N_x increase. The influence of axial load would be more significant when the stiffness of the Winkler foundation is coming close to zero. However, the effect of the axial load would be less significant when the value of the stiffness of the Winkler foundation is coming close to $\pi^4 \frac{EI}{L^2}$.

4.3 The Influence of the Spring Supports on the Frequency of the Timoshenko Beam Resting on the Partial Winkler Foundation

For verification of the efficacy of the present method, the influence of the spring supporting the behavior is evaluated based on free vibration characteristics of Timoshenko beam partially supported on a Winkler foundation (Figure 3). For this purpose, a Timoshenko beam is assumed with general boundary conditions, K_T and K_R. The stiffness of the rotational and the translational restraint are taken as having the same values at both of the supports. The beam characteristics are as follows:

$K_{WC} = \pi^4 \frac{EI}{L^2}$ $K_{WL} = K_{WR} = 0$ $\beta_L = \beta_C = \beta_R = L/3$

$N_x = 0$ $\upsilon = 0.30$ $\kappa = 0.85$

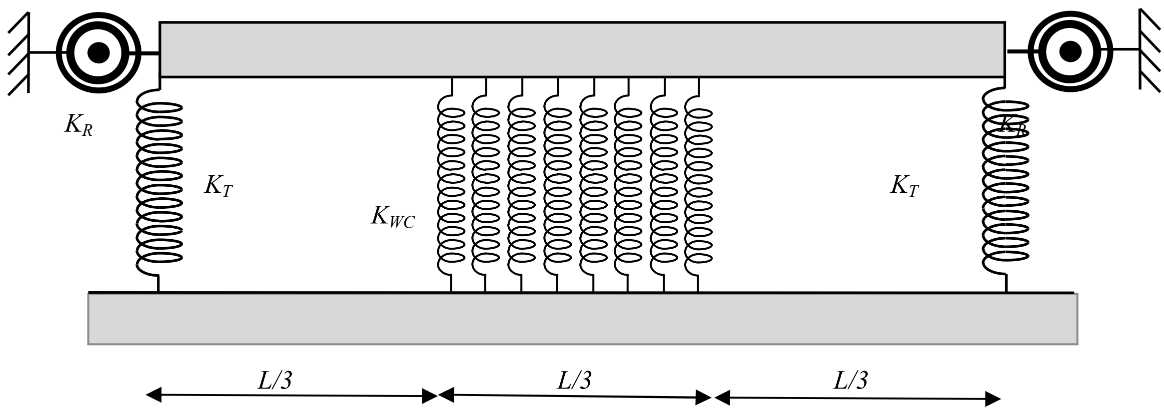


Figure 3: Timoshenko beam with general boundary conditions partially supported on Winkler foundation.

$N_x\left(\pi^2 \frac{EI}{L^2}\right)$	$K_W(\pi^4 \frac{EI}{L^2})$										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
-1.0	0.000	1.687	3.519	4.682	5.609	6.403	7.108	7.750	8.343	8.896	9.417
-0.9	1.687	3.519	4.682	5.609	6.403	7.108	7.750	8.343	8.896	9.417	9.910
-0.8	3.519	4.682	5.609	6.403	7.108	7.750	8.343	8.896	9.417	9.910	10.380
-0.7	4.682	5.609	6.403	7.108	7.750	8.343	8.896	9.417	9.910	10.380	10.830
-0.6	5.609	6.403	7.108	7.750	8.343	8.896	9.417	9.910	10.380	10.830	11.262
-0.5	6.403	7.108	7.750	8.343	8.896	9.417	9.910	10.380	10.830	11.262	11.677
-0.4	7.108	7.750	8.343	8.896	9.417	9.910	10.380	10.830	11.262	11.677	12.079
-0.3	7.750	8.343	8.896	9.417	9.910	10.380	10.830	11.262	11.677	12.079	12.467
-0.2	8.343	8.896	9.417	9.910	10.380	10.830	11.262	11.677	12.079	12.467	12.844
-0.1	8.896	9.417	9.910	10.380	10.830	11.262	11.677	12.079	12.467	12.844	13.210
0	9.417	9.910	10.380	10.830	11.262	11.677	12.079	12.467	12.844	13.210	13.566
0.1	9.910	10.380	10.830	11.262	11.677	12.079	12.467	12.844	13.210	13.566	13.913
0.2	10.380	10.830	11.262	11.677	12.079	12.467	12.844	13.210	13.566	13.913	14.252
0.3	10.830	11.262	11.677	12.079	12.467	12.844	13.210	13.566	13.913	14.252	14.583
0.4	11.262	11.677	12.079	12.467	12.844	13.210	13.566	13.913	14.252	14.583	14.906
0.5	11.677	12.079	12.467	12.844	13.210	13.566	13.913	14.252	14.583	14.906	15.223
0.6	12.079	12.467	12.844	13.210	13.566	13.913	14.252	14.583	14.906	15.223	15.533
0.7	12.467	12.844	13.210	13.566	13.913	14.252	14.583	14.906	15.223	15.533	15.837
0.8	12.844	13.210	13.566	13.913	14.252	14.583	14.906	15.223	15.533	15.837	16.135
0.9	13.210	13.566	13.913	14.252	14.583	14.906	15.223	15.533	15.837	16.135	16.428
1.0	13.566	13.913	14.252	14.583	14.906	15.223	15.533	15.837	16.135	16.428	16.715

Table 3: Variations of the first modal frequency parameter of free vibration of Timoshenko beam on uniform Winkler elastic foundation and under axial load

$\frac{r^2}{L^2} = 0.01$ $\bar{\phi} = \omega\sqrt{\frac{\rho A L^4}{EI}}$

$K_{TL} = K_{TR} = K_T$ $K_{RL} = K_{RR} = K_R$

The frequency parameter ($\bar{\phi}$) of free vibration of Timoshenko beam with and without partially supported on Winkler foundation is demonstrated in Table 4. It is observed that the beam on foundation can be considered as fixed-fixed at both ends when the values of K_T/EI and K_R/EI are greater than 100000.

K_T/EI	Mode	K_R/EI						
		0	1	10	100	1000	10000	100000
0	1	1.772 (0)	4.486 (0)	5.491 (0)	5.510 (0)	5.512 (0)	5.512 (0)	5.512 (0)
	2	5.381 (0)	5.425 (4.0682)	7.625 (7.3142)	8.578 (8.2742)	8.695 (8.3914)	8.707 (8.4033)	8.708 (8.4045)
	3	17.746 (16.819)	19.610 (18.794)	24.181 (23.332)	26.036 (25.193)	26.276 (25.434)	26.301 (25.459)	26.303 (25.461)
1	1	2.919 (1.3990)	5.016 (1.4022)	5.699 (1.4070)	5.712 (1.4085)	5.713 (1.4087)	5.714 (1.4087)	5.714 (1.4087)
	2	5.622 (2.3093)	5.653 (4.6405)	5.699 (7.5930)	8.799 (8.5013)	8.911 (8.6127)	8.922 (8.6241)	8.923 (8.6253)
	3	17.884 (16.984)	19.801 (18.934)	24.262 (23.423)	26.102 (25.267)	26.341 (25.506)	26.365 (25.530)	26.367 (25.533)
10	1	7.168 (4.0329)	7.172 (4.1188)	7.178 (4.2550)	7.180 (4.2968)	7.181 (4.3018)	7.181 (4.3023)	7.181 (4.3024)
	2	7.40535 (7.1547)	8.275 (8.0262)	9.933 (9.6741)	10.527 (10.263)	10.602 (10.337)	10.610 (10.345)	10.611 (10.346)
	3	19.147 (18.450)	20.895 (20.179)	24.999 (24.240)	26.699 (25.926)	26.920 (26.145)	26.942 (26.168)	26.945 (26.170)
100	1	10.262 (7.4030)	10.753 (8.0912)	11.851 (9.5749)	12.304 (10.173)	12.363 (10.252)	12.369 (10.260)	12.370 (10.261)
	2	18.669 (18.442)	18.673 (18.444)	18.680 (18.449)	18.683 (18.451)	18.683 (18.451)	18.683 (18.451)	18.683 (18.451)
	3	29.049 (28.802)	29.664 (29.373)	31.181 (30.783)	31.843 (31.402)	31.931 (31.484)	31.940 (31.493)	31.941 (31.493)
1000	1	11.110 (8.2925)	11.940 (9.3196)	14.146 (11.939)	15.333 (13.224)	15.497 (13.405)	15.514 (13.424)	15.516 (13.426)
	2	25.008 (24.685)	25.437 (25.117)	26.607 (26.295)	27.194 (26.887)	27.276 (26.971)	27.285 (26.979)	27.286 (26.980)
	3	42.730 (42.414)	42.809 (42.505)	43.003 (42.728)	43.092 (42.830)	43.104 (42.844)	43.106 (42.845)	43.106 (42.845)
10000	1	11.204 (8.3933)	12.077 (9.4637)	14.487 (12.247)	15.748 (13.643)	15.930 (13.842)	15.949 (13.863)	15.951 (13.865)
	2	25.718 (25.385)	26.282 (25.948)	27.886 (27.556)	28.728 (28.401)	28.848 (28.522)	28.860 (28.534)	28.862 (28.536)
	3	44.557 (44.218)	44.821 (44.499)	45.521 (45.245)	45.859 (45.605)	45.906 (45.655)	45.911 (45.660)	45.912 (45.660)
100000	1	11.213 (8.4035)	12.090 (9.4784)	14.518 (12.278)	15.791 (13.687)	15.974 (13.887)	15.993 (13.908)	15.995 (13.910)
	2	25.788 (25.454)	26.365 (26.031)	28.017 (27.685)	28.888 (28.559)	29.012 (28.684)	29.025 (28.697)	29.027 (28.699)
	3	44.723 (44.382)	45.011 (44.687)	45.780 (45.504)	46.154 (45.900)	46.206 (45.955)	46.211 (45.961)	46.212 (45.961)

Note: Values in parentheses are the frequency parameter for Timoshenko beam without foundation.

Table 4: Variations of the frequency parameter of free vibration of Timoshenko beam with general boundary conditions partially supported on Winkler foundation.

From Table 4, it illustrates that in the Timoshenko beam with $\frac{K_T}{EI} = 100000$ and $\frac{K_R}{EI} = 100000$, the first, second and third frequency parameter for beam on partial Winkler foundation are 15.995, 29.027 and 46.212, respectively. Also, the first, second and third frequency parameter for Timoshenko beam without the Winkler foundation are 13.910, 28.699 and 45.961, respectively. In comparison with the Timoshenko beam with $\frac{K_T}{EI} = 100000$ and $\frac{K_R}{EI} = 100000$, the maximum difference of the first, second and third frequency parameter for the beam with $\frac{K_T}{EI} = 100000$ and $\frac{K_R}{EI} = 0$ with and without the partial Winkler foundation are approximately 33.43%, 1.31% and 0.77%, respectively. Also, the Euler-Bernoulli beam model can be obtained from the Timoshenko beam model by setting r^2 to zero (that is, if the rotational effect is ignored) and α^2 to zero (that is, if the shear effect is ignored) (Mei and Mace, 2005). Therefore, the non-dimensional frequency parameter ($\overline{\phi}$) of free vibration of Euler-Bernoulli beam with and without resting on the partial Winkler foundation is demonstrated in Table 5. In comparison with the Timoshenko beam with $\frac{K_T}{EI} = 100000$ and $\frac{K_R}{EI} = 100000$, the maximum difference of the first, second and third frequency parameter for the Euler-Bernoulli beam with $\frac{K_T}{EI} = 100000$ and $\frac{K_R}{EI} = 100000$ with and without the partial Winkler foundation are approximately 2.21%, 0.37% and 0.07%, respectively. Table 5 clearly shows that the values of the first frequency parameters are almost the same when the stiffness of the rotational springs is larger than 100.

4.4 The Mode Shapes of the Fixed-free Timoshenko Beam on Winkler Foundation

The influence of the elastic coefficient of uniform Winkler foundation on the mode shapes of fixed-free Timoshenko beam is evaluated. Thus the beam is considered with the following characteristics:

$\upsilon = 0.3$ $\kappa = 0.85$

L=10 m $r^2 = \frac{1}{300}$

$\alpha^2 = \frac{2(1+\upsilon)}{\kappa} r^2$ $N_x = 0$

$K_{WL} = K_{WC} = K_{WR} = K$

The first four mode shapes of the fixed-free Timoshenko beam on uniform Winkler foundation and without axial load are shown in Figure 4. It is observed that there are no slight differences between the results for K=5EI and K=0 along with the maximum difference being less than 39% for the maximum magnitude of the 3rd mode shape.

K_T/EI	Mode	K_R/EI						
		0	1	10	100	1000	10000	100000
0	1	1.8925 (0)	4.8213 (0)	5.5948 (0)	5.6135 (0)	5.6157 (0)	5.6159	5.6160
	2	5.4882 (0)	5.5314 (4.3931)	8.6295 (8.3363)	9.9583 (9.6778)	10.129 (9.8499)	10.147	10.149
	3	23.229 (22.373)	26.261 (25.490)	34.733 (34.097)	39.318 (38.724)	39.989 (39.400)	40.059	40.066
1	1	3.0980 (1.4025)	5.3712 (1.4058)	5.7872 (1.4106)	5.8005 (1.4122)	5.8021 (1.4122)	5.8023	5.8024
	2	5.7125 (2.4466)	5.7424 (4.9877)	8.8899 (8.6046)	10.161 (9.8859)	10.326 (10.051)	10.343	10.344
	3	23.390 (22.552)	26.395 (25.635)	34.807 (34.176)	39.369 (38.778)	40.037 (39.451)	40.108	40.114
10	1	7.2186 (4.1304)	7.2230 (4.2190)	7.2301 (4.3608)	7.2324 (4.4047)	7.2327 (4.4100)	7.2328	7.2328
	2	7.9041 (7.6541)	8.8466 (8.6041)	10.937 (10.699)	11.821 (11.580)	11.940 (11.698)	11.952	11.953
	3	24.842 (24.141)	27.598 (26.933)	35.486 (34.892)	39.837 (39.269)	40.478 (39.912)	40.545	40.551
100	1	10.941 (8.2757)	11.582 (9.1789)	13.167 (11.298)	13.895 (12.234)	13.994 (12.361)	14.005	14.006
	2	21.905 (21.751)	21.907 (21.753)	21.913 (21.757)	21.917 (21.760)	21.917 (21.760)	21.917	21.917
	3	37.187 (36.920)	38.308 (38.017)	42.123 (41.764)	44.601 (44.268)	44.989 (44.590)	45.031	45.035
1000	1	12.324 (9.6787)	13.620 (11.251)	18.139 (16.344)	21.450 (19.887)	22.015 (20.482)	22.075	22.081
	2	36.667 (36.446)	37.852 (37.367)	42.798 (42.604)	47.229 (47.051)	48.055 (47.879)	48.145	48.154
	3	73.540 (73.369)	73.852 (73.687)	75.238 (75.096)	76.581 (76.460)	76.842 (76.724)	76.870	76.873
10000	1	12.492	13.884	18.959	22.921	23.617	23.692	23.699
	2	39.405	41.162	49.381	58.286	60.140	60.345	60.366
	3	87.417	89.085	98.289	111.35	114.55	114.92	114.96
100000	1	12.509	13.911	19.044	23.076	23.786	23.862	23.870
	2	39.687	41.508	50.106	59.538	61.507	61.727	61.747
	3	88.852	90.715	101.15	116.25	119.96	120.38	120.39

Note: Values in parentheses are reported from reference (Wang and Wang, 2013) for Euler–Bernoulli beam without foundation.

Table 5: Variations of the frequency parameter of free vibration of Euler–Bernoulli beam with general boundary conditions partially supported on Winkler foundation

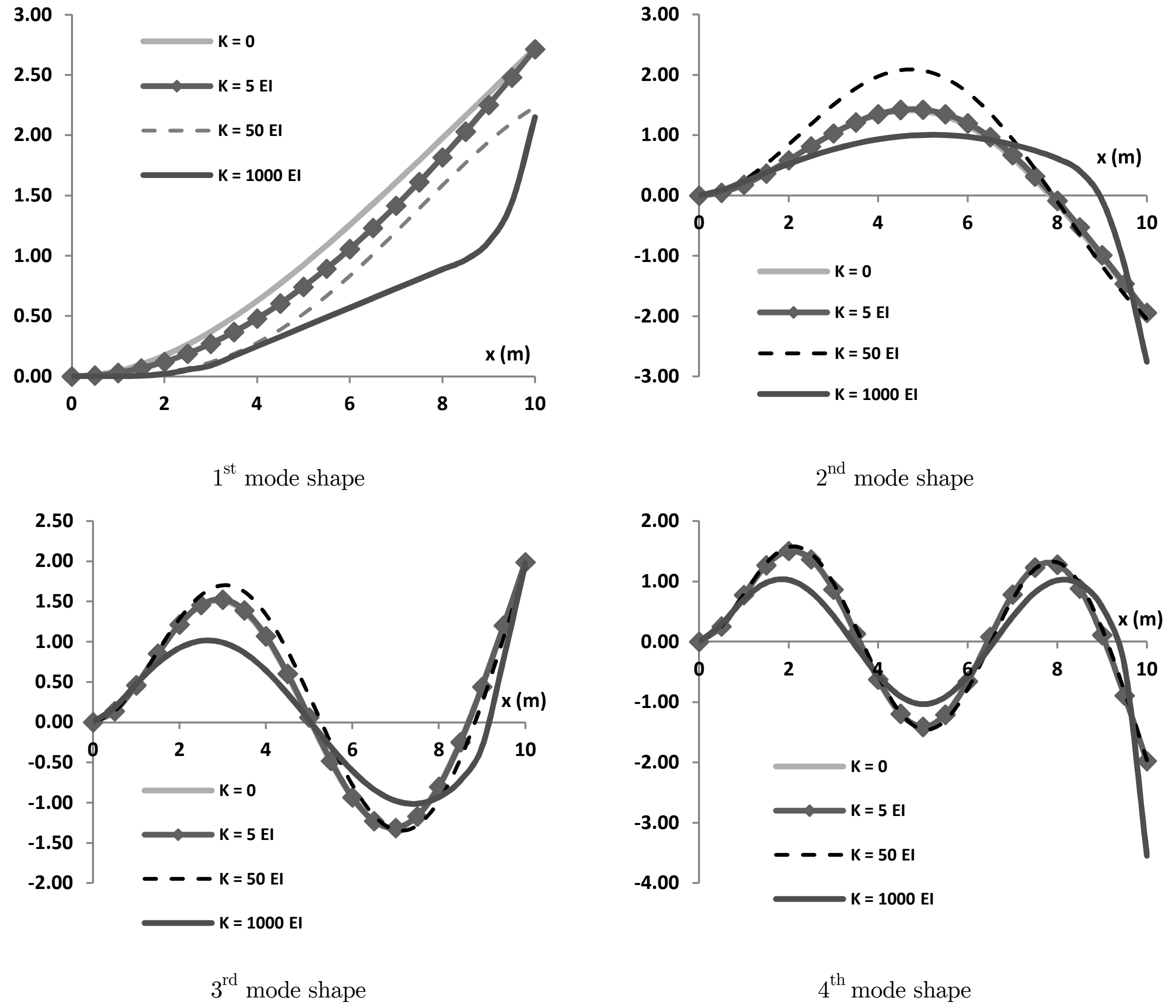


Figure 4: The mode shapes of the fixed-free Timoshenko beam on uniform Winkler foundation and without axial load.

5 CONCLUSIONS

This paper presents the free vibration of elastically restrained Timoshenko beam on a partially Winkler foundation using dynamic green function. An accurate and direct modeling technique is stated for modeling beam structures with various boundary conditions. This technique is based on the Green function. The method of Green functions is more efficient and simplistic when compared with other methods (e.g. series method) due to the Green function yielding precise solutions in closed forms. In addition, the boundary conditions are embedded in the Green functions by the Green function method. The effect of different boundary condition, the elastic coefficient of Winkler foundation, as well as, other parameters are determined. Finally, some numerical examples are shown to illustrate the efficiency and simplicity of the new formulation based on the Green function.

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Appendix

The coefficient matrix $[A_{i,j}]$ in Eq. (32) is given for general boundary conditions by:

$A_{1,1} = \frac{\lambda_{1L}\phi}{1-r^2\alpha^2\phi^2} D_{1,L}$ $\quad A_{1,2} = \frac{K_{TL}}{EI}$ $\quad A_{1,3} = \frac{\lambda_{2L}\phi}{1-r^2\alpha^2\phi^2} D_{2,L}$ $\quad A_{1,4} = \frac{K_{TL}}{EI}$

$A_{2,1} = \frac{-1}{1-r^2\alpha^2\phi^2}\frac{K_{RL}}{EI} D_{3,L}$ $\quad A_{2,2} = D_{1,L} - r^2\phi^2$ $\quad A_{2,3} = \frac{-1}{1-r^2\alpha^2\phi^2}\frac{K_{RL}}{EI} D_{4,L}$ $\quad A_{2,4} = D_{2,L} - r^2\phi^2$

$A_{3,9} = \frac{\lambda_{1R}\phi}{1-r^2\alpha^2\phi^2} D_{1,R}\, cos(\phi\lambda_{1R}L) - \frac{K_{TR}}{EI}\, sin(\phi\lambda_{1R}L)$ $\quad A_{3,10} = -\frac{\lambda_{1R}\phi}{1-r^2\alpha^2\phi^2} D_{1,R}\, sin(\phi\lambda_{1R}L) - \frac{K_{TR}}{EI}\, cos(\phi\lambda_{1R}L)$

$A_{3,11} = \frac{\lambda_{2R}\phi}{1-r^2\alpha^2\phi^2} D_{2,R}\, cosh(\phi\lambda_{2R}L) - \frac{K_{TR}}{EI}\, sinh(\phi\lambda_{2R}L)$ $\quad A_{3,12} = \frac{\lambda_{2R}\phi}{1-r^2\alpha^2\phi^2} D_{2,R}\, sinh(\phi\lambda_{2R}L) - \frac{K_{TR}}{EI}\, cosh(\phi\lambda_{2R}L)$

$$A_{4,9} = \left(D_{1,R} - r^2\phi^2\right) sin(\phi\lambda_1 L) - \frac{K_{RR}}{EI}\frac{D_{3,R}}{1-r^2\alpha^2\phi^2} cos(\phi\lambda_{1R}L)$$

$$A_{4,10} = \left(D_{1,R} - r^2\phi^2\right) cos(\phi\lambda_{1R}L) - \frac{K_{RR}}{EI}\frac{D_{3,R}}{1-r^2\alpha^2\phi^2} sin(\phi\lambda_{1R}L)$$

$$A_{4,11} = \left(D_{2,R} - r^2\phi^2\right) sinh(\phi\lambda_{2R}L) + \frac{K_{RR}}{EI}\frac{D_{4,R}}{1-r^2\alpha^2\phi^2} cosh(\phi\lambda_{2R}L)$$

$$A_{4,12} = \left(D_{2,R} - r^2\phi^2\right) cosh(\phi\lambda_{2R}L) + \frac{K_{RR}}{EI}\frac{D_{4,R}}{r^2\alpha^2\phi^2-1} sinh(\phi\lambda_{2R}L)$$

$A_{5,1} = sin(\phi\lambda_{1L}\beta_L L)$ $\quad A_{5,2} = cos(\phi\lambda_{1L}\beta_L L)$ $\quad A_{5,3} = sinh(\phi\lambda_{2L}\beta_L L)$ $\quad A_{5,4} = cosh(\phi\lambda_{2L}\beta_L L)$

$A_{5,5} = -sin(\phi\lambda_{1C}\beta_L L)$ $\quad A_{5,6} = -cos(\phi\lambda_{1C}\beta_L L)$ $\quad A_{5,7} = -sinh(\phi\lambda_{2C}\beta_L L)$ $\quad A_{5,8} = -cosh(\phi\lambda_{2C}\beta_L L)$

$A_{6,1} = D_{3,L}\, A_{5,2}$ $A_{6,2} = D_{3,L}\, A_{5,5}$ $A_{6,3} = D_{4,L}\, A_{5,4}$ $A_{6,4} = D_{4,L}\, A_{5,3}$

$A_{6,5} = D_{3,C}\, A_{5,6}$ $A_{6,6} = D_{3,C}\, A_{5,1}$ $A_{6,7} = D_{4,C}\, A_{5,8}$ $A_{6,8} = D_{4,C}\, A_{5,7}$

$A_{7,1} = A_{2,2}\, A_{5,1}$ $A_{7,2} = A_{2,2}\, A_{5,2}$ $A_{7,3} = A_{2,4}\, A_{5,3}$ $A_{7,4} = A_{2,4} A_{5,4}$

$A_{7,5} = \left(D_{1,C} - r^2\phi^2\right) A_{5,5}$ $A_{7,6} = \left(D_{1,C} - r^2\phi^2\right) A_{5,6}$ $A_{7,7} = \left(D_{2,C} - r^2\phi^2\right) A_{5,7}$ $A_{7,8} = \left(D_{2,C} - r^2\phi^2\right) A_{5,8}$

$A_{8,1} = \lambda_{1L}\phi D_{1,L} A_{5,2}$ $A_{8,2} = \lambda_{1L}\phi D_{1,L} A_{5,5}$ $A_{8,3} = \lambda_{2L}\phi D_{2,L} A_{5,4}$ $A_{8,4} = \lambda_{2L}\phi D_{2,L} A_{5,3}$

$A_{8,5} = \lambda_{1C}\phi D_{1,C} A_{5,6}$ $A_{8,6} = \lambda_{1C}\phi D_{1,C} A_{5,1}$ $A_{8,7} = \lambda_{2C}\phi D_{2,C} A_{5,8}$ $A_{8,8} = \lambda_{2C}\phi D_{2,C} A_{5,7}$

$A_{9,5} = sin\left((\beta_L + \beta_C)\phi\lambda_{1C} L\right)$ $A_{9,6} = cos\left((\beta_L + \beta_C)\phi\lambda_{1C} L\right)$ $A_{9,7} = sinh\left((\beta_L + \beta_C)\phi\lambda_{2C} L\right)$

$A_{9,8} = cosh\left((\beta_L + \beta_C)\phi\lambda_{2C} L\right)$ $A_{9,9} = -sin\left((\beta_L + \beta_C)\phi\lambda_{1R} L\right)$ $A_{9,10} = -\,cos\left((\beta_L + \beta_C)\phi\lambda_{1R} L\right)$

$A_{9,11} = -\,sinh\left((\beta_L + \beta_C)\phi\lambda_{2R} L\right)$ $A_{9,12} = -cosh\left((\beta_L + \beta_C)\phi\lambda_{2R} L\right)$

$A_{10,5} = D_{3,C} A_{9,6}$ $A_{10,6} = D_{3,C}\, A_{9,9}$ $A_{10,7} = D_{4,C} A_{9,8}$ $A_{10,8} = D_{4,C} A_{9,7}$

$A_{10,9} = D_{3,R} A_{9,10}$ $A_{10,10} = D_{3,R}\, A_{9,5}$ $A_{10,11} = D_{4,R} A_{9,12}$ $A_{10,12} = D_{4,R} A_{9,11}$

$A_{11,5} = \left(D_{1,C} - r^2\phi^2\right) A_{9,5}$ $A_{11,6} = \left(D_{1,C} - r^2\phi^2\right) A_{9,6}$ $A_{11,7} = \left(D_{2,C} - r^2\phi^2\right) A_{9,7}$ $A_{11,8} = \left(D_{2,C} - r^2\phi^2\right) A_{9,8}$

$A_{11,9} = \left(D_{1,R} - r^2\phi^2\right) A_{9,9}$ $A_{11,10} = \left(D_{1,R} - r^2\phi^2\right) A_{9,10}$ $A_{11,11} = \left(D_{2,R} - r^2\phi^2\right) A_{9,11}$ $A_{11,12} = \left(D_{2,R} - r^2\phi^2\right) A_{9,12}$

$A_{12,5} = \lambda_{1C}\phi D_{1,C} A_{9,6}$ $A_{12,6} = \lambda_{1C}\phi D_{1,C} A_{9,9}$ $A_{12,7} = \lambda_{2C}\phi D_{2,C} A_{9,8}$ $A_{12,8} = \lambda_{2C}\phi D_{2,C} A_{9,7}$

$A_{12,9} = \lambda_{1R}\phi D_{1,R} A_{9,10}$ $A_{12,10} = \lambda_{1R}\phi D_{1,R} A_{9,5}$ $A_{12,11} = \lambda_{2R}\phi D_{2,R} A_{9,12}$ $A_{12,12} = \lambda_{2R}\phi D_{2,R} A_{9,11}$

where:

$D_{1,i} = \left(\left(r^2 + \alpha^2 - (1 + \alpha^2\gamma){\lambda_{1i}}^2\right)\phi^2 - \alpha^2\eta_i\right)$ $D_{2,i} = \left(\left(r^2 + \alpha^2 + (1 + \alpha^2\gamma){\lambda_{2i}}^2\right)\phi^2 - \alpha^2\eta_i\right)$

$D_{3,i} = \phi\lambda_{1i}(\alpha^2\phi^2\left(\alpha^2 - {\lambda_{1i}}^2\right) - \alpha^4\left(\eta_i + \gamma{\lambda_{1i}}^2\phi^2\right) + 1)$ $D_{4,i} = \phi\lambda_{2i}(1 + \alpha^2\phi^2(\alpha^2 + {\lambda_{2i}}^2) - \alpha^4(\eta_i - \gamma{\lambda_{2i}}^2\phi^2))$