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A Novel Methodology Using Simplified Approaches for Identification of Cracks in Beams

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Abstract

In this paper, natural frequency based forward and inverse methods are proposed for identifying multiple cracks in beams. Forward methods include simplified definition of the natural frequency drops caused by the cracks. The ratios between natural frequencies obtained from multi-cracked and un-cracked beams are determined by an approach that uses the local flexibility model of cracks. This approach does not consider nonlinear crack effects that can be easily neglected when the number of cracks is not excessive. In addition, an expression, which removes the necessity of repeating natural frequency analyses, is given for identifying the connection between the crack depths and natural frequency drops. These simplified approaches play crucial role in solving inverse problem using constituted crack detection methodology. Solution needs a number of measured modal frequency knowledge two times more than the number of cracks to be detected. Efficiencies of the methods are verified using the natural frequency ratios obtained by the finite element package. The crack detection methodology is also validated using some experimental natural frequency ratios given in current literature. Results show that the locations and depths ratios of cracks are successfully predicted by using the methods presented.

Keywords

Cracked beam vibration, local flexibility model, simplified approaches, natural frequency ratios, crack detection methodology.

1 INTRODUCTION

Inspection and detection of the damages are crucial for the systems having beam type mechanical or structural components. A damage known as crack may be invisible in many applications of visual inspection methods which do not guarantee the detection in early stage. At result, advances of crack in a short time may cause catastrophic failures. However, existence of any damages leads to the alteration of the dynamic behaviour of the systems. Therefore, vibration based crack identification methods take great attention throughout the researchers studying non-destructive testing and evaluation methods.

Changes in vibration characteristics can be observed by using modal parameters such as natural frequency and mode shape. In contrast to the mode shapes, natural frequencies can be measured easily, and they are not seriously affected by the experimental errors. In connection with this, crack detection systems, which use frequencies, can be inexpensive, non-invasive and automated for avoiding subjective operator differences. Therefore, frequency based inverse methods supported by the theoretical vibration models have been frequently proposed up to date. Dimarogonas (1996) and Doebling et al. (1998) present overviews to the methods examining the theoretical changes in dynamic behaviours and their agreement with measured vibration responses. Frequency based crack identification methods are summarised by Salawu (1997). Detailed review on vibration based identification of multiple cracks is given by Sekhar (2008). Recently, Jassim et al. (2013) review the studies on vibration analysis of damaged cantilever beams.

The identification of a single transverse crack in a beam is popularly studied using the lowest three natural frequencies which can be easily obtained (Chen et al. 2005; Chinchalkar 2001; Kim and Stubbs 2003; Liang et al. 1991; Nandwana and Maiti 1997; Owolabi et al. 2003). However, simultaneous detection of crack parameters is much more involved and complex than the identification of single crack. In many cases, supports of theoretical vibration analyses are inevitable for the detection. In addition, multiple crack detection can require the knowledge of additional parameters and thus many measurements in several test conditions. Number of unknown parameters is generally decreased by removing some of them which can have negligibly small effects. Theories of most studies are based upon the analytical method including local flexibility model which uses the conditions of compatibility and continuity at crack locations (Bakhtiari-Nejad et al. 2014; Caddemi and Calio 2009; Dado 1997; Khiem and Lien 2001; Nandwana and Maiti 1997; Ostachowicz and Krawczuk 1991; Shifrin and Ruotolo 1999). Local flexibility model neglects the distributed energy effects of crack and accumulates the effects of dynamical changes into rotational springs considered at the crack locations. In classical solution method that use local flexibility model, the equation set satisfying the boundary conditions at two ends of the beam is expanded with four new equations of continuity and compatibility conditions for each crack. With this method, to construct linear system is not a simple task for the beams with n cracks. Shifrin and Ruotolo (1999) extend this base approach by defining $n+2$ equations for simplifying the vibration analysis of the beams with n cracks. However, their method can have the lack of exactness due to the some simplifications and linearization done for decreasing the matrix size. Some other methods, including exponentially decaying crack disturbance functions, have been proposed to develop vibration equations for continuous models (Christides and Barr 1984; Mazanoglu et al. 2009; Yang et al. 2001). When these methods are used in the inverse analyses, interaction effects of the cracks can be required to be neglected for being capable of detecting cracks. Accuracy of the vibration models is significantly important for the success of inverse methods presented for crack detection.

In the last two decades, several papers have been published to solve inverse problem of detecting multiple cracks with the knowledge of natural frequency changes. However, the number of these studies is still less than the analyses addressing the forward problem. Mazanoglu and

Sabuncu (2012) present an algorithm that uses natural frequencies for detection of cracks in beams and a process which minimises the errors in experimental results. Gillich and Praisach (2014) study on a crack detection method based upon natural frequency changes and a procedure for determining accurate natural frequencies in measurement. Khiem and Toan (2014) present Rayleigh quotient based explicit expression providing a tool for calculating natural frequencies of the beam with arbitrary number of cracks. A method that combines the frequency measurements and vibration modelling using transfer matrix method is presented by Patil and Maiti (2003). They observe notable amounts of errors like 10% in the prediction of locations and sizes of cracks. Khiem and Lien (2004) use the natural frequencies obtained by the dynamic stiffness matrix method and formulate the multiple crack detection as a non-linear optimization problem. Morassi and Rollo (2001) present a technique, which uses the changes in the first three natural frequencies, for a simply supported beam with two cracks having equal severity. Although the method is verified numerically, notable errors are seen in their predictions. Douka *et al.* (2004) use the antiresonance changes, complementary with natural frequency changes, in a prediction scheme for identification of two cracks in beams. In their work, crack detection is based upon the availability of many experimental results. Lee (2009) presents a simple method for detecting n cracks using $2n$ natural frequencies by means of the finite element and the Newton–Raphson methods. Their method reveals quite satisfactory results provided proper initial guesses are made for convergence of the solution. Ruotolo and Surace (1997) propose a solution procedure employing a genetic algorithm and the results of the finite element model for the detection of multiple cracks in beams. Krawczuk (2002) uses the wave propagation approach combined with an iterative searching strategy including two methods for damage detection in beam-like structures.

This paper presents simplified methods for determining natural frequency drops due to the multiple cracks in beams and a methodology for identifying location and depth ratio parameters of the cracks. First of all, theoretical background about the vibration of cracked beam is outlined. Theoretical simplifications are demonstrated to find natural frequencies of cracked beams. After that, methodology of crack identification, which employs the natural frequencies obtained by the forward analyses, is detailed. Then, several cracked beam scenarios are considered for discussing the results that are verified using the natural frequency ratios obtained by both the commercial finite element package (ANSYS©) and some experiments given by Kim and Stubbs (2003), Mazanoglu and Sabuncu (2012), Ruotolo and Surace (1997). Finally, achievements and shortcomings of the methods are clearly stated.

2 THEORETICAL BACKGROUND

Free bending vibration of a uniform Euler-Bernoulli beam is identified by following well known differential equation.

$$EI\frac{\partial^4 w(z,t)}{\partial z^4}+\rho A\frac{\partial^2 w(z,t)}{\partial t^2}=0\,, \tag{1}$$

where, E, I, A, and ρ represent modulus of elasticity, area moment of inertia, cross-section area, and density respectively. Flexural displacement is symbolised by w, and variables z, t are the position along the beam length and the time respectively. When this equation is solved by separating the variables of z and t, mode shape of the beam, $W(z)$, is observed in the following solution form (Rao 1995).

$$W(z) = C_1 \cos\beta z + C_2 \sin\beta z + C_3 \cosh\beta z + C_4 \sinh\beta z \,, \tag{2}$$

where C_1, C_2, C_3, and C_4 are the coefficients of harmonic and hyperbolic terms. Frequency parameter, β, which depends upon the natural frequency, ω, is written as:

$$\beta = \sqrt[4]{\frac{\rho A \omega^2}{EI}} \,. \tag{3}$$

An equation set is formed by using mode shape function and its derivatives satisfying the end conditions of the beam. The equations appropriate for some general end conditions are given in Table 1.

Ends	Equations	
Free	$W''=0$,	$W'''=0$
Fixed	$W=0$,	$W'=0$
Pinned	$W=0$,	$W''=0$

Table 1: Equations for classical end conditions..

In example, for a cantilever beam, the matrix obtained by the harmonic and hyperbolic terms of functions is formed as follows:

$$M_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -\cos\beta L & -\sin\beta L & \cosh\beta L & \sinh\beta L \\ \sin\beta L & -\cos\beta L & \sinh\beta L & \cosh\beta L \end{bmatrix} \tag{4}$$

where L is the length of beam. Zero determinant of the matrix, M_0, gives singular values as uncracked beam's natural frequencies, ω_0.

2.1 Vibration of the Beam with Cracks

When the cracks exist on a beam, local flexibility changes should be identified at the crack locations. Local flexibility changes are simulated by the rotational springs, which are joints of the sections separated by the cracks, as shown in Figure 1. Cracks are modelled as slots whose depths are assumed to be unchanged along the width of the beam. Crack widths are considered to be negligibly small and thus the cracked beam's vibration identification problem is handled by the assumption of no mass loss from the beams. In classical approaches, the existence of n cracks requires the expression of n local flexibility changes for connecting $n+1$ sections. Vibration form of each section can be expressed by appropriate harmonic and hyperbolic terms of the function written in Eq. (2). Connection at the crack location is provided by the continuity conditions with negligible effects of crack width. Deflection, bending moment and shear force are assumed to be equal at right hand and left hand sides of the crack as follow:

$$W_i(z) = W_{i+1}(z), \quad i = 1,....,n \tag{5a}$$

$$W_i''(z) = W_{i+1}''(z), \tag{5b}$$

$$W_i'''(z) = W_{i+1}'''(z), \tag{5c}$$

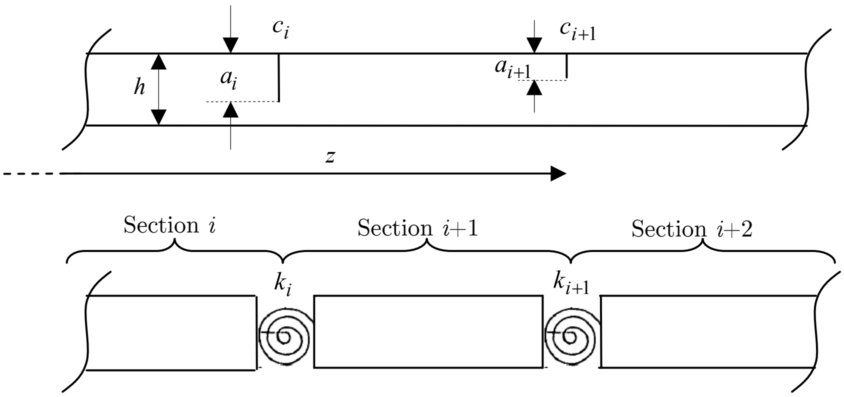

Figure 1: Cracked beam model.

In addition, compatibility condition relates bending moment with the difference of slopes between both sides of the crack as represented in following equation.

$$W_i''(z) = \alpha_i \left[W_{i+1}'(z) - W_i'(z)\right], \quad i = 1,....,n . \tag{6}$$

where

$$\alpha_i = \frac{k_i}{EI} . \tag{7}$$

Here, k_i represents the local rotational stiffness caused by i^{th} crack, and it is described by the theoretical and experimental expressions of the fracture mechanics (Ostachowicz and Krawczuk 1991; Tada et al. 1973) as follows:

$$k_i = \frac{Ebh^2}{72\pi \left(a_i/h\right)^2 f(a_i)}, \tag{8}$$

where, b, h and a symbolise width and height of the beam and depth of the crack respectively. $f(a_i)$ is known as flexibility compliance function of i^{th} crack that is formulated for the rectangular beam as follows:

$$f(a_i) = 0.6384\left(\frac{a_i}{h}\right)^2 - 1.035\left(\frac{a_i}{h}\right)^3 + 3.7201\left(\frac{a_i}{h}\right)^4 - 5.1773\left(\frac{a_i}{h}\right)^5 + 7.553\left(\frac{a_i}{h}\right)^6 - 7.3324\left(\frac{a_i}{h}\right)^7 + 2.4909\left(\frac{a_i}{h}\right)^8 \quad (9)$$

In conventional procedure, the equation set having size, $4n+4$, is formed by $4n$ equations of continuity and compatibility conditions and 4 equations of the end conditions (Nandwana and Maiti 1997; Ostachowicz and Krawczuk 1991). Large sized matrix shaped by harmonic and hyperbolic terms of the equation set must be singular for determining natural frequencies. Instead, a simplified approach is presented in linear manner to find natural frequencies of the beams with cracks.

2.2 An Approach to Obtain Multi-cracked Beam's Natural Frequencies

Suppose that a cantilever beam considered as an example has a crack at the location z_1. 8×8 matrix, M_1, is formed by the terms of functions providing continuity and compatibility conditions given in Eqs. (5, 6) and boundary conditions tabulated in Table 1. Here, subscript "$_1$" is used for defining the parameters in cases of beams with one crack. Reduced form of the matrix is constructed as follows:

$$M_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \cos\beta z_1 & \sin\beta z_1 & \cosh\beta z_1 & \sinh\beta z_1 & -\cos\beta z_1 & -\sin\beta z_1 & -\cosh\beta z_1 & -\sinh\beta z_1 \\ \begin{matrix}\alpha\sin\beta z_1\\+\beta\cos\beta z_1\end{matrix} & \begin{matrix}-\alpha\cos\beta z_1\\+\beta\sin\beta z_1\end{matrix} & \begin{matrix}-\alpha\sinh\beta z_1\\-\beta\cosh\beta z_1\end{matrix} & \begin{matrix}-\alpha\cosh\beta z_1\\-\beta\sinh\beta z_1\end{matrix} & -\alpha\sin\beta z_1 & \alpha\cos\beta z_1 & \alpha\sinh\beta z_1 & \alpha\cosh\beta z_1 \\ -\cos\beta z_1 & -\sin\beta z_1 & \cosh\beta z_1 & \sinh\beta z_1 & \cos\beta z_1 & \sin\beta z_1 & -\cosh\beta z_1 & -\sinh\beta z_1 \\ \sin\beta z_1 & -\cos\beta z_1 & \sinh\beta z_1 & \cosh\beta z_1 & -\sin\beta z_1 & \cos\beta z_1 & -\sinh\beta z_1 & -\cosh\beta z_1 \\ 0 & 0 & 0 & 0 & -\cos\beta L & -\sin\beta L & \cosh\beta L & \sinh\beta L \\ 0 & 0 & 0 & 0 & \sin\beta L & -\cos\beta L & \sinh\beta L & \cosh\beta L \end{bmatrix} \quad (10)$$

Since the functions are continuous, infinite number of β can be available for producing zero determinant of M_1. As is known, each β is correspond of a different natural frequency whose number is now represented by the superscript $^{(m)}$. For each mode, single cracked beam's natural frequency, $\omega_{c(1)}^{(m)}$ is smaller than un-cracked beam frequency, $\omega_0^{(m)}$ obtained from the matrix M_0.

Change in frequency can be examined by natural frequency ratio, $r_1^{(m)} = \omega_{c(1)}^{(m)} \big/ \omega_0^{(m)}$, or natural frequency drop ratio, $d_1^{(m)} = 1 - r_1^{(m)}$. When the second crack initiates from the location, z_2, m^{th} mode natural frequency starts to decrease from $\omega_{c(1)}^{(m)}$ to $\omega_{c(1,2)}^{(m)}$. This proves that individual effect of the second crack should be equal to $r_2^{(m)} = \omega_{c(1,2)}^{(m)} \big/ \omega_{c(1)}^{(m)}$. Thus, the natural frequency ratio in case of the existence of two cracks can be computed for m^{th} vibration mode as follows (Mazanoglu and Sabuncu 2012):

$$\left\{ r_{1,2} = \frac{\omega_{c(1,2)}}{\omega_0} \cong r_1\, r_2 \right\}^{(m)} \tag{11}$$

Minor errors arising from the linear system assumption are indicated by the approximate equivalence in the equation. If the crack at the location, z_2, initiates first, individual effect of this crack should remain same. It is for this reason that the following expression is accurate.

$$\left\{ r_2 = \frac{\omega_{c(1,2)}}{\omega_{c(1)}} = \frac{\omega_{c(2)}}{\omega_0} \right\}^{(m)} \tag{12}$$

In general aspect, Eqs. (11, 12) can be arranged for the cases of n cracked beams as follow:

$$\left\{ r_{1,2,\ldots,n} = \frac{\omega_{c(1,2,\ldots,n)}}{\omega_0} \cong r_1\, r_2 \ldots r_n \right\}^{(m)} \tag{13}$$

$$\left\{ r_n = \frac{\omega_{c(1,2,\ldots,n)}}{\omega_{c(1,2,\ldots,n-1)}} = \frac{\omega_{c(n)}}{\omega_0} \right\}^{(m)} \tag{14}$$

where $\omega_{c(1,2,\ldots,n)}$ and $\omega_{c(n)}$ symbolise the natural frequencies for the beam with n cracks and solely n^{th} crack respectively. As a result of the substitution of Eq. (14) into Eq. (13), following expression is obtained for the natural frequency ratio of a beam with n cracks.

$$\left\{ r_{1,2,\ldots,n} = \frac{\omega_{c(1)}}{\omega_0} \cdot \frac{\omega_{c(2)}}{\omega_0} \cdot \ldots \cdot \frac{\omega_{c(n)}}{\omega_0} \right\}^{(m)} \tag{15}$$

Each natural frequency ratio multiplied in Eq. (15) is obtained by making singularity analysis of 8×8 matrix shaped for every single cracked beam case. It is assumed here that each crack has a specific capacity of energy storing varying with the location and depth ratio. Since the energies consumed due to the cracks are locally stored on the springs in local flexibility model, each crack has independent influences on beams vibration. In the present work, this assumption is verified up to three cracks that need natural frequency ratios for six vibration modes. Existence of more cracks ($n > 3$) requiring more natural frequency ratios ($2n$) is not taken into consideration since the errors of the frequency ratio multiplication increase as the order of natural frequency increases (Gillich et al. 2012). This is because the following crack detection methodology is effective for detecting three cracks at most.

3 A METHODOLOGY FOR DETECTING CRACKS

Novel crack detection procedure is now introduced step by step.

Step 1: Experimentally found modal frequency drop ratios, $d_{inp}^{(m)}$, are employed as input data of proposed crack detection method. Instead of direct use of the natural frequency ratios, $r_{inp}^{(m)}$, natural frequency drop ratios are employed, since the methodology works better with this parameter.

Step 2: Theoretical natural frequency drop ratios $d_1^{(m)}$ should be calculated to have a prior knowledge about the frequency of the beam with a crack. Conventionally, this may require repeating singularity analysis for all crack depths to prepare prediction scheme of corresponding natural frequency drop ratios. This is quite inconvenient and time-consuming procedure. To avoid this, a simplified formula is proposed here to calculate the changes in natural frequency drop ratios reasoned by various crack depths at one position. For small cracks, fractional changes in modal strain energy are equal to the natural frequency drop ratios (Gillich and Praisach 2014; Gudmunson 1982; Kim and Stubbs 2003). By neglecting minor effects of crack based changes in second derivative of mode shape, approximate strain energy changes can directly be expressed by the flexibility increases which are the function of $f(a)(a/h)^2$ as given in Eq. (8). Once the drop ratio, $d_{1ref}^{(m)}$, is calculated for one reference crack depth, a_{ref}, at the position of z_{ref}, the ratios, $d_1^{(m)}$, arising from the crack with different depth, a, can be calculated by the following expression.

$$d_1^{(m)} = d_{1ref}^{(m)} \frac{f(a)a^2}{f(a_{ref})\,a_{ref}^2} \tag{16}$$

Step 3: For the reference crack depth, theoretical natural frequency drop ratios are determined by shifting the crack location along the z coordinate axis of beam. By fitting a curve on these natural frequency drop ratios, reference polynomial functions, $p_{ref}^{(m)}(a_{ref}, z)$, are constituted in following form:

$$p_{ref}^{(m)}(a_{ref}, z) = \kappa_1^{(m)} z^k + \kappa_2^{(m)} z^{k-1} + ... + \kappa_k^{(m)} z + \kappa_{k+1}^{(m)}, \qquad m = 1,2,....,6 \tag{17}$$

Eq. (17) is the set of reference polynomials constructed using natural frequency drop ratios for six modes of vibration ($m = 1,2,....,6$) that are necessary to inspect three cracks at most. κ_k symbolises the coefficients of polynomial terms indexed by k. These coefficients can vary with the depth of crack and the geometric properties of beam. Values of coefficients and number of polynomial terms also depend upon the mode of vibration considered. Higher vibration modes require larger number of terms in the function.

Step 4: Reference polynomial functions should be generalised for determining natural frequency drop ratios caused by the crack with any depth and location. For this purpose, Eq. (16) is rearranged in the form of polynomial function as follows:

$$p^{(m)}(a,z) = p_{ref}^{(m)}(a_{ref},z)\frac{f(a)a^2}{f(a_{ref})a_{ref}^2}, \qquad m = 1,2,....,6 \tag{18}$$

where $p^{(m)}(a,z)$ symbolises the function set of natural frequency drop ratios varying with the location and depth of a crack. This set of polynomials satisfy the natural frequency drop ratios ($d_i^{(m)} = p^{(m)}(a_i, z_i)$ for $i = 1,2,3$) when the unknown crack parameters, locations (z_i) and depths (a_i), are accurately substituted.

Step 5: It is definite that measured modal frequency ratios employed as input of the method should be equal to the theoretical natural frequency ratios. Accordingly, following equation set should be satisfied.

$$r_{inp}^{(m)} - r_{1,2,3}^{(m)} = 0, \quad m = 1,2,....,6 \tag{19}$$

Eq. (19) is written in the form of the set of polynomial functions using the relations of $r_{1,2,3}^{(m)} = r_1^{(m)} r_2^{(m)} r_3^{(m)}$, $r_i^{(m)} = 1 - d_i^{(m)}$, and $d_i^{(m)} = p^{(m)}(a_i, z_i)$ as follows:

$$r_{inp}^{(m)} - \left(1 - p^{(m)}(a_1,z_1)\right)\cdot\left(1 - p^{(m)}(a_2,z_2)\right)\cdot\left(1 - p^{(m)}(a_3,z_3)\right) = 0, \quad m = 1,2,....,6 \tag{20}$$

Eq. (20) presenting the nonlinear equation set for six vibration modes allows to find unknown crack parameters, z_1, z_2, z_3 and a_1, a_2, a_3.

Step 6: In the final step, unknown crack parameters are initially predicted as $a_{1(0)}, a_{2(0)}, a_{3(0)}$ and $z_{1(0)}, z_{2(0)}, z_{3(0)}$, and then an optimisation problem of finding accurate crack parameters is solved by minimising following residual function set $R^{(m)}$.

$$R^{(m)} = r_{inp}^{(m)} - \left(1 - p^{(m)}(a_1,z_1)\right)\cdot\left(1 - p^{(m)}(a_2,z_2)\right)\cdot\left(1 - p^{(m)}(a_3,z_3)\right), \quad m = 1,2,....,6 \tag{21}$$

In this work, trust-region dogleg algorithm introduced in the package (Matlab©) is carried out as a solution method of the equation set (Nocedal and Wright 1999).

4 RESULTS AND DISCUSSION

4.1 Properties of the Beam Tested Numerically

Uniform aluminium alloy cantilever beam is simulated for checking the efficiency of the methods presented. Dimensions of the beam are given as 10×10 mm^2 cross-section and $L = 0.36$ m length. The modulus of elasticity, density, and Poisson ratio of the beam are $E = 69$ GPa, $\rho = 2678$ kg/m^3 and $\nu = 0.3$ respectively. Geometric properties of a cantilever beam model with two cracks are shown in Figure 2. As a result of the theoretical computations, un-cracked beam natural frequencies are found as $\omega_0^{(1)} = 397.5341$ rad/s, $\omega_0^{(2)} = 2491.304$ rad/s, $\omega_0^{(3)} = 6975.722$ rad/s,

$\omega_0^{(4)} = 13669.63\,\mathrm{rad/s}$ $\omega_0^{(5)} = 22596.88\,\mathrm{rad/s}$ $\omega_0^{(6)} = 33755.83\,\mathrm{rad/s}$. Several examples of cracked beam cases are considered for the theoretical verifications given in following subsection.

The beam is also modelled by the finite element package (ANSYS©). In the program, cracks with the thicknesses, 0.4 mm, are modelled as slots formed by subtracting thin transverse blocks from the beam. Solid element called ‘Solid95’, which includes 20 dynamic nodes, is used as meshing element. Default edge length of the element is set to 5 mm using the “esize” command. Good convergence to exact results requires frequent meshing in the vicinity of any discontinuities. This necessity is provided by using the “smrtsize1” command that is the most refined mesh option of the free meshing procedure. Figure 3 shows meshing view around a slot considered as a crack. At result, natural frequencies are obtained by using “modal analysis” as the analysis type. It should be noted that variations of crack location and crack size, which lead to change of total number of elements, have negligible effects on sensitivity of the computations.

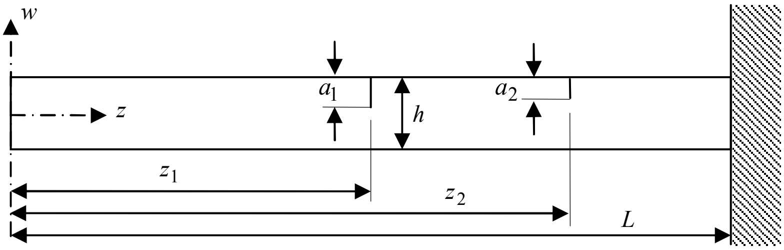

Figure 2: Geometric properties of the cantilever beam with cracks.

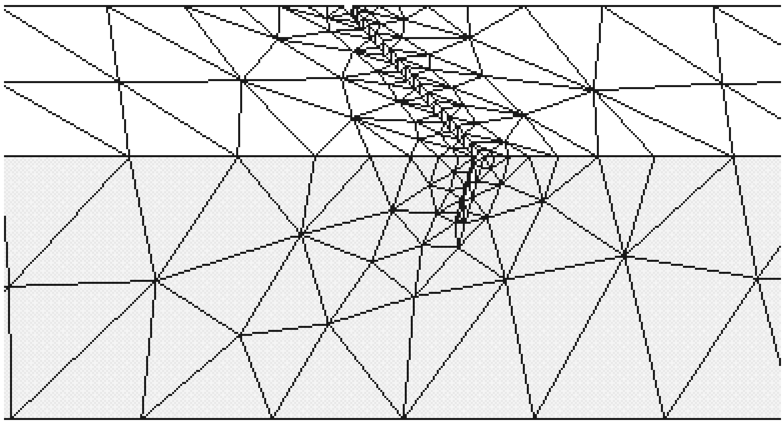

Figure 3: Meshing view of a cracked region of the beam modelled in the finite element package.

4.2 Verification of Forward Methods

The approaches formulated by Eq. (15) and Eq. (16) are verified by considering the example cases given in Table 2. It is seen that the crack modelled in Case 1 is fixed in Case 4 as one of the two cracks. Triple cracked beam scenario including these cracks is also considered in Case 10. Similar relationships are established throughout Cases 2,5,11 and Cases 3,6,12. Additionally, extraordinary cases like the existence of small or large distance between two cracks with incipient or advanced severity are handled in Cases 7,8,9. Eq. (15) is validated throughout Cases 4-12.

Table 3 demonstrates the agreement between the natural frequency ratios computed by large sized matrix solution and the finite element package. In addition, for the beam cases with two and three cracks, comparison of the results obtained by large sized matrix solution and frequency ratio multiplication method are given in Table 4. Fractal numbers equal or very close to unity show the success of frequency ratio multiplication method. The largest deviations from unity are obtained in Case 7 due to the rise of nonlinearity with two cracks having advanced depth ratio as 0.4. The maximum deviation observed as 1.048 is still admissible for approximate detection of the cracks. It should be noted as an advantage of frequency ratio multiplication that the determination of singular values for the 8×8 matrix is easier than the solution of larger sized matrix constructed for multi-cracked beams.

Cases	Normalised locations	Depth ratios	Cases	Normalised locations	Depth ratios
1	0.20	0.20	7	0.30 0.70	0.40 0.40
2	0.45	0.35	8	0.1 0.90	0.40 0.10
3	0.80	0.15	9	0.45 0.55	0.20 0.40
4	0.20 0.40	0.20 0.20	10	0.20 0.40 0.50	0.20 0.20 0.20
5	0.25 0.45	0.15 0.35	11	0.25 0.45 0.75	0.15 0.35 0.30
6	0.55 0.80	0.25 0.15	12	0.55 0.65 0.80	0.25 0.25 0.15

Table 2: Considered cracked beam scenarios..

Cases	Method	Natural frequency ratios ($r_{1,2,3}^{(m)}$)					
		Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
1	Analytical (Numerical)	0.9999 (0.9999)	0.9979 (0.9978)	0.9916 (0.9918)	0.9874 (0.9878)	0.9907 (0.9907)	0.9976 (0.9975)
2	Analytical (Numerical)	0.9940 (0.9939)	0.9635 (0.9640)	0.9940 (0.9940)	0.9756 (0.9762)	0.9863 (0.9845)	0.9863 (0.9875)
3	Analytical (Numerical)	0.9932 (0.9931)	0.9999 (0.9999)	0.9980 (0.9981)	0.9947 (0.9950)	0.9955 (0.9947)	0.9988 (0.9988)
4	Analytical (Numerical)	0.9987 (0.9988)	0.9874 (0.9876)	0.9859 (0.9862)	0.9848 (0.9853)	0.9802 (0.9809)	0.9974 (0.9972)
5	Analytical (Numerical)	0.9939 (0.9939)	0.9615 (0.9620)	0.9878 (0.9881)	0.9691 (0.9700)	0.9853 (0.9832)	0.9854 (0.9866)

Cases (cont.)	Method (cont.)	Natural frequency ratios ($r_{1,2,3}^{(m)}$)(cont.)					
		Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
6	Analytical (Numerical)	0.9876 (0.9875)	0.9841 (0.9845)	0.9959 (0.9960)	0.9820 (0.9827)	0.9886 (0.9887)	0.9916 (0.9921)
7	Analytical (Numerical)	0.9640 (0.9638)	0.9642 (0.9643)	0.9115 (0.9126)	0.9688 (0.9686)	0.9918 (0.9932)	0.9295 (0.9387)
8	Analytical (Numerical)	0.9956 (0.9961)	0.9975 (0.9977)	0.9942 (0.9942)	0.9842 (0.9841)	0.9691 (0.9689)	0.9560 (0.9573)
9	Analytical (Numerical)	0.9821 (0.9819)	0.9479 (0.9485)	0.9920 (0.9920)	0.9617 (0.9619)	0.9759 (0.9737)	0.9793 (0.9808)
10	Analytical (Numerical)	0.9961 (0.9963)	0.9765 (0.9770)	0.9859 (0.9862)	0.9743 (0.9754)	0.9801 (0.9808)	0.9862 (0.9871)
11	Analytical (Numerical)	0.9719 (0.9720)	0.9610 (0.9617)	0.9704 (0.9713)	0.9519 (0.9539)	0.9815 (0.9783)	0.9826 (0.9834)
12	Analytical (Numerical)	0.9781 (0.9780)	0.9769 (0.9776)	0.9835 (0.9840)	0.9819 (0.9826)	0.9770 (0.9743)	0.9782 (0.9797)

Table 3: Natural frequency ratios, $r_{1,2,3}^{(m)}$, obtained by large sized matrix solution (Analytical) and the finite element package (Numerical).

Cases	$\omega_{analytical}^{(m)} / \omega_{approximate}^{(m)}$					
	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
4	1	1	1.0003	0.9996	1	1
5	1	1.0002	1.0005	0.9993	1	1
6	1.0001	1	1.0001	0.9998	1.0002	0.9999
7	1.0001	1.0014	1.0004	0.9981	1.0005	1.0048
8	1	1	1	1	1.0001	1.0001
9	1	1.0013	0.9997	1.0022	0.9983	1.0016
10	1	1.0004	1.0003	1	0.9999	0.9997
11	1.0003	1.0005	0.9999	1	0.9993	1.0010
12	1.0003	1.0005	1.0007	0.9998	1.0011	0.9983

Table 4: Ratios between the natural frequencies obtained by analytical large size matrix solution and present approximate approach.

Eq. (16), which states the change of natural frequency drop ratios caused by various depths of the cracks, also needs verification. Figure 4 shows natural frequency drop ratios caused by the crack whose location is shifted along the beam. It is seen that the results obtained by proposed formulation employing only the knowledge of reference crack depth ratio, 0.25, present good agreement with the results of matrix solution repeatedly done for the depth ratios, 0.1 and 0.4. When the crack depth ratio is 0.4, some minor discrepancies appear between the results of the matrix solution and presented formulation. These deviations increase with rising crack depth differences. This is because the selection of the reference at the average level of crack depth ratio.

4.3 Application of Crack Detection Procedure

In this subsection, proposed crack detection procedure is implemented on the cantilever beam whose properties are given in Section 4.1. First of all, a database including the natural frequency drop ratios due to the reference crack should be generated for all considered modes of vibration. For this aim, a crack having reference depth ratio, 0.25, is modelled and it is shifted along the beam by taking proper spacing between two neighbour sample locations. Minimum normalised distances, which accurately reveal the shapes of natural frequency drop functions, are determined as 0.1 for the first and second vibration modes, 0.05 for the third and fourth vibration modes, and 0.025 for the fifth and sixth vibration modes of the cantilever beam model. Curve fitting procedure is applied through reference frequency drop ratios found at the sample crack locations. As a result, reference polynomial functions are generated for all vibration modes. The first mode coefficients ($\kappa_k^{(1)}$) of the fractal natural frequency drop polynomial are obtained as follows:

$$\kappa_k^{(1)} = [-0.0324 \quad 0.0888 \quad -0.0242 \quad 0.0024 \quad -0.00002], \qquad k = 1,...,5$$

Here, 4^{th} order polynomial function is adequate to define the change of natural frequency. As the vibration mode considered increases, shift of the crack along the beam result in more frequent modulation of the fractal natural frequency drops. Therefore, more sample locations and polynomial terms are required to fit a curve on drop ratios of higher natural frequencies. Following coefficients of the reference functions with the order of 8, 12, 16, 20, and 24 are employed to identify the natural frequency drop ratios obtained for the second, third, fourth, fifth, and sixth vibration modes respectively.

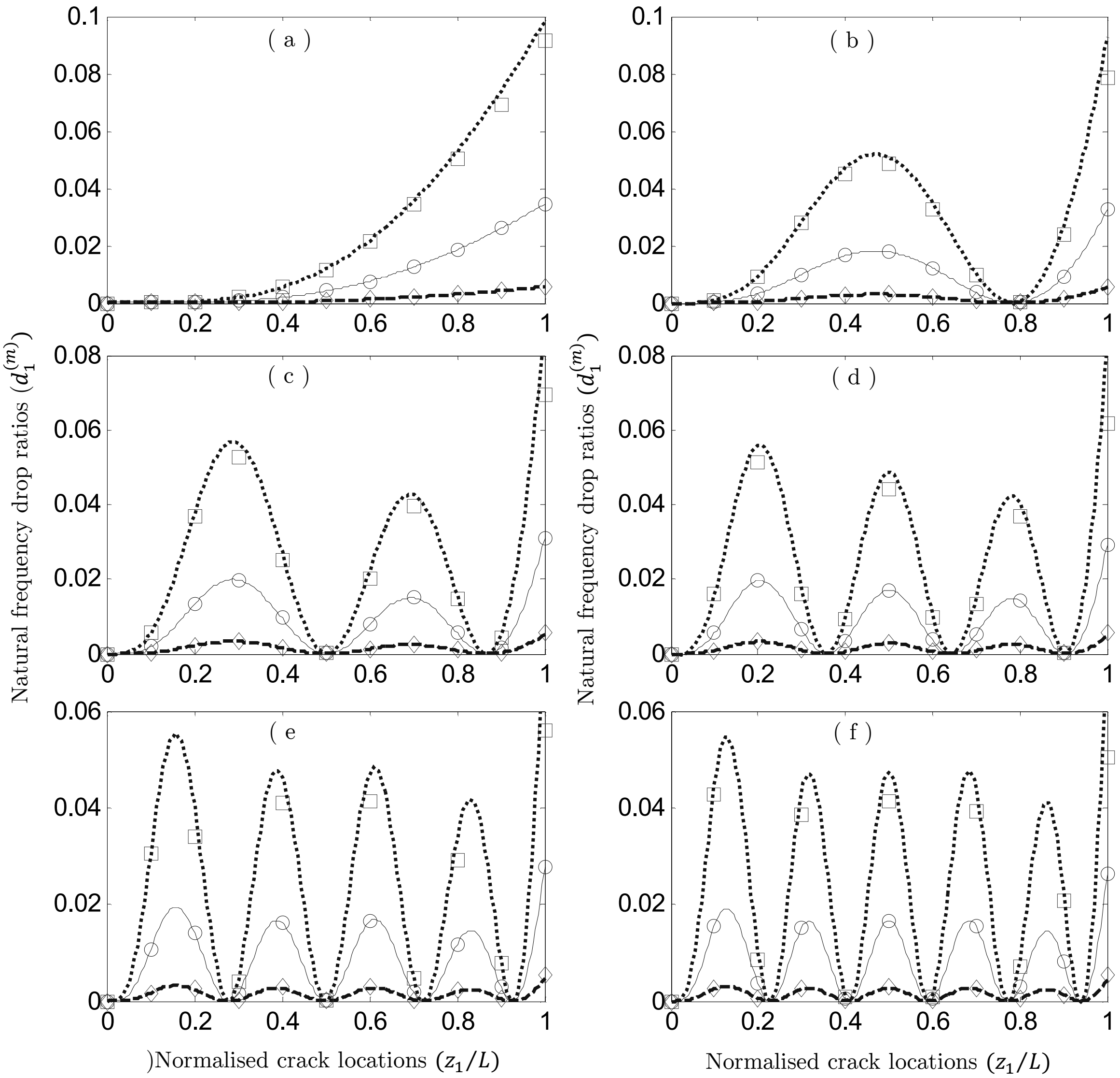

Figure 4: Natural frequency drop ratios obtained by the matrix solution for the crack depth ratios 0.1 (◊), 0.25 (o), 0.4 (□), and the drop ratios obtained by Eq. (16) for the crack depth ratios 0.1 (– –), 0.25 (—), 0.4 (- - -). The ratios are illustrated on the (a) first, (b) second, (c) third, (d) fourth, (e) fifth, and (f) sixth mode of vibrations.

$$\kappa_k^{(2)} = [6.098 \;\; -25.21 \;\; 38.98 \;\; -26.89 \;\; 7.376 \;\; -0.348 \;\; 0.019 \;\; -0.0004 \;\; 0.0000002], \quad k = 1,...,9$$

$$\kappa_k^{(3)} = \begin{matrix}[-1118 \;\; 6816 \;\; -17748 \;\; 25693 \;\; -22568 \;\; 12375 \;\; -4267 \;\; 955.9 \\ -158.3 \;\; 20.16 \;\; -1.116 \;\; 0.024 \;\; -0.0000006]\end{matrix}, \quad k = 1,...,13$$

$$\kappa_k^{(4)} = 10^3 [466\ \ -3707\ \ 13310\ \ -28499\ \ 40514\ \ -40310\ \ 28860\ \ -15076\ \ 5778\ \ -1622\ \ 330.3\ -47.44\ \ 4.512\ \ -0.257\ \ 0.0083\ -0.00011\ \ -0.0000], \quad k = 1,...,17$$

$$\kappa_k^{(5)} = 10^5 [67.3\ -707.4\ \ 3590\ \ -11618\ \ 26595\ \ -45181\ \ 58353\ \ -57984\ \ 44557\ -26491\ \ 12146\ -4267\ \ 11377\ -227.1\ \ 33.26\ -3.459\ \ 0.241\ \ -0.010\ \ 0.00024\ \ -0.000002\ -0.00000], \quad k = 1,...,21$$

$$\kappa_k^{(6)} = 10^7 [-181.2\ \ 2142\ -11935\ \ 41654\ -102125\ \ 186741\ \ -263653\ \ 293260\ -259632\ \ 183410\ -102867\ \ 45163\ -15073\ \ 3580\ -489.4\ \ -15.52\ \ 26.23\ -6.938\ \ 1.072\ -0.108\ \ 0.007\ -0.0003\ \ 0.000006\ \ -0.00000005\ \ 0.00000], \quad k = 1,...,25$$

Polynomial functions, which have coefficients proportional to the reference coefficients, are defined for unknown crack depths by using the expression given in Eq. (18). Thus, the equation set, given in Eq. (21), is generated to find unknown crack parameters.

The cracked beam cases given in Table 2 are also considered as the scenarios of the inverse analysis. The results obtained by the theoretical and numerical natural frequency inputs are represented in Table 5. Theoretical natural frequency inputs tabulated by Table 3 are employed for checking the amount of errors caused by the simplifications given in Eq. (15) and Eq. (16). Results show that the cracks are revealed with some minor deviations even if the beam model includes three cracks which are inspected by the natural frequency knowledge of six vibration modes. It is seen that the deviations generally rise as the number of cracks increases. Similar results are also obtained when the same cracked beam cases are handled by using the finite element inputs. As one would expect, the deviations generally increase due to the numerical errors originated from the finite element calculations.

Cases	Results of theoretical frequency inputs				Results of numerical frequency inputs			
	Normalised locations	Errors (%)	Depth ratios	Errors (%)	Normalised locations	Errors (%)	Depth ratios	Errors (%)
1	0.198	0.2	0.202	0.2	0.201	0.1	0.198	0.2
2	0.449	0.1	0.345	0.5	0.449	0.1	0.339	1.1
3	0.799	0.1	0.152	0.2	0.803	0.3	0.151	0.1
4	0.185	1.5	0.209	0.9	0.152	4.8	0.237	3.7
	0.398	0.2	0.204	0.4	0.389	1.1	0.210	1.0
5	0.269	1.9	0.157	0.7	0.282	3.2	0.159	0.9
	0.456	0.6	0.341	0.9	0.459	0.9	0.336	1.4
6	0.548	0.2	0.250	0.0	0.550	0.0	0.247	0.3
	0.798	0.2	0.152	0.2	0.804	0.4	0.152	0.2
7	0.311	1.1	0.393	0.7	0.314	1.4	0.393	0.7
	0.714	1.4	0.383	1.7	0.718	1.8	0.381	1.9
8	0.099	0.1	0.410	1.0	0.102	0.2	0.393	0.7
	0.904	0.4	0.099	0.1	0.901	0.1	0.093	0.7

Cases (cont.)	Results of theoretical frequency inputs (cont.)				Results of numerical frequency inputs (cont.)			
	Normalised locations	Errors (%)	Depth ratios	Errors (%)	Normalised locations	Errors (%)	Depth ratios	Errors (%)
9	0.449	0.1	0.178	2.2	0.450	0.0	0.159	4.1
	0.554	0.4	0.396	0.4	0.554	0.4	0.400	0.0
10	0.196	0.4	0.203	0.3	0.198	0.2	0.199	0.1
	0.400	0.0	0.198	0.2	0.400	0.0	0.195	0.5
	0.500	0.0	0.201	0.1	0.494	0.6	0.197	0.3
11	0.260	1.0	0.155	0.5	0.260	1.0	0.135	1.5
	0.449	0.1	0.342	0.8	0.448	0.2	0.341	0.9
	0.753	0.3	0.297	0.3	0.751	0.1	0.298	0.2
12	0.521	2.9	0.216	3.4	0.528	2.2	0.208	4.2
	0.639	1.1	0.269	1.9	0.631	1.9	0.268	1.8
	0.801	0.1	0.160	1.0	0.800	0.0	0.169	1.9

Table 5: The cracks detected by present methodology using frequency inputs obtained by theoretical and finite element analyses.

It should be noted that increasing number and depth of cracks brings some difficulties arise from the growth of the errors in linear modelling. The errors in natural frequency drop ratios obtained by simplified model may still be negligible in respect of natural frequency determination. However, large errors, which stem from the excessive number and depth ratio of cracks, lead deviations in polynomial coefficients which make difficult to converge towards accurate crack parameters. Besides, excessive number of cracks needs the use of higher modes of vibration frequencies which frequently modulate due to the crack shifted along the beam. Rise of the number of modulations in fractal natural frequency drops makes the method convergence difficult and necessitates suitable initial predictions for the crack parameters. Based upon these effects, the inverse method can be suggested to be efficiently applied up to three cracks with depth ratios less than 0.5.

It is also worth noting that, when three cracks are detected using six polynomial functions, it is not judged about the possibility of existence of more cracks. In other words, six polynomial functions are able to detect three cracks, while giving definite knowledge up to two cracks. If there are two cracks that are inspected using six polynomial functions, methodology detects the third crack at random position with negligibly small depth ratio. From a different point of view, the use of larger number of polynomial functions is helpful in assessment of cracks near the free end. Because, especially in measurement, lower mode natural frequencies may have insufficient sensitivity to the effects of a crack near the free end unless the crack has significant depth ratio. As recognised from Figure 4, changes of the first two natural frequencies may be inadequate to distinguish the effects of the crack having normalised distance less than 0.1 from the free end. This inadequacy can be diminished to 0.05 and 0.025 by considering the first four and six natural frequencies respectively. Note that all of these limitations and thus success of the method can vary with the sensitivity and accuracy of measurements.

4.4 Testing the method on data from literature

Simplified theories and inverse method used are also validated by using several tests given in current literature. At first, selected experiments of single cracked free-free beam scenarios reported by Kim and Stubbs (2003) are utilised. The test beam has the material properties as $E = 206$ GPa, $\rho = 7650$ kg/m^3 and $\nu = 0.29$. Dimensions of the beam are given as $h = 0.032$ m height, $b = 0.016$ m width and $L = 0.72$ m long. Secondly, double crack scenarios of the aluminium alloy cantilever beam experimented by Mazanoglu and Sabuncu (2012), which is also specified in Section 4.1, is employed for testing the methods. Finally, cantilever beam experimented by Ruotolo and Surace (1997) is considered with double crack scenarios. Their tests were conducted on C30 steel beam with the geometric properties as $A = 0.02 \times 0.02$ m^2 cross-section and 0.8 m long. Measured natural frequency ratios inputted to the methodology are given in Table 6.

Cases	Natural frequency ratios ($r_{1,2}^{(m)}$)			
	First mode	Second mode	Third mode	Fourth mode
1 (Kim and Stubbs 2003)	0.9969	0.9847	0.9582	0.9396
2 (Kim and Stubbs 2003)	0.9972	0.9926	0.9940	0.9970
3 (Kim and Stubbs 2003)	0.8485	0.9219	0.9955	0.9010
4 (Kim and Stubbs 2003)	0.9601	1.0000	0.9725	0.9993
5 (Mazanoglu and Sabuncu 2012)	0.9750	0.9647	0.9874	0.9580
6 (Ruotolo and Surace 1997)	0.9925	0.9907	0.9814	0.9966
7 (Ruotolo and Surace 1997)	0.9946	0.9814	0.9643	0.9926
8 (Ruotolo and Surace 1997)	0.9895	0.9894	0.9692	0.9952

Table 6: Measured natural frequency ratios used as inputs of the methodology.

For all scenarios, actual and detected parameters of the cracks are tabulated in Table 7. Results show that the cracks are approximately identified for all cases. Applications using free-free and cantilever beam conditions demonstrate that boundary changes do not clearly affect the success of the methodology if the polynomial function is prepared for each different boundary condition. It is also shown that the methodology works well even if the crack is quite close to the free end as in the first case and its depth is advanced as in the third case. The largest error ratios are obtained as 1.6% for the location parameter of second and sixth scenarios and 4.6% for the depth ratio parameter of the eighth scenario. It can be observed that most of the errors are smaller than the errors given in the references.

Cases	Actual cracks		Detected cracks			
	Locations	Depth ratios	Locations	Errors (%)	Depth ratios	Errors (%)
1 (Kim and Stubbs 2003)	0.125	0.375	0.140	1.5	0.350	2.5
2 (Kim and Stubbs 2003)	0.250	0.125	0.234	1.6	0.123	0.2
3 (Kim and Stubbs 2003)	0.375	0.500	0.382	0.7	0.490	1.0
4 (Kim and Stubbs 2003)	0.500	0.250	0.486	1.4	0.270	2.0
5 (Mazanoglu and Sabuncu 2012)	0.750 0.550	0.250 0.350	0.754 0.538	0.4 1.2	0.236 0.353	1.4 0.3
6 (Ruotolo and Surace 1997)	0.682 0.319	0.200 0.200	0.697 0.335	1.5 1.6	0.199 0.200	0.1 0.0
7 (Ruotolo and Surace 1997)	0.682 0.319	0.200 0.300	0.680 0.312	0.2 0.7	0.162 0.321	3.8 2.1
8 (Ruotolo and Surace 1997)	0.682 0.319	0.300 0.200	0.683 0.309	0.1 1.0	0.254 0.234	4.6 3.4

Table 7: Cracked beam scenarios tested by several researchers and the cracks detected by present methodology.

5 CONCLUSIONS

This paper presents forward and inverse approaches to determine natural frequencies of cracked beams and to identify cracks. In forward analyses, two theoretical simplifications are introduced to find natural frequency drop ratios caused by cracks. One of them is helpful to compute natural frequency ratios of multiple cracked / un-cracked beams. The other one sets relation between the natural frequency drop ratios for various crack depths. Results are theoretically validated using well-known analytical approach. Proposed theoretical simplifications are also engaged in novel methodology of crack identification. It is demonstrated here that researchers do not have to use complex theoretical formulations to express crack effects and to detect cracks with acceptable proximity. In presented methodology, a set of polynomials, which have coefficients varying with unknown crack parameters, is produced using reference drop ratios obtained along the beam. Each crack to be detected needs two polynomial functions corresponding to two different natural frequency drop ratios. The function set is solved using the numerical optimisation to get singular values exhibiting the crack parameters. Proposed crack detection methodology is verified by numerically obtained frequency ratios and the experimental frequency changes given in literature. It is seen that locations and depth ratios of the cracks are identified with admissible deviations. The methodology is powerful in the detection of one and two cracks. Furthermore, it can efficiently be employed up to three cracks if the initial estimations done for the crack parameters are not far from the accurate parameters. Limitation on crack number mainly stems from the growth of nonlinearity and rising number of modulations in fractal natural frequency drops. Here, it must be borne in mind that the identification of single and double cracked beams satisfies the people in most applications of non-destructive testing.

Several achievements can be remarked when the methodology is evaluated in respect of non-destructive evaluation. First of all, it only needs a dynamic data acquisition system for determining natural vibration frequencies. A point to consider is requirement of the un-cracked beam's natural frequencies which are used as base data for zero setting procedure. This provides the robustness of method that is indispensable for the automated crack detection system. Simplified approaches for determining natural frequencies reduce the computational work load by eliminating integrations and differentiations. Simple relations also make the inverse method available and increase the adaptation speed of the method applied to the beams having different properties. In addition, if there is no measurement fail, the method does not depend on to the user experiences. As a result, it can be concluded that present methodology can be proposed due to its inexpensiveness, practicability, robustness, user independence, and convenience for automation.

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