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INPUT ALLOCATION WITH THE ELLIPSOIDAL FRONTIER MODEL

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ABSTRACT. This work aims at complementing the development of the EFM (Ellipsoidal Frontier Model) proposed by Milioni *et al.* (2011a). EFM is a parametric input allocation model of constant sum that uses DEA (Data Envelopment Analysis) concepts and ensures a solution such that all DMUs (Decision Making Units) are strongly CCR (Constant Returns to Scale) efficient. The degrees of freedom obtained with the possibility of assigning different values to the ellipsoidal eccentricities bring flexibility to the model and raises the interest in evaluating the best distribution among the many that can be generated. We propose two analyses named as local and global. In the first one, we aim at finding a solution that assigns the smallest possible input value to a specified DMU. In the second, we look for a solution that assures the lowest data variability.

Keywords: Date Envelopment Analysis, DEA-Parametric Models, Ellipsoidal Frontier Model.

1 INTRODUCTION

DEA (Data Envelopment Models) Models of Constant Sum refer to problems in which a new (or already existing) input or output variable has to be assigned (or reassigned) to a group of DMUs (Decision Making Units) such that the total sum of this new (or existing) variable across all DMUs has to remain constant.

Such models may be parametric or nonparametric. Examples of nonparametric DEA Models of Constant Sum are Cook & Kress (1999), Wei at al. (2010), Beasley (2003), Lins *et al.* (2003) and Gomes & Lins (2008). Parametric DEA models were first proposed by Kozyreff & Milioni (2004). Further publications on Parametric DEA Models are Avellar (2004, 2010), Avellar *et al.* (2005, 2007 and 2010), Milioni *et al.* (2011a and 2011b), Silva & Milioni (2012), Guedes (2007) and Guedes *et al.* (2012).

Parametric DEA models are characterized by the assumption of the geometrical shape or locus of points of the production frontier. This may be considered a strong assumption, but parametric DEA models are also the only ones for which it is possible to prove a desirable property known

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as coherent property, within the context of sensitivity analysis (see, for instance, Milioni *et al.*, 2011a and Guedes *et al.*, 2012).

This work aims to complement the development of an input allocation model of constant sum, parametric, adapted to the characteristics of Data Envelopment Analysis (DEA) and that guarantees a strongly efficient solution to all the Decision Making Units (DMUs), in models with constant return of scale. This is the Ellipsoidal Frontier Model (EFM), based on an efficiency frontier with ellipsoidal shape that is capable of distributing inputs taking into account the problem.

The EFM provides several possibilities for different solutions due to its flexibility derived from the degrees of freedom (eccentricities) of the model. For this reason, it became interesting to guide the decision maker to select the best solution, which is the purpose of this work. Therefore, two different types of analyzes are proposed, classified as: Local (LA) and Global (GA). The first one (LA) has the objective of finding the lowest possible input value associated to a specific DMU. The second (GA) searches for a solution with the lowest total variability of the data set: input and output values.

2 EFM MODEL

According to Avellar (2010) and Milioni *et al.* (2011a), the Ellipsoidal Frontier Model (EFM) is a parametric model of constant sum such that the efficiency frontier has an ellipsoidal shape. It assures several solutions that are CCR strongly efficient for all DMUs, by distributing (redistributing) a new (already existing) input variable among all DMUs, taking into account all other input and output variables involved in the problem.

Model construction is presented in three different cases:

(i) two outputs and a single input,

(ii) s Outputs and a single input and

(iii) s Outputs and $m + 1$ Inputs.

According to Avellar (2010): *"Consider $y_r j\,(> 0)$ the measured value of output $r(r = 1, \ldots, s)$ for DMU $j\,(j = 1, \ldots, n)$; $F(> 0)$ the total fixed input (or cost) to be distributed to all DMUs, i.e., $F = \sum_{j=1}^{n} f_j$, where f_j is the input value to be allocated to each DMU j."*

Thus, the coordinate values and the values of the new Input (f_j) to be distributed for each case will be:

Case (i):

$$f_j = \frac{F.\sqrt{\left(\frac{y_{1j}}{\sum\limits_{k=1}^{n} y_{1k}}\right)^2 - e^2.\left(\frac{y_{1j}}{\sum\limits_{k=1}^{n} y_{1k}}\right)^2 + \left(\frac{y_{2j}}{\sum\limits_{k=1}^{n} y_{2k}}\right)^2}}{\sum\limits_{i=1}^{n}\sqrt{\left(\frac{y_{1l}}{\sum\limits_{k=1}^{n} y_{1k}}\right)^2 - e^2.\left(\frac{y_{1l}}{\sum\limits_{k=1}^{n} y_{1k}}\right)^2 + \left(\frac{y_{2l}}{\sum\limits_{k=1}^{n} y_{2k}}\right)^2}} \tag{1}$$

The value assigned to e refers to the eccentricity of the ellipse. Thus, the model allows a different solution (frontier) for each different value of e. It is noteworthy that an ellipse with zero eccentricity is a sphere (thus, spherical frontier is a particular case of this model).

Case (ii):

$$f_j = \frac{F.\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rj}}{\sum_{k=1}^{n} y_{rk}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2.\left(\frac{y_{rj}}{\sum_{k=1}^{n} y_{rk}}\right)^2\right]}}{\sum_{r=1}^{n}\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rl}}{\sum_{k=1}^{n} y_{rk}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2.\left(\frac{y_{rl}}{\sum_{k=1}^{n} y_{rk}}\right)^2\right]}} \tag{2}$$

Case (iii):

$$f_j = \frac{1}{m}\left[\frac{(2m)\,.\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rj}}{\sum_{k=1}^{n} y_{rl}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2.\left(\frac{y_{rj}}{\sum_{k=1}^{n} y_{rl}}\right)^2\right]}}{\sum_{p=1}^{n}\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rp}}{\sum_{k=1}^{n} y_{rl}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2.\left(\frac{y_{rp}}{\sum_{k=1}^{n} y_{rl}}\right)^2\right]}} - \sum_{i=1}^{m}\left(\frac{x_{ij}}{\sum_{k=1}^{n} x_{ik}}\right)\right] \tag{3}$$

In order that we assure that $f_j > 0$, we must have:

$$\frac{\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rj}}{\sum_{k=1}^{n} y_{rl}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2.\left(\frac{y_{rj}}{\sum_{k=1}^{n} y_{rl}}\right)^2\right]}}{\sum_{r=1}^{n}\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rp}}{\sum_{k=1}^{n} y_{rl}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2.\left(\frac{y_{rp}}{\sum_{k=1}^{n} y_{rl}}\right)^2\right]}} = \frac{1}{2m}.\sum_{i=1}^{m}\left(\frac{x_{ij}}{\sum_{i=1}^{n} x_{ik}}\right) \tag{4}$$

As it is shown in Avellar (2010), EFM solution can be obtained with the use of a Linear Programming Problem (LPP) presented in the following set of equations:

$$\begin{aligned} \text{Min} \quad & W_{\max} - W_{\min} \\ \text{Subject to :} \quad & W_{\max} = \frac{f_j}{\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rj}}{\sum_{l=1}^{n} y_{rl}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2.\left(\frac{y_{rj}}{\sum_{l=1}^{n} y_{rl}}\right)^2\right]}} \end{aligned}$$

$$
\begin{aligned}
& W_{\min} = \frac{f_j}{\sqrt{\sum_{r=1}^{s}\left(\frac{y_{rj}}{\sum_{l=1}^{n} y_{rl}}\right)^2 - \sum_{r=1}^{s-1}\left[(e_r)^2 . \left(\frac{y_{rj}}{\sum_{l=1}^{n} y_{rl}}\right)^2\right]}} \\
& \sum_{j=1}^{n} f_j = 100 \\
& \sum_{k=1}^{n} \frac{u_{rk} y_{rk} = 1}{f_j} \\
& f_j > 0 \\
& W_{\max}, W_{\min} = 0 \\
& j = 1, \ldots, n; r = 1, \ldots, s
\end{aligned} \tag{5}
$$

According to the author, their properties and characteristics make EFM a Cooperative, Competitive and Flexible model. Namely:

- Frontier Homogeneity property: replaces the original piece-wise linear DEA frontier by a smooth frontier.
- DEA control weights Effective solutions generating property (flexibility model): gives the decider the possibility of obtaining a weight distribution for each combination of eccentricity, strongly efficient solutions CCR (characteristic competitive);
- Coherent Distribution Ownership (cooperation characteristics): Inputs to distribute special consistently in the presence of errors;
- Input distribution characteristic considering input and output values existing in the current problem.
- No DMU has to increase input value to become efficient.

Further details of the model can be found in Avellar (2010) e Milioni *et al.* (2011a).

3 METHOD

EFM model is in essence flexible due to its many degrees of freedom. By using different eccentricities values one can generate many different solutions. Moreover, in the illustrative example presented in Milioni *et al.* (2011a) the authors show that by choosing different values to the eccentricities one can gain control on the weights assigned to each input and output variable in the DEA solution.

In their example, they propose two solutions, analyze the characteristics of each one of them and then claim:

"So, for both strongly efficient solutions, the decision maker can choose which weight distribution is more adequate to his or her reality (...)".

Further ahead they point out that:
"The kind of procedure could provide a guideline for how one chooses specific parameters based on the ellipsoidal shape of the frontier, an issue so important and dense that we intend to address it in another paper".

This is precisely our goal in this paper. We propose two different analyses denominated as Local (LA) and Global (GA):

(i) in the first one (LA), by varying the values assigned to the eccentricities ($0 \leq e \leq 1$), we investigate the solution that achieves the smallest possible input value to a specific and previously chosen DMU;

(ii) in the second (GA), we suppose that there is a prior solution (*i.e.*, the input variable with constant sum is already somehow distributed among all DMUs) and seek the values of eccentricities for which one has the smallest total variability, considering the current existing distribution.

For this variability, it is used the Euclidean Distance as a metric. We mean the total sum of squares of the differences between prior (currently existing) and new (provided by the solution) input variable for each DMU.

4 EXAMPLE

We use the same real data presented in Gomes & Lins (2008). In their case, DMUs are countries and the problem is to fairly distribute a single input, which is emission of CO_2 (carbon equivalent ton[3]) considering three outputs: population (in million), energy (million BTU) and Gross Domestic Product (GDP, in billions of dollars).

Since this is a three outputs problem, the model formulation has two degrees of freedom, *i.e.*, there are two eccentricities to be chosen. The 64 countries regarded as DMUs are:

(01) Argentina	(02) Australia	(03) Austria	(04) Belgium
(05) Bolivia	(06) Brazil	(07) Bulgaria	(08) Canada
(09) Chile	(10) China	(11) Costa Rica	(12) Croatia
(13) Czech Republic	(14) Denmark	(15) Egypt	(16) El Salvador
(17) Estonia	(18) Finland	(19) France	(20) Germany
(21) Greece	(22) Guatemala	(23) Honduras	(24) Indonesia
(25) Ireland	(26) Israel	(27) Italy	(28) Japan
(29) Kazakhstan	(30) Latvia	(31) Lithuania	(32) Luxembourg
(33) Malaysia	(34) Maldives	(35) Malta	(36) Mexico

(37) Netherlands (38) New Zealand (39) Nicaragua (40) Norway
(41) Panama (42) Paraguay (43) Peru (44) Philippines
(45) Poland (46) Portugal (47) Republic of Korea (48) Romania
(49) Russian Federation (50) Seychelles (51) Slovakia (52) Slovenia
(53) Spain (54) Sweden (55) Switzerland (56) Thailand
(57) Turkmenistan (58) Ukraine (59) United Kingdom (60) United States
(61) Uruguay (62) Uzbekistan (63) Vietnam (64) Zambia

Input and output real data values for each country are presented in Table 1.

Table 1 – Inputs × Outputs from the example.

DMUs	CO_2	Population	Energy	GDP	DMUs	CO_2	Population	Energy	GDP
1	34.85	37.52	2664.87	280.05	**33**	36.15	23.63	2274.95	112.21
2	99.03	19.49	4974.21	453.26	**34**	0.13	0.28	6.77	0.54
3	18.19	8.08	1419.42	268.65	**35**	1.07	0.39	51.41	3.99
4	39.36	10.26	2773.55	321.57	**36**	96.05	101.75	6004.00	372.41
5	2.62	8.47	161.63	8.04	**37**	67.52	16.04	4231.06	502.58
6	95.77	172.39	8782.13	771.45	**38**	9.61	3.85	844.12	70.98
7	15.48	7.87	927.93	12.59	**39**	1.02	5.21	58.12	2.38
8	156.19	31.08	12513.07	718.13	**40**	11.45	4.51	1906.09	172.91
9	14.75	15.40	1060.30	81.93	**41**	2.26	2.86	138.46	9.40
10	831.74	1285.00	39665.26	1113.59	**42**	0.96	5.64	110.93	9.59
11	1.39	3.87	154.08	15.10	**43**	7.19	26.35	550.33	60.89
12	5.69	4.66	429.16	23.35	**44**	18.62	77.13	1254.27	91.24
13	29.01	10.29	1530.56	57.09	**45**	78.61	38.64	3536.04	165.27
14	16.24	5.33	895.23	207.44	**46**	16.25	10.02	1088.21	131.88
15	34.29	67.89	2132.60	80.80	**47**	120.80	47.34	8058.12	639.24
16	1.53	6.40	114.66	11.24	**48**	25.97	22.41	1637.66	34.92
17	1.94	1.38	95.67	4.81	**49**	440.26	144.40	28197.17	366.90
18	14.41	5.19	1326.01	173.57	**50**	0.17	0.08	8.45	0.62
19	108.13	59.19	10521.36	1812.35	**51**	10.83	5.40	832.04	23.81
20	223.24	82.36	14351.56	2701.90	**52**	4.06	1.99	305.56	23.86
21	28.08	10.60	1393.20	144.77	**53**	82.72	40.27	5699.31	723.24
22	2.52	11.68	158.70	18.19	**54**	14.58	8.83	2221.20	281.29
23	1.27	6.58	86.47	4.68	**55**	12.27	7.23	1304.67	340.28
24	87.13	214.84	4629.78	215.93	**56**	48.49	62.91	2903.94	174.97
25	11.15	3.84	609.29	112.91	**57**	7.68	4.88	477.26	6.97
26	16.32	6.45	792.02	107.30	**58**	96.58	49.11	6076.24	36.43
27	121.50	57.95	8110.68	1225.57	**59**	154.33	59.54	9810.06	1334.92
28	315.83	127.34	21921.99	5651.49	**60**	1565.31	283.97	97049.88	9039.46
29	33.37	14.83	1734.57	21.81	**61**	1.69	3.36	157.36	20.79
30	2.65	2.36	205.87	6.03	**62**	30.16	25.56	2075.01	12.80
31	4.33	3.49	329.19	7.51	**63**	12.56	79.18	760.13	30.99
32	2.47	0.44	203.10	25.47	**64**	0.56	10.65	89.46	4.08

In the Local Analysis (LA) we seek the values of eccentricities that assign the lowest possible input value for a single chosen DMU, still assuring the existence of a solution in which all DMUs are strongly CCR efficient.

Table 2 shows the results obtained for the analysis carried out for each DMU.

Table 2 – Local Analysis.

DMU	f_j^*	e_1	e_2	DMU	f_j^*	e_1	e_2
1	41.81	0.90	0.00	**33**	19.20	0.99	0.99
2	49.84	0.50	0.99	**34**	0.10	0.99	0.50
3	26.18	0.00	0.00	**35**	0.55	0.50	0.99
4	34.19	0.70	0.99	**36**	64.38	0.99	0.99
5	2.18	0.99	0.99	**37**	53.42	0.70	0.99
6	127.83	0.99	0.80	**38**	8.11	0.50	0.99
7	3.24	0.99	0.99	**39**	1.00	0.99	0.00
8	80.09	0.50	0.99	**40**	18.22	0.50	0.99
9	13.67	0.99	0.99	**41**	1.64	0.99	0.99
10	328.17	0.99	0.99	**42**	1.78	0.99	0.10
11	2.41	0.99	0.00	**43**	9.73	0.99	0.00
12	3.96	0.99	0.99	**44**	20.05	0.99	0.00
13	9.92	0.99	0.99	**45**	28.59	0.99	0.99
14	19.42	0.00	0.00	**46**	16.10	0.00	0.00
15	19.49	0.99	0.99	**47**	78.88	0.50	0.99
16	1.97	0.99	0.00	**48**	8.01	0.99	0.99
17	0.84	0.99	0.99	**49**	88.66	0.90	0.99
18	18.34	0.50	0.90	**50**	0.10	0.70	0.99
19	180.81	0.00	0.00	**51**	4.34	0.99	0.99
20	263.72	0.00	0.00	**52**	3.06	0.50	0.99
21	17.73	0.50	0.99	**53**	82.13	0.00	0.70
22	3.13	0.99	0.00	**54**	29.84	0.50	0.99
23	1.41	0.99	0.10	**55**	31.18	0.00	0.00
24	56.71	0.99	0.99	**56**	31.41	0.99	0.99
25	11.13	0.00	0.00	**57**	1.80	0.99	0.99
26	12.12	0.00	0.00	**58**	17.30	0.99	0.99
27	130.16	0.00	0.00	**59**	144.90	0.00	0.00
28	519.47	0.00	0.00	**60**	963.10	0.50	0.99
29	5.90	0.99	0.99	**61**	2.90	0.99	0.00
30	1.16	0.99	0.99	**62**	7.12	0.99	0.99
31	1.56	0.99	0.99	**63**	14.33	0.99	0.00
32	2.63	0.50	0.99	**64**	1.85	0.99	0.00

If we look at the result for DMU 1 presented in Tables 1 and 2 we conclude that Argentina's current CO2 emission, which 34.85 carbon equivalent ton[3] has to climb to a minimum of 41.81 such that there will still be a solution for which global total CO_2 emission remains constant and all DMUs (countries) are strong CCR efficient.

On the global analysis we investigated in all distributions (varying eccentricity values), the less variability on the data conjunct. The results are presented in Table 3:

Table 3 – Global Analysis.

GA	$e_2 = 0.0$	$e_2 = 0.1$	$e_2 = 0.2$	$e_2 = 0.3$	$e_2 = 0.4$	$e_2 = 0.5$
$e_1 = 0.0$	5346	5346	1867	1893	1929	1979
$e_1 = 0.1$	1842	1847	1862	1887	1924	1974
$e_1 = 0.2$	1826	1831	1845	1871	1908	1958
$e_1 = 0.3$	1799	1804	1819	1844	1880	1930
$e_1 = 0.4$	1761	1766	1780	1805	1842	1892
$e_1 = 0.5$	1708	1713	1727	1752	1789	1839
$e_1 = 0.6$	1636	1640	1655	1680	1717	1767
$e_1 = 0.7$	1541	1546	1561	1586	1623	1674
$e_1 = 0.8$	1520	1523	1534	1553	1581	1619
$e_1 = 0.9$	1577	1581	1594	1615	1647	1691
$e_1 = 0.99$	1728	1734	1750	1778	1819	1878
GA	$e_2 = 0.5$	$e_2 = 0.6$	$e_2 = 0.7$	$e_2 = 0.8$	$e_2 = 0.9$	$e_2 = 0.99$
$e_1 = 0.0$	1979	2045	2129	2239	2384	2559
$e_1 = 0.1$	1974	2039	2124	2234	2379	2555
$e_1 = 0.2$	1958	2023	2108	2219	2365	2542
$e_1 = 0.3$	1930	1996	2081	2193	2341	2521
$e_1 = 0.4$	1892	1958	2043	2155	2305	2489
$e_1 = 0.5$	1839	1906	1992	2105	2256	2444
$e_1 = 0.6$	1767	1834	1921	2037	2195	2393
$e_1 = 0.7$	1674	1742	1831	1950	2113	2322
$e_1 = 0.8$	1619	1669	1737	1829	1997	2220
$e_1 = 0.9$	1691	1750	1830	1938	2092	2307
$e_1 = 0.99$	1878	1957	2066	2220	2450	2811

As we can see from Table 3, the pair of eccentricities that minimizes total variability is the pair (0.8, 0.0).

For this study, all Local Analysis (Table 2) of the EFM model (Equation 2) were conducted using MSExcel. The Global Analysis (Table 3) was implemented on Matlab.

5 RESULTS AND CONCLUSION

In the LA, the largest differences between current and minimum possible values were observed for DMUs 10 (China), 28 (Japan), 49 (Russian Federation) and 60 (United States). For these Countries, the differences are, 503.57, 203.64, 351.60 and 602.23 ton^3 CO_2 found in eccentricities $(e_1, e_2) = (0.99, 0.99)$, $(0.00, 0.00)$, $(0.90, 0.99)$ and $(0.50, 0.99)$, respectively. Among these countries, only DMU 28 (Japan) has a minimum value that is greater than its current value of emission.

Regarding Global Analysis (GA), it was possible to realize once more the merits of the EFM model to provide various distributions in which the decision maker can choose how to redistribute their data with strongly efficient solutions. The values in Table 3 illustrate these possibilities ranging from 1520 ton^3 CO_2 (minimum) – found in the eccentricities values $(e_1, e_2) = (0.80, 0.00)$ – which represents 28% of the total F value, to 5346 ton^3 CO_2 (maximum). The average was 1990 ton^3 CO_2.

While observing both analysis results (LA and GA) the minimum value found on GA distribution do not reveal any of the cases of LA distribution. However, the merit of this study is attained by indication of distributions with eccentricities values that occurs the solution of analyses LA and GA, according to each problem. The guidance while choosing one solution from many possibilities is now feasible, since it requires minimum data treatment. This is a relevant fact mainly due to the nature of input associated with many resources quantity.

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