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FROM WEIERSTRASS TO KY FAN THEOREMS AND EXISTENCE RESULTS ON OPTIMIZATION AND EQUILIBRIUM PROBLEMS

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ABSTRACT. In this work, we study coerciveness notions and their implications to existence conditions. We start with the presentation of classical ideas of coerciveness in the framework of Optimization Theory, and, then, using a classical technical result, introduced by Ky Fan in 1961, we extend these ideas first to Optimization Problems and then to Equilibrium Problems. We point out the importance of related conditions to the introduced coerciveness notion in order to obtain existence results for Equilibrium Problems, without using monotonicity or generalized monotonicity assumptions.

Keywords: Equilibrium Problems, Recession techniques, Minimization Problems.

1 INTRODUCTION

In this work, we consider two problems: Minimization and Equilibrium Problems. The Minimization Problem (MP) is defined by a function $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a nonempty subset C of $\mathbb{R}^n$, and it is formulated as

$$\min_{x \in C} h(x). \tag{1}$$

The Equilibrium Problem (EP) is defined by a nonempty subset K of $\mathbb{R}^n$ and a function $f\colon K \times K \to \mathbb{R}$, and it is formulated as

$$\text{find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0 \text{ for all } y \in K. \tag{2}$$

The Equilibrium Problem appeared for the first time with this name, in 1994, in the paper of Blum and Oettli (see [6]). The interesting aspect about EP is that it has as particular cases the following problems (for more details see [6], [12], [13] and references therein):

a) The minimization problem,

b) The fixed point problem,

c) The saddle point problem,

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d) The complementarity problem,

e) The Nash equilibria problem in non cooperative games,

f) The variational inequality problem,

g) The vector minimization problem.

The solution set of MP and EP, denoted as $S(MP)$ and $S(EP)$, can be represented respectively as the intersection of a family of sets:

$$S(MP) = \bigcap_{y \in \mathbb{R}^n} L_h(h(y)),$$

where, for each $\lambda \in \mathbb{R}$, $L_h(\lambda) = \{x \in \mathbb{R}^n : h(x) \leq \lambda\}$ denotes the λ-level set of h, and

$$S(EP) = \bigcap_{y \in K} L_{f^y}(0), \tag{3}$$

where, for each $y \in K$, the function $f^y : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^y(x) = \begin{cases} -f(x, y) & \text{if} \quad x \in K, \\ +\infty & \text{if} \quad x \notin K. \end{cases} \tag{4}$$

In 1961, Ky Fan introduced in a technical Lemma (see Lemma 1 in [9]) two sufficient conditions in order to guarantee non emptiness of the intersection of a closed set family. Also known as the KKM condition, the first condition establishes that for all finite subset $\{x^1, \ldots, x^m\}$ of the index set for the set family, its convex hull $co\{x^1, \ldots, x^m\}$ is a subset of the union of the corresponding sets indexed by $\{x^1, \ldots, x^m\}$. The second condition establishes the existence of at least one compact set in the family.

Regarding minimization problems, considering lower semi-continuous functions over compact sets, the Weierstrass Theorem gives the main classical existence result. In Section 3, we shall verify that, under the assumptions of h being lower semi-continuous (lsc) and the existence of at least one compact level set of h, minimization problems have solutions. In order to deal with non compact sets, recession techniques have been introduced. In this sense, several conditions have appeared in the literature such as coerciveness conditions. We observe that coerciveness conditions of a lsc function h are based on compactness of all its level sets. It can be verified, without using Ky Fan's Lemma, that the solution set of the corresponding minimization problem is exactly the intersection of all nonempty level sets of h, and, since the level sets are compact, the intersection is nonempty. It is worth mentioning that some of the results that are presented here for completeness are also in the working paper "Semi-continuous programming: existence conditions and duality scheme" [15].

With respect to equilibrium problems, Ky Fan introduced a technical Lemma to prove his famous Minimax Theorem (see [8]) that gives the main classical existence result for Equilibrium Problems, provided K is compact (2). In order to obtain existence results for equilibrium problem considering non compact sets, some conditions using recession techniques have been introduced in the literature, but all of them require generalized monotonicity assumptions, see for example ([5], [6], [12] and the reference there in).

The objective of this paper is to introduce the notion of coerciveness for the Equilibrium Problem as an extension of the classical coerciveness ideas for the Minimization Problem. In this way, we use the Ky Fan's Lemma, replacing the compactness assumption in the second sufficient condition by another one that is related to a new coerciveness notion here introduced, making use of recession techniques in order to guarantee non emptiness of the solution set of the Equilibrium Problem without using generalized monotonicity assumptions.

The outline of paper is as follows. In Section 2, we recall some basic concepts and recession tools in order to present known coerciveness ideas developed in the context of Optimization Theory. In Section 3, we introduce a new coerciveness idea for Minimization Problems, which is related to other existence conditions already appeared in the literature, considering at least one bounded level set. Some interesting results are commented and an application to the minimization quadratic functions over polyhedral sets is considered in order to recuperate the famous Frank Wolf Theorem. In Section 4, we present theoretical results on Equilibrium Problems and introduce the coerciveness notion for Equilibrium Problems. Also, some existence results are introduced and an application to saddle points for quadratic Lagrangian is considered. For this, we transform the problem of finding a saddle point of a Lagrangian function into an Equilibrium Problem in order to apply our results and so to guarantee the existence of saddle points of a quadratic Lagrangian function over polyhedral sets.

2 PRELIMINARIES

In this section, we introduce some basic concepts and tools in order to introduce a new coerciveness notion in the context of Optimization Theory.

2.1 Some basic concepts and notations

Now, we introduce some basic concepts as well as their respective notations that will be used hereafter. Denote by $\overline{\mathbb{R}}$ the whole extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$. Given a function $h\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, the effective domain of h is defined as $dom(h) = \{x \in \mathbb{R}^n : h(x) < +\infty\}$. The function h is proper if $dom(h) \neq \emptyset$ and $h(x) > -\infty\ \forall x \in \mathbb{R}^n$. For each $\lambda \in \overline{\mathbb{R}}$, the epigraph of h is defined by $epi(h) = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}\colon h(x) \leq \lambda\}$. The function h is convex if $epi(h)$ is convex in $\mathbb{R}^n \times \mathbb{R}$. The function h is quasi convex if $L_h(\lambda)$ is convex for all $\lambda \in \overline{\mathbb{R}}$. The function h is lower semi-continuous (lsc in short) if $L_h(\lambda)$ is closed for all $\lambda \in \mathbb{R}$, while h is upper semi-continuous (usc in short) if $-h$ is lsc. The function h is upper hemi-continuous if h is usc in the segment lines of $dom(h)$.

2.2 Some recession tools

The key for developing recession techniques is the concept of recession sets (see Definition 2.1.1 [3]). Given a set $K \subset \mathbb{R}^n$, the recession set of K is defined by:

$$K^\infty = \Big\{x \in \mathbb{R}^n : \exists\, t_n \downarrow 0,\ \ \exists\, x_n \in K,\ \ t_n x_n \to x\Big\}, \tag{5}$$

where $\downarrow$ means that the sequence converges to zero and $t_n > 0\ \forall n$.

By convention, the recession set of the empty set is the singleton $\{0\}$ (i.e. $(\emptyset)^\infty = \{0\}$). Thus, we have the following key property (see Proposition 2.1.2 in [3]):

$$A \text{ is bounded if and only if } A^\infty = \{0\}. \tag{6}$$

In case K is also a nonempty closed convex subset of $\mathbb{R}^n$, it is known that for any $x_0 \in K$

$$K^\infty = \Big\{x \in \mathbb{R}^n : \ x_0 + tx \in K\ \forall\, t > 0\Big\}. \tag{7}$$

Note that definition (7) considers only the vectorial structure where as definition (5) considers vectorial and topological structures.

Now, we enumerate some basic results on recession sets that will be useful in the sequel. Let K, K_1 and K_2 be subsets in $\mathbb{R}^n$.

1. $K_1 \subset K_2$ implies $K_1^\infty \subset K_2^\infty$;
2. $(K + x)^\infty = K^\infty$ for all $x \in \mathbb{R}^n$;
3. Let (K_i), $i \in I$, be any family of nonempty sets in $\mathbb{R}^n$, then

$$\Big(\bigcap_{i\in I} K_i\Big)^\infty \subset \bigcap_{i\in I}(K_i)^\infty.$$

   If, in addition, each set is convex and closed and $\cap_{i\in I} K_i \neq \emptyset$, then we obtain an equality in the previous inclusion.

Additional properties of the recession set C^∞ of a set C are the following (see Proposition 2.1.1 [3]):

1. C^∞ is nonempty closed cone.
2. $(\overline{C})^\infty = C^\infty$.
3. If C is a cone, then $C^\infty = \overline{C}$

If C is a nonempty polyhedral set, Auslender [3] proved that if $\{x^k\} \subset C$ is a sequence such that $\|x^k\| \to +\infty$ and $\frac{x^k}{\|x^k\|} \to d$ when $k \to +\infty$, then for any $t > 0$, there exists $k \in \mathbb{N}$ such that $x^k - td \in C$ (Proposition 2.3 in [2]). It is easy to see that, for each $t > 0$, there exists $k \in \mathbb{N}$, such that $x^k - td \in C$ as well as $\|x^k - td\| < \|x^k\|$ (for more details see [2], [3] and [4]). This facts will be used later when we consider quadratic functions over polyhedral sets.

Remark 2.1. Note that the definition of K^∞ is related to the theory of set convergence of Painleve-Kuratowski. Indeed, for a family $\{K_t\}_{t>0}$ of subsets of $\mathbb{R}^n$, the outer limit is the set

$$\limsup_{t\to+\infty} K_t := \{x : \liminf_{t\to+\infty} d(x, K_t) = 0\}. \tag{8}$$

It can then be verified that

$$K^{\infty} = \limsup_{t \to +\infty} t^{-1} K. \tag{9}$$

Given $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the recession function of h denoted by h^{∞} is defined as the function whose epigraph is the recession set of the epigraph of h, i.e., $epi(h^{\infty}) = (epi(h))^{\infty}$.

It is well known that

$$h^{\infty}(d) = \lim_{t \to +\infty} \left[\inf_{d' \to d} \frac{h(td')}{t} \right] \tag{10}$$

or equivalently

$$h^{\infty}(d) = \inf\{\lim \inf_{k \to +\infty} \frac{h(t_k d_k)}{t_k} : t_k \to +\infty, d_k \to d\}. \tag{11}$$

The relation between the level set of h^{∞} for $\lambda = 0$ and any level set of h is as follows (see Proposition 2.5.3 in [3]): for any proper function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and any $\lambda \in \mathbb{R}$, we have

$$(L_h(\lambda))^{\infty} \subset L_{h^{\infty}}(0), \tag{12}$$

where the equality holds if h is proper, convex and lsc.

Note that if $L_{h^{\infty}}(0) = \{0\}$, then from (12) we have that all level sets of the function h are bounded. Moreover if h is lsc, then all the level set are compact.

2.3 Some classical coerciveness ideas in Optimization Theory

Here we collect the main ideas about coerciveness in Optimization Theory which can be found in related books, such as, for example, [3] and [16].

Definition 2.2. *[3] The function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is strong coercive if*

$$h^{\infty}(d) > 0 \text{ for all } d \neq 0. \tag{13}$$

It is evident that if h is strong coercive and lsc, then for each $\lambda \in \mathbb{R}$, $L_h(\lambda)$ is compact. The converse is not true, for example, consider $h(x) := \sqrt{|x|}$.

Definition 2.3. *[3] The function $h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is Inf-compact if $\forall\lambda \in \mathbb{R}$, $L_h(\lambda)$ is compact.*

It follows directly that, h is Inf-compact if and only if h is lsc and all level sets are bounded.

Definition 2.4. *[3] The function $h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is level bounded if $\forall\lambda \in \mathbb{R}$, $L_h(\lambda)$ is bounded.*

Note that if h is level bounded but not lsc, then h is not Inf-compact.

Definition 2.5. *[3] The function $h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, is zero-coercive if*

$$\lim_{||x|| \to +\infty} h(x) = +\infty \tag{14}$$

It is well known that zero-coercive and level bounded notions are equivalent. Moreover, if h is lsc, then Inf-compact, zero-coercive and level bounded notions are equivalent.

Note that all these coerciveness ideas require that all the level sets are bounded. Before exploiting the condition that at least one of the level set is bounded, we need to consider some simple ideas using recession techniques.

3 COERCIVENESS FOR MINIMIZATION PROBLEMS

In this section, we introduce some theoretical coerciveness ideas using recession techniques for minimization problems. At the end, we show an application of these ideas to the minimization of quadratic functions over polyhedral sets.

Let us consider first the minimization of a function $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and the corresponding solution set denoted by $S(h) := \{x \in \mathbb{R}^n \colon h(x) \leq h(y)\ \forall y \in \mathbb{R}^n\}$. Note that $S(h) = \bigcap_{x\in\mathbb{R}^n} L_h(h(x))$, moreover, if $\lambda = \inf\{h(x)\colon x \in \mathbb{R}^n\}$, then $S(h) = L_h(\lambda)$.

Lemma 3.1. *Consider the minimization of $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. It follows that*

$$(S(h))^\infty \subset \bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty.$$

Proof. Since $S(h) = \bigcap_{x\in\mathbb{R}^n} L_h(h(x)) \subset L_h(h(x)) \quad \forall x \in \mathbb{R}^n$, we have that

$$(S(h))^\infty \subset \big(L_h(h(x))\big)^\infty \quad \forall x \in \mathbb{R}^n,$$

and so $(S(h))^\infty \subset \bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty$. □

Lemma 3.2. *Let $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any function and let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to+\infty} \lambda_k = \lambda := \inf\{h(x)\colon x \in \mathbb{R}^n\}$. If $L_h(\lambda) = \emptyset$ and $L_h(\lambda_k) \neq \emptyset\ \forall k \in \mathbb{N}$, then $\bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty = \bigcap_{k\in\mathbb{N}} \big(L_h(\lambda_k)\big)^\infty$.*

Proof. First, we show that

$$\bigcap_{k\in\mathbb{N}} \big(L_h(\lambda_k)\big)^\infty \subset \bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty.$$

Indeed, take $u \in \bigcap_{k\in\mathbb{N}}(L_h(\lambda_k))^\infty$, thus $u \in (L_h(\lambda_k))^\infty\ \forall k \in \mathbb{N}$. For each $x \in \mathbb{R}^n$ (arbitrarily fixed), we have that $\lambda < h(x)$, because $L_h(\lambda) = \emptyset$. Since $\lim_{k\to+\infty} \lambda_k = \lambda$ and $L_h(\lambda_k) \neq \emptyset$ $\forall k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\lambda_m \leq h(x)$. So, $L_h(\lambda_m) \subset L_h(h(x))$, which implies that $(L_h(\lambda_m))^\infty \subset (L_h(h(x)))^\infty$, and so $u \in (L_h(h(x)))^\infty$. The statement follows.

Finally, we show that

$$\bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty \subset \bigcap_{k\in\mathbb{N}} \big(L_h(\lambda_k)\big)^\infty.$$

Indeed, take $u \in \bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty$, thus $u \in \big(L_h(h(y))\big)^\infty$ $\forall y \in \mathbb{R}^n$. For each $k \in \mathbb{N}$ (arbitrarily fixed), we have, by assumption, that $\lambda < \lambda_k$. Since $L_h(\lambda_k) \neq \emptyset$, take $x \in L_h(\lambda_k)$. So $h(x) \leq \lambda_k$, it implies that $L_h(h(x)) \subset L_h(\lambda_k)$ and then $\big(L_h(h(x))\big)^\infty \subset \big(L_h(\lambda_k)\big)^\infty$. Here, $u \in \big(L_h(\lambda_k)\big)^\infty$. The statement follows. □

Theorem 3.3. *Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. $S(h) \neq \emptyset$ if and only if $(S(h))^\infty = \bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty$.*

Proof. Consider $\lambda := \inf_{x\in\mathbb{R}^n} h(x)$. If $S(h) \neq \emptyset$ then $L_h(\lambda) = S(h) \neq \emptyset$. So, we have that

$$\bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty \subset \big(L_h(\lambda)\big)^\infty = \big(S(h)\big)^\infty,$$

and the statement follows.

Conversely, if $S(h) = \emptyset$, then $(S(h))^\infty = \{0\}$ and so $L_h(h(y))$ are unbounded $\forall y \in \mathbb{R}^n$. Now, take $u^k \in (L_h(\lambda_k))^\infty$ with $||u^k|| = 1$ $\forall k \in \mathbb{N}$, where $\lambda_k \to \lambda$ with $\lambda_k \geq \lambda_{k+1} > \lambda$ and $\lambda = \inf_{y\in\mathbb{R}^n} f(y)$. Since $\lambda_i \leq \lambda_k$ $\forall k \in \mathbb{N}$ and $\forall i \geq k$, we have that

$$(L_h(\lambda_i))^\infty \subset (L_h(\lambda_k))^\infty \quad \text{and} \quad \{u^i\}_{i\geq k} \subset (L_h(\lambda_k))^\infty \;\; \forall k \in \mathbb{N}$$

and so any cluster point of $\{u^k\}_{k\in\mathbb{N}}$ belongs to $(L_h(\lambda_k))^\infty$ $\forall k \in \mathbb{N}$. From Lemma 3.2, we have that any cluster point of $\{u^k\}_{k\in\mathbb{N}}$ belongs to $\bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty$, and so, the statement follows. □

The following theorem states necessary and sufficient conditions for the existence of at least one nonempty bounded level set. Here $\overline{co}$ means the closed convex hull.

Theorem 3.4. *For any given function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the following statements are equivalent:*

(a) $\bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty = \{0\}$.

(b) *There exists $y \in \mathbb{R}^n$ such that $L_h(h(y))$ is bounded.*

(c) *There exists $y \in \mathbb{R}^n$ such that $\overline{co} L_h(h(y))$ is bounded.*

Proof. We need to prove only that (a) implies (b), because the other equivalences follow from classical results of classical convex analysis. From Lemma 3.1, we have that

$$\{0\} \subset (S(h))^\infty \subset \bigcap_{y\in\mathbb{R}^n} \big(L_h(h(y))\big)^\infty = \{0\}$$

and from Theorem 3.3 we have that $S(h) \neq \emptyset$. Finally, the statement follows from the fact that $S(h) = L_h(\lambda)$, where $\lambda = \inf_{x\in\mathbb{R}^n} h(x)$. □

Now, we are able to introduce a new coerciveness notion for Optimization Theory.

Definition 3.5. *We say that* $h\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ *is a coercive function if there exists* $\lambda \in \mathbb{R}$ *such that* $L_h(\lambda)$ *is nonempty and bounded. From Theorem 3.4, this is equivalent to say that*

$$R(h) := \bigcap_{y\in\mathbb{R}^n} \left(L_h(h(y))\right)^{\infty} = \{0\} \tag{15}$$

We notice that, level boundedness implies coerciveness, but the converse is not true; consider, for example, the function $h\colon \mathbb{R}\to\mathbb{R}$ defined by $h(x) = -exp(-x^2)$.

Corollary 3.6. *If* $h\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ *is a coercive and lsc function, then h reaches its minimum value over* $\mathbb{R}^n$. *Moreover,* $S(h)$ *is nonempty and compact.*

In the convex case, all the previous ideas about coerciveness are equivalent under the lsc assumption on the function, which we can verify in the following result.

Proposition 3.7. *If* $h\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ *is a proper convex and lsc function, then the following statements are equivalent:*

(1) *h is a strong coercive function.*

(2) *h is a Inf-compact function.*

(3) *h is a level bounded function.*

(4) *h is a zero-coercive function.*

(5) *h is a coercive function.*

(6) $S(h)$ *is nonempty, convex and compact.*

Proof. The equivalence between (2) and (3) is evident from the lsc assumption. The equivalence between (3) and (5) follows from the fact that if a convex function has a nonempty bounded level set, then all the level sets are bounded. The remaining equivalences follow from Proposition 3.1.3 in [3]. □

Now, given $h\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, we consider the following assumptions:

(H1) $\forall\{x^k\}_{k\in\mathbb{N}} \subset dom(h)$ with $\lim ||x^k|| = +\infty$ and $\lim \frac{x^k}{||x^k||} = u \in R(h)$, $\exists m \in \mathbb{N}$ large enough such that $L_h(h(x^m)) \bigcap B(0, ||x^m||) \neq \emptyset$.

(H2) $\forall\{x^k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ with $\lim ||x^k|| = +\infty$, $\exists m \in \mathbb{N}$ large enough such that

$$L_h(h(x^m)) \bigcap B(0, ||x^m||) \neq \emptyset.$$

Conditions related to assumptions **H1** or **H2** can be found, for instance, in [1], [2], [4], [12], [13], [14] and references therein.

The following result is given without proof because it is evident.

Lemma 3.8. *Given $h \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. If h is coercive, then* **H1** *holds.*

Theorem 3.9. *If $h \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is proper and lsc, then* **H1** *holds if and only if the solution set of the minimization of h is nonempty, i.e., $S(h) \neq \emptyset$.*

Proof. If **H1** holds, for each $k \in \mathbb{N}$, consider $h_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h_k(x) = \begin{cases} h(x) & \text{if} \quad \|x\| \leq k \\ +\infty & \text{if} \quad \|x\| > k \end{cases}.$$

Without loss of generality, consider h_k proper $\forall k \in \mathbb{N}$. So, from Corollary 3.6 $S(h_k)$ is nonempty and compact $\forall k$. Now, take $x^k \in \text{argmin}\{\|x\| \colon x \in S(h_k)\} \subset dom(h)$. Suppose that the sequence is unbounded, without loss of generality we can consider $\|x^k\| \to \infty$ when $k \to \infty$ and $\frac{x^k}{\|x^k\|} \to u$. We see that $u \in R(h)$. Indeed, take $x \in \mathbb{R}^n$ arbitrarily fixed, then for all $m \geq \|x\|$, we have that $x^m \in L_h(h(x))$ and making $t_m = \frac{1}{\|x^m\|}$, we have that $t_m x^m \to u$ when $m \to \infty$ and so $u \in (L_h(h(x)))^\infty$. Since x is arbitrary, then $u \in R(h)$. From **H1**, $\exists m \in \mathbb{N}$ large enough such that $L_h(h(x^m)) \bigcap B(0, ||x^m||) \neq \emptyset$. Now, taking $y \in L_h(h(x^m)) \bigcap B(0, ||x^m||)$, then $h(y) \leq h(x^m)$ and $\|y\| < \|x^m\|$, and so $y \in S(h_m)$, which is a contradiction. Then the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded. If $\{x^k\}$ is bounded, the statement follows, because any cluster point of $\{x^k\}$ is a minimizer of h.

If $S(h) \neq \emptyset$, take $\{x^k\}_{k \in \mathbb{N}} \subset dom(h)$ with $\lim_{k \to +\infty} ||x^k|| = +\infty$ and $\lim \frac{x^k}{||x^k||} = u \in R(h)$. The statement follows taking $\bar{x} \in S(h)$ and $k \in \mathbb{N}$ such that $\|x^k\| > \|\bar{x}\|$. □

Theorem 3.10. *If $h \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper and lsc, then* **H1** *is equivalent to* **H2**.

Proof. We need to prove only that **H1** implies **H2**. Take $\{x^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ with $\lim_{k \to +\infty} ||x^k|| = +\infty$. Since **H1** holds and h is lsc, then from Theorem 3.9 we have that $S(h) \neq \emptyset$. The statement follows taking $\bar{x} \in S(h)$ and $k \in \mathbb{N}$ such that $\|x^k\| > \|\bar{x}\|$. □

Since **H1** and **H2** are equivalent when h is proper and lsc, from now on, we shall refer to one of them as just H.

3.1 The application to linear constrained quadratic minimization

We finish this section with an application of the introduced coerciveness ideas to the minimization of a quadratic function over nonempty polyhedral set.

Here, we want to minimize $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined as:

$$h(x) = \begin{cases} \frac{1}{2}x^T Ax + a^T x + \alpha & \text{if} \quad x \in C := \{y : By \leq b\}, \\ +\infty & \text{if} \quad x \notin C \end{cases}, \tag{16}$$

where A is a $n \times n$ matrix, B is a $p \times n$ matrix, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^p$ and $\alpha \in \mathbb{R}$ are known given data.

Note that in (16), $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is proper and lsc on $\mathbb{R}^n$.

The following result will be used in the sequel.

Lemma 3.11. *If* $\inf_{x\in\mathbb{R}^n} h(x) = \beta > -\infty$*, then for each* $x \in C$ *and each* $u \in R(h)$ *fixed, the application* $l : [0, +\infty[\to \mathbb{R}$ *defined by* $l(t) = h(x + tu)$ *is linear and* $l(t) \geq l(0)$ *for all* $t \in [0, +\infty[$.

Proof. Take $d \in R(h)$ arbitrarily fixed, then $d \in \big(L_h(h(x))\big)^\infty$ $\forall x \in \mathbb{R}^n$. Now, take $x \in C$ arbitrarily fixed, then $\exists\{y^k\}_{k\in\mathbb{N}} \subset L_h(h(x))$, $\exists\{t_k\}_{k\in\mathbb{N}} \downarrow 0$ such that $t_k y^k \to u$ when $k \to +\infty$. So, $u \in C^\infty$ and $-\infty < \beta \leq h(y^k) = \frac{1}{2}(y^k)^T Ay^k + a^T y^k + \alpha \leq h(x) < +\infty$ $\forall k \in \mathbb{N}$, implying $u^T Au = 0$. Since $(x+tu) \in C$ $\forall t > 0$, then $l(t) = h(x+tu) = h(x)+t\big[\frac{1}{2}x^T Au+\frac{1}{2}u^T Ax+a^T u\big]$ $\forall t > 0$. So, the application l is linear on $[0, +\infty[$. Here, $l(t) \geq l(0)$ $\forall t \geq 0$, because $l(t) = h(x + tu) \geq \beta > -\infty$ $\forall t \geq 0$. □

Theorem 3.12. *Consider the minimization of h as defined in (16). Then*

$$\inf_{x\in\mathbb{R}^n} h(x) = \beta > -\infty$$

if and only if the assumption H holds.

Proof. If $dom(h)$ is bounded, then H holds. If not, take $\{x^m\}_{m\in\mathbb{N}} \subset C$ such that $\|x^m\| \to +\infty$ and $\frac{x^m}{\|x^m\|} \to u \in R(h)$ when $m \to +\infty$. So, from Lemma 3.11, we have that $h(x) \leq h(x + tu)$ $\forall t \geq 0$. From Proposition 2.3 in [2], we have that for any $t > 0$, there exists $k \in \mathbb{N}$ large enough such that $(x^k - tu) \in C$. Taking $y = x^k - tu$, we have that $h(y) \leq h(y + tu) = h(x^k)$. Note that, for k large enough we have that $\|y\| = \|x^k - tu\| < \|x^k\|$, and so H holds.

Conversely, the statement follows from the lsc of h, H and Theorem 3.9. □

Finally, we state that the well known Frank-Wolf Theorem is a consequence of Theorems 3.9 and 3.12.

Corollary 3.13. *(Frank-Wolf Theorem). Consider the minimization of h as defined in (16). If h is bounded below, then* $S(h) \neq \emptyset$.

4 COERCIVENESS FOR EQUILIBRIUM PROBLEMS

In the Introduction, we showed that the solution set of EP can be written as an intersection of a set family, where the index set is the set K, and, for each $y \in K$, the respective set is the 0-level set of the function f^y (4). In other words, it follows that $\bar{x} \in S(EP)$ (3) if and only if $\bar{x} \in \bigcap_{y\in K} L_{f^y}(0)$.

Now, we present the contents of the Ky Fan's Lemma.

Lemma 4.1. *(Lemma 1 in [9], Ky Fan's Lemma). Given* $Y \neq \emptyset$ *and for each* $y \in Y$ *consider* $C(y)$ *as a closed set. If the following two conditions hold:*

(C1) *For any finite subset* $\{x^1, \dots, x^q\} \subset Y$*:* $co\{x^1, \dots, x^q\} \subset \bigcup_{i=1}^{q} C(x^i)$,

(C2) $C(x)$ *is compact for at least some* $x \in Y$,

then $\bigcap_{y \in Y} C(y) \neq \emptyset$.

In the sequel, let us denote EP by EP(f,K) and consider the following assumptions for the function f.

(f0) f^y is lsc $\forall y \in K$,

(f1) For any finite set $\{x^1, \dots, x^q\} \subset K$ and for any $x \in co\{x^1, \dots, x^q\}$:

$$\max_{i=1,\dots,q} f(x, x^i) \geq 0.$$

Remark 4.2. We point out that assumptions **f0** and **f1** are related to the Ky Fan's Lemma. Indeed, taking $Y = K$ and $C(y) = L_{f^y}(0)$ for each $y \in Y$

1. Assumption f1 is equivalent to condition C1.
2. Assumption f0 implies that C(y) is a closed for each $y \in Y$.
3. If K is convex, $f(x, x) \geq 0$ for each $x \in K$, and the function $f_x : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by
$$f_x(y) = \begin{cases} f(x, y) & \text{if } \ y \in K \\ +\infty & \text{if } \ y \notin K \end{cases} \tag{17}$$
is quasi-convex, then f1 holds, as well as C1.

Given EP(f,K), we introduce the coerciveness notion for it, using the notation of the previous section, as follow.

Definition 4.3. *We say that EP(f,K) is:*

1. *Strong coercive if there exist* $y \in K$ *such that* f^y *is strong coercive.*
2. *Inf-compact if there exist* $y \in K$ *such that* f^y *is Inf-compact.*
3. *Level bounded if there exist* $y \in K$ *such that* f^y *is level bounded.*
4. *Coercive if there exist* $y \in K$ *such that* $L_{f^y}(0)$ *is bounded.*

Remark 4.4. We can find in the literature some extensions of classical coerciveness ideas that were introduced in the setting of the Variational Inequality Problem (VIP), which is defined by the set K and the operator T, and consists of

$$\text{VIP(T,K): Find } x \in K \quad \text{such that} \quad \langle T(x), y - x \rangle \geq 0 \quad \text{for all} \quad y \in K.$$

We cited a couple of the these developed coerciveness ideas by applying recession techniques to the operator $T: K \subset X \to X^*$ (here, X^* denote the dual topological space of X):

1. Harker and Pang [11] consider the following condition for some $\bar{y} \in K$
$$\lim_{\|x\|\to+\infty,\, x\in K} \frac{\langle T(x), x-\bar{y}\rangle}{\|x\|} = +\infty \tag{18}$$

2. Guo and Yao [10] consider another condition for some $\bar{y} \in K$
$$\liminf_{\|x\|\to+\infty,\, x\in K} \langle T(x), x-\bar{y}\rangle > 0 \tag{19}$$

The following trivial result is given without proof.

Lemma 4.5. *Given VIP(T,K), consider conditions (18) and (19). Take* $f(x, y) = \langle T(x), y - x\rangle$*, it follows that*

1. *Condition (18) implies that EP(f,K) is strong coercive.*

2. *Condition (19) implies that EP(f,K) is level bounded.*

Existence results for EP(f,K) with K compact have been introduced for the first time in 1972, with the formulation of the famous Minimax Theorem by Ky Fan (see [8]). In the same year, Brezis, Nirenberg and Stampacchia extend the Minimax Theorem to non compact sets (see [7]) using the following assumption regarding $f: K \times K \to \mathbb{R}$:

(BNS) there exists a nonempty compact subset L of $\mathbb{R}^n$ and $\bar{y} \in L \cap K$ such that for every $x \in K \setminus L$, $f(x, \bar{y}) < 0$.

As a consequence, we have the following result.

Lemma 4.6. *Given EP(f,K).*

1. *If K is compact, then EP(f,K) is strong coercive.*

2. *If BNS holds, then EP(f,K) is coercive.*

Recall that $R(h)$ (15) was set as an upper bound to the recession cone of the solution set of the minimization problem. As the solution set of EP(f,K) is the intersection of $L_{f^y}(0)$, for all $y \in K$, analogous to $R(h)$ we define
$$RE(f) = \bigcap_{y\in K} \left(L_{f^y}(0)\right)^{\infty}. \tag{20}$$

The following Lemma is given for completeness.

Lemma 4.7. *Given EP(f,K). If f0 is fulfilled, then the following statements hold.*

1. *If EP(f,K) is strong coercive, then EP(f,K) is Inf-compact.*
2. *If EP(f,K) is Inf-compact, then EP(f,K) is level bounded.*
3. *If EP(f,K) is level bounded, then EP(f,K) is coercive.*
4. *If EP(f,K) is coercive, then* $RE(f) = \{0\}$.

Theorem 4.8. *Given EP(f,K) with K nonempty and closed, and assume that f0 and f1 hold. If* $RE(f) = \{0\}$*, then the solution set of EP(f,K), denoted by S(EP), is compact and nonempty.*

Proof. Since $S(EP) = \bigcap_{y\in K} L_{f^y}(0)$, $\left(\bigcap_{y\in K} L_{f^y}(0)\right)^{\infty} \subset R = \{0\}$ and f0 hold, we have that $S(EP)$ is closed and bounded and so it is compact. In order to prove that $S(EP)$ is nonempty, we define for each $i \in \mathbb{N}$, $K_i = \{x \in K : \|x\| \le i\}$. Without loss of generality, we can consider $K_i \ne \emptyset$ for all $i \in \mathbb{N}$. The compactness of K_i, f0 and f1 imply that $EP(f, K_i)$ satisfies C1 and C2 of Ky Fan's Lemma and so we can take $x^i \in S(EP(f, K_i))$. Suppose that the sequence $\{x^i\}$ is unbounded, without loss of generality, consider that $\|x^i\| \to +\infty$ and $\frac{x^i}{\|x^i\|} \to u$ when $i \to +\infty$. Now take $y \in K$ arbitrarily fixed, then

$$f(x^i, y) \ge 0 \ \ \forall i \ge \|y\|, \quad \text{and so} \quad \{x^i\}_{i\ge\|y\|} \subset L_{f^y}(0).$$

It implies that $u \in (L_{f^y}(0))^{\infty}$. Since $y \in K$ is arbitrary, then $0 \ne u \in R$. This contradiction implies that the sequence $\{x^i\}$ is bounded. Finally, any cluster point of $\{x^i\}$ is a solution of EP(f,K). □

Before extending the assumption H to the setting of the Equilibrium Problems, we consider the definition of function $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$h(x) = \sup_{y\in K} f^y(x). \tag{21}$$

Note that $-h(x) = \inf_{y\in K} f(x, y)$ and $-\inf_{x\in\mathbb{R}^n} h(x) = \sup_{x\in K} \inf_{y\in K} f(x, y)$.

Proposition 4.9. *Given EP(f,K) and the definition of h as in (21). If K is compact and f satisfies f0, then*

$$EP(f, K) \text{ has solution if and only if } \inf_{x\in\mathbb{R}^n} h(x) \le 0.$$

Proof. Let $\bar{x}$ be a solution of EP(f,K), then we have that $f(\bar{x}, y) \ge 0$. Thus, $\inf_{y\in K} f(\bar{x}, y) \ge 0$ and so $-\inf_{x\in\mathbb{R}^n} h(x) = \sup_{x\in K} \inf_{y\in K} f(x, y) \ge 0$.

Conversely, from f0 the application h is lsc. On the other hand, K compact implies that exists $\bar{x}$ such that $0 \le -\inf_{x\in\mathbb{R}^n} = -h(\bar{x}) = \inf_{y\in K} f(\bar{x}, y)$ and so $\bar{x}$ is solution of EP(f,K). □

If we replace the compactness of K by the assumption H for the function h, we have the following result that is given without proof, since it is a direct consequence of Proposition 4.9.

Theorem 4.10. *Given EP(f,K). If f satisfies f0 and h defined as in (21) satisfies H, then EP(f,K) has solution if and only if* $\inf_{x\in\mathbb{R}^n} h(x) \le 0$.

Given $f: K \times K \to \mathbb{R}$, we consider for each $x \in K$ the following sets, called Lower Diagonal level sets of the function f.

$$LD_f(x) := \{y \in K : f(x, y) \leq 0\}.$$

Note that, if $x \notin L_{f^y}(0)$, then $y \in LD_f(x)$.

Now, we are able to extend the assumptions H (introduced for Optimization Problems) to an assumption for Equilibrium problems as follows:

(F1) $\forall \{x^k\}_{k\in\mathbb{N}} \subset K$ with $\lim ||x^k|| = +\infty$ and $\lim \frac{x^k}{||x^k||} = u \in RE(f)$, there exists $m \in \mathbb{N}$ such that $LD_f(x^m) \bigcap B(0, ||x^m||) \neq \emptyset$.

(F2) $\forall \{x^k\}_{k\in\mathbb{N}} \subset K$ with $\lim ||x^k|| = +\infty$, there exists $m \in \mathbb{N}$ such that

$$LD_f(x^m) \bigcap B(0, ||x^m||) \neq \emptyset.$$

Theorem 4.11. *Given a nonempty set K and $f : K \times K \to \mathbb{R}$. The assumptions F1 and F2 are equivalent.*

Proof. We need only to show that F1 implies F2. Indeed, take $\{x^k\}_{k\in\mathbb{N}} \subset K$ with $\lim ||x^k|| = +\infty$. Without loss of generality, we can consider that $\lim \frac{x^k}{||x^k||} = u$. If $u \in RE(f)$, then the statement follows from F1. In other case, $u \notin RE(f)$, so there exists $y \in K$ such that $u \notin (L_{f^y}(0))^\infty$. It implies that there exists $N \in \mathbb{N}$ such that $\forall k \geq N$, $x^k \notin L_{f^y}(0)$. Finally, we can take k large enough such that $\|y\| < \|x^k\|$ and $y \in DL_f(x^k)$. □

Since F1 and F2 are equivalent, then hereafter we refer to one of them as just F.

The notion of pseudo-convexity for functions in the setting of real vectorial spaces (without topological structure) appears in [13] as follows, given a function $h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we say that f is pseudo-convex if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $t \in]0, 1[$: $h(tx + (1-t)y) \geq f(x)$ implies $h(tx + (1-t)y) \leq f(y)$. So, now we consider the following assumption for f.

(f1') For all $x \in K$, f_x is pseudo-convex and $f(x, x) \geq 0$.

It is easy to see that convexity implies pseudo-convexity and pseudo-convexity implies quasi-convexity. But, the converses are not true.

Theorem 4.12. *Given f satisfying* **f0**, **f1'** *and let K be a nonempty closed convex set. If F holds then EP(f,K) has solutions.*

Proof. From **f0**, **f1'** and the compactness of $K_i := \{x \in K : \|x\| \leq i\}$ we have from Ky Fan's Lemma that the Equilibrium Problem defined by f and K_i ($EP(f, K_i)$) has solutions for all $i \in \mathbb{N}$ (without loss of generality, we can consider that $K_i \neq \emptyset$). Take a solution x^i of $EP(f, K_i)$.

If the sequence $\{x^i\}$ is unbounded, without loss of generality, we can consider $\|x^i\| \to +\infty$, when $i \to +\infty$. So, from assumption F, there exists $m \in \mathbb{N}$ such that $LD_f(x^m) \bigcap B(0, ||x^m||)$

$\neq \emptyset$. Take $y \in LD_f(x^m) \bigcap B(0, ||x^m||)$, then $f(x^m, y) \leq 0$ and $\|y\| < \|x^m\|$. Now taking $w \in K$ arbitrary fixed, there exists $t \in]0, 1[$ such that $w_t = tw + (1-t)y \in K_m$. Then, $f(x^m, w_t) \geq 0 \geq f(x^m, y)$, from f1' we have $0 \leq f(x^m, w_t) \leq f(x^m, w)$. Since $w \in K$ is arbitrary, we have that x^m is a solution of EP(f,K).

If $\{x^i\}$ is bounded, from f0, we have that any cluster point of $\{x^i\}$ is solution of EP(f,K). □

Finally consider the following problem:

Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be two nonempty convex polyhedral sets; let $L : U \times V \to \mathbb{R}$ be the Lagrangian function defined by

$$L(x, y) = \frac{1}{2}\langle x, Px\rangle - \frac{1}{2}\langle y, Qy\rangle - \langle y, Sx\rangle, \tag{22}$$

where S is a real matrix of order (n, m), and P (resp. Q) is a real symmetric matrix of order (m, m) (resp. (n, n)). We are interested in solving the Saddle Point Problem (SPP) formulated as follows:

$$\text{find } (\bar{x}, \bar{y}) \in U \times V \quad \text{such that} \quad L(\bar{x}, y) \leq L(\bar{x}, \bar{y}) \leq L(x, \bar{y})\ \forall (x, y) \in U \times V. \tag{23}$$

SPP(L,U,V) is a particular case of the Equilibrium Problem. Indeed, let us consider $K = U \times V$, $u = (x, y)$ and $v = (w, z)$, and let us define the function $f\colon K \times K \to \mathbb{R}$ by

$$f(u, v) = L(w, y) - L(x, z).$$

We now consider the following equilibrium problem

$$\text{find } \bar{u} \in K \quad \text{such that} \quad f(\bar{u}, v) \geq 0 \quad \text{for all} \quad v \in K. \tag{24}$$

Proposition 4.13. *$(\bar{x}, \bar{y})$ is a solution of SPP(L,U,V) (23) if, and only if, $\bar{u} = (\bar{x}, \bar{y})$ is a solution of EP(K,f) (24).*

Take

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & S^t \\ -S & 0 \end{pmatrix}$$

We notice that A is a symmetric matrix and $B^t = -B$. One can verify that K is a polyhedral set and

$$f(u, v) = \frac{1}{2}\langle v, Av\rangle - \frac{1}{2}\langle u, Au\rangle + \langle u, Bv\rangle. \tag{25}$$

Lemma 4.14. *Given L as defined in (22) and f as defined in (25). If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are nonempty convex polyhedral sets and $K = U \times V \subset \mathbb{R}^{n+m}$, then the following statements are equivalent.*

1. $\inf_{x \in U} \sup_{y \in V} L(x, y) = \sup_{y \in V} \inf_{x \in U} L(x, y) \in \mathbb{R}$.
2. $\inf_{v \in K} \sup_{u \in K} f(u, v) = 0 = \sup_{u \in K} \inf_{v \in K} f(u, v)$.

3. $\sup_{u \in K} \inf_{v \in K} f(u, v) = 0$.
4. $\inf_{v \in K} \sup_{u \in K} f(u, v) = 0$.

Theorem 4.15. *Given EP(f,K) as defined in (24). It follows that*

$$\textit{EP(f,K) has solutions if and only if } \sup_{u \in K} \inf_{v \in K} f(u, v) = 0.$$

Proof. We only need to prove that $\sup_{u \in K} \inf_{v \in K} f(u, v) = 0$ implies that EP(f,K) has solutions. Indeed, take $h\colon \mathbb{R}^{n+m} \to \mathbb{R} \cup \{+\infty\}$ defined by $h(u) = \sup_{v \in K} f^v(u)$. Note that $\inf_{u \in \mathbb{R}^{n+m}} h(u) = -\sup_{u \in K} \inf_{v \in K} f(u, v) = 0$. The statement follows from Theorem 4.10, if we prove that H holds. By contradiction, there exists $\{u^k\} \subset dom(h)$ with $\|u^k\| \to +\infty$ and $\frac{u^k}{\|u^k\|} \to d \in R(h)$, such that for all k we have that $L_h(h(u^k) \cap B(0, \|u^k\|) = \emptyset$, i.e., for each k we have that $h(u) > h(u^k)$ for all $\|u\| < \|u^k\|$. From Proposition 2.3 in [2], take $t > 0$ such that for k large enough $\|u^k - td\| < \|u^k\|$, $u^k - td \in K$, then $h(u^k - td) > h(u^k)$. Since $h(v) = \sup_{u \in K} f(u, v)$, then from the continuity of f, there exists $\bar{v} \in dom(h)$ such that $h(u^k - td) \geq f(\bar{v}, u^k - td) > h(u^k) \geq f(\bar{v}, u^k)$. Taking $u = u^k - td$, we have that $f(\bar{v}, u) > f(\bar{v}, u + td)$, but this is a contradiction to Lemma 3.11. □

Corollary 4.16. *The Lagrangian function L (22) has saddle points on a polyhedral set $U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ if and only if*

$$\inf_{x \in U} \sup_{y \in V} L(x, y) = \sup_{y \in V} \inf_{x \in U} L(x, y) \in \mathbb{R}.$$

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REFERENCES

[1] ADLER S, GOELEVEN D & THÉRA M. 1996. Recession mappings and noncoercive variational inequalities. *Nonlinear Analysis, Theory, Methods and Applications*, **26**: 1573–1603.

[2] AUSLENDER A. 1996. Noncoercive optimization problems. *Mathematics of Operation Research*, **21**: 769–782.

[3] AUSLENDER A & TEBOULLE M. 2003. Asymptotic cones and functions in Optimization and Variational Inequalities. Springer Monographs in Mathematics.

[4] BAIOCCHI C, BUTTAZO G, GASTALDI F & TOMARELLI F. 1988. General existence theorem for unilateral problems in continuum mechanics. *Archive for Rational Mechanics and Analysis*, **100**: 149–189.

[5] BIANCHI M & PINI R. 2005. Coercivity conditions for equilibrium problems. *Journal of Optimization Theory and Applications*, **124**: 79–92.

[6] Blum E & Oettli W. 1994. From optimization and variational inequalities to equilibrium problems. *The Mathematics Student*, **63**: 123–145.

[7] Brezis H, Nirenberg L & Stampacchia G. 1972. A remark on Ky Fan's minimax principle. *Bolletino della Unione Matematica Italiana*, **6**: 293–300.

[8] Fan K. 1972. A minimax inequality and applications, in O. Shisha (Ed.), Inequality III, Academic Press, New York, pp. 103–113.

[9] Fan K. 1961. A generalization of Tychonoff's fixed point theorem, *Mathematische Annalen*, **142**: 305–310.

[10] Guo JS & Yao JC. 1994. Variational inequalities with monotone operators. *Journal of Optimization Theory and Applications*, **80**: 63–74.

[11] Harker PT & Pang JS. 1990. Finite dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Mathematical Programming*, **48**: 161–220.

[12] Iusem AN & Sosa W. 2003. New existence results for equilibrium problems. *Nonlinear Analysis*, **52**: 621–635.

[13] Iusem AN, Kassay G & Sosa W. 2009. On certain conditions for the existence of solutions of equilibrium problems. *Journal of Mathematical Programming*, serie B **116**: 259–273.

[14] Iusem AN, Kassay G & Sosa W. 2009. An existence result for equilibrium problems with some surjectivity consequences. *Journal of Convex Analysis*, **16**: 807–826.

[15] Raupp FMP & Sosa W. 2012. Semi-continuous programming: existence conditions and duality scheme, working paper of 2012.

[16] Rockafellar RT & Wets RJ-B. 1998. Variational Analysis, Springer Verlag, New York.