Pesquisa Operacional (2014) 34(1): 73-89

Printed version ISSN 0101-7438 / Online version ISSN 1678-5142
www.scielo.br/pope

A METHOD FOR SOLVING LINEAR PROGRAMMING MODELS WITH INTERVAL TYPE-2 FUZZY CONSTRAINTS

Juan Carlos Figueroa-García[1*] and Germán Hernández[2]

Received December 20, 2012 / Accepted January 18, 2014

ABSTRACT. This paper shows a method for solving linear programming problems that includes Interval Type-2 fuzzy constraints. The proposed method finds an optimal solution in these conditions using convex optimization techniques. Some feasibility conditions are presented, and some interpretation issues are discussed. An introductory example is solved using the proposed method, and its results are described and discussed.

Keywords: Fuzzy Linear Programming, Interval Type-2 fuzzy sets, Fuzzy Optimization.

1 INTRODUCTION

Some practical applications such as financial, logistics, Markov chains, control, etc, include non-probabilistic uncertainty. This way, decision making has to deal with uncertainty using appropriate methods and models to find a solution of the problem. Linear Programming (LP) is within the most useful tools in decision making, and its application to non-deterministic problems has proved to be efficient (stochastic programming, fuzzy linear programming, etc). This opens the possibility of involving new uncertainty sources to problems where statistical information is not reliable or is absent through new theories such as fuzzy sets.

In some cases, availability of the resources of a system cannot be measured in an exact way (*i.e.* mathematical precision), due to different issues. Moreover, available information usually comes from the experts of the system, so decision making is intimately related to their perceptions and expertise. This information represents the knowledge of the experts about the availability of the resources of the system (a.k.a as constraints), which can be measured using fuzzy sets.

A special kind of LP models known as Fuzzy Linear Programming (FLP) models include fuzzy constraints. Roughly speaking, fuzzy constrained problems deal with non-probabilistic uncertainty, which is a common practical issue that needs special methods at different complexity levels.

*Corresponding author
[1] Universidad Distrital Francisco José de Caldas – Universidad Nacional de Colombia.
[2] Universidad Nacional de Colombia.
E-mails: jcfigueroag@udistrital.edu.co; gjhernandezp@gmail.com

Some FLP models have been proposed by Ghodousiana & Khorram [8], Guu & Wu [9], Tanaka & Asai [29], Tanaka, Asai & Okuda [30], Inuiguchi & Ramík [11], and Inuiguchi & Sakawa [12, 13] who proposed solutions for several fuzzy sets, all of them considering only Type-1 fuzzy sets. An intuitionistic fuzzy optimization approach has been presented by Angelov [2] and Dubey *et al.* [3], based on the idea of using two measures $\mu_A(x)$ and $\upsilon_A(x)$ to represent both membership and non-membership degrees of x regarding a concept A, constrained to $0 \leq \mu_A(x) + \upsilon_A(x) \leq 1$, which is similar to an Interval Type-2 fuzzy set in the sense that the distance between $\mu_A(x)$ and $\upsilon_A(x)$ can be shown as an interval.

In this paper, we propose an extension of the FLP method proposed by Zimmermann [32, 33] (originally designed for Type-1 fuzzy constrained problems) to an Interval Type-2 FLP (IT2FLP) with linear membership functions. Our proposal uses Type-2 fuzzy numbers instead of pure intervals or intuitionistic fuzzy sets (even when they are uncertainty measures as well) to address non-probabilistic information coming from multiple experts.

The paper is divided into six sections. In Section 1, the Introduction and Motivation is presented. In Section 2, the classical LP model with fuzzy constraints is presented. In Section 3, some elements of linguistic uncertainty, in particular Type-2 fuzzy constraints are introduced. In Section 4, a formal definition of an Interval Type-2 FLP model is provided, and Section 5 presents an optimization method. In Section 6, an illustrative application is introduced, and finally Section 7 presents some concluding remarks.

1.1 Motivation of using Type-2 FLP sets

Many decision making problems involve uncertainty, so the analyst has to deal with it (probabilistic and/or possibilistic) in different ways. In LP problems, all its parameters (costs, technological coefficients and constraints) can contain uncertainty, so different approaches can be used when different uncertainty sources appear. As usual, as more uncertainty sources are involved, more complex its modeling is (including the algorithms for finding a solution). A typical uncertainty source comes from the concept of a constraint, which is commonly assumed as deterministic (in some cases, probabilistic). But, what if the constraints of the problem are defined by experts, or are they based on non-probabilistic information?

A specific kind of linguistic uncertainty can be considered when having experts' judgements. This uncertainty appears when different perceptions of a concept are provided by different people, like the one arising when multiple experts (with equally valuable opinions) are defining the constraints of an LP problem. To do so, Interval Type-2 Fuzzy Sets (IT2FS) seem to be an appropriate representation of this uncertainty, so we propose its use to deal with the perception of multiple experts who define the constraints of an FLP problem.

A Type-2 fuzzy set is a more complex uncertainty measure, so it needs a specific mathematical framework. This way, we propose a method for reducing its complexity using a Type-reduction strategy that consists on finding a fuzzy set embedded into a Type-2 fuzzy set, in order to apply convex optimization techniques, which is highly desirable by decision makers since it ensures interpretability and simplicity.

2 THE ZIMMERMANN'S SOFT CONSTRAINTS MODEL

Zimmermann [32] and [33] proposed the following LP model which includes Type-1 fuzzy constraints:

$$\begin{aligned} \max_{x \in X} \; & z = c'x + c_0 \\ & s.t. \\ & Ax \lesssim B \\ & x \geqslant 0 \end{aligned} \tag{1}$$

where $x, c \in \mathbb{R}^m$, $c_0 \in \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$. B is a vector of $i \in \mathbb{N}_m$ fuzzy numbers, where a fuzzy number is a fuzzy set defined over the real numbers (see Fig. 1), and $\lesssim$ is a fuzzy partial order.[1] $\mathbb{N}_m$ is the amount of constraints of the problem.

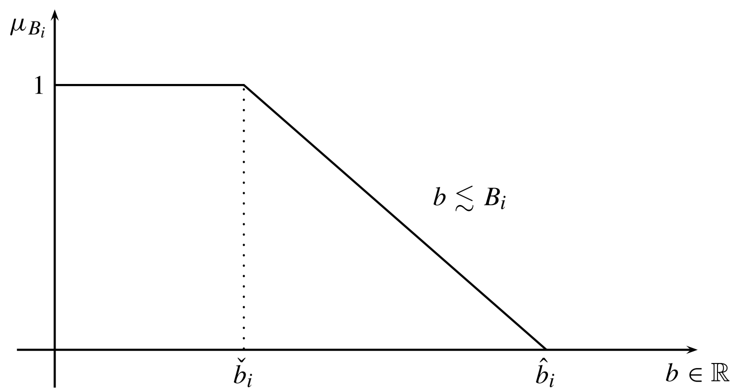

Figure 1 – Fuzzy set B_i.

Zimmermann proposed a method for solving this fuzzy constrained problem based on two requirements: B is defined as a vector of $\mathbb{N}_m$ L-R fuzzy numbers with linear membership functions $\tilde{B}_i$, $i \in \mathbb{N}_m$, and B is characterized by a single membership function. B is defined by two parameters $\check{b}_i$ and $\hat{b}_i$ (see Figure 1), and the remaining parameters are constants. The method is as follows:

Algorithm 1

1. Define a fuzzy set Z with parameters $\check{z}$ and $\hat{z}$.
2. Compute $\check{z} = \max\{c'x \mid Ax \leqslant \check{b}, x \geqslant 0\}$ as lower boundary of Z.

[1] Usually B is a linear fuzzy number, but there is the possibility of using nonlinear shapes.

3. Compute $\hat{z} = \max\{c'x \mid Ax \leqslant \hat{b}, x \geqslant 0\}$ as upper boundary of Z. Given a maximizing goal, then the membership function of $Z(x)$ is:

$$\mu_Z(x; \check{z}, \hat{z}) = \begin{cases} 1, & c'x \geqslant \hat{z} \\ \dfrac{c'x - \check{z}}{\hat{z} - \check{z}}, & \check{z} \leqslant c'x \leqslant \hat{z} \\ 0, & c'x \leqslant \check{z} \end{cases} \tag{2}$$

Its graphical representation is:

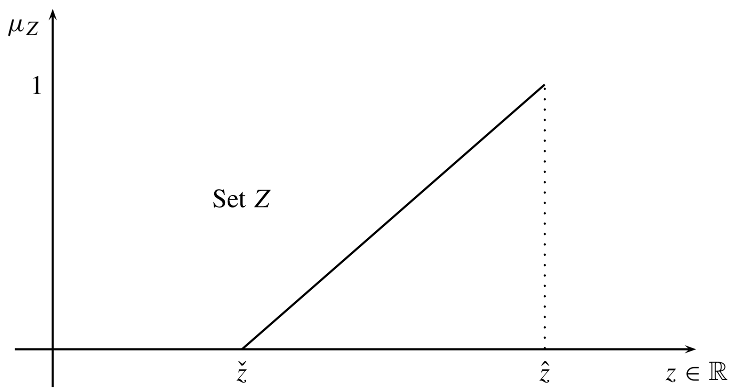

Figure 2 – Fuzzy set Z.

4. Create an auxiliary variable α and solve the following model:

$$\begin{gathered} \max\ \{\alpha\} \\ s.t. \\ c'x + c_0 - \alpha(\hat{z} - \check{z}) = \check{z} \\ Ax + \alpha(\hat{b} - \check{b}) \leqslant \hat{b} \\ x \geqslant 0,\ \alpha \in [0, 1] \end{gathered} \tag{3}$$

This method sets α as a *global satisfaction degree* of all constraints regarding the fuzzy set of optimal solutions Z. In fact, α operates as a balance point between the use of the resources (denoted by the constraints of the problem) and the desired profits (denoted by z), since more resources usage imply higher profits, at different uncertainty degrees. Then, the main idea of this method is to find an overall satisfaction degree of both goals (profits vs. resource usage) that maximizes the global satisfaction degree, *i.e.* minimizing the global uncertainty.

Note that the set $Z(x^*)$ represents the set of all optimal solutions regarding the goal. In other words, a thick solution of the fuzzy problem (see Kall & Mayer [14] and Mora [26]) where $[\check{z}, \hat{z}]$ are the boundaries of $Z(x^*)$.

3 INTERVAL TYPE-2 FUZZY CONSTRAINTS

As mentioned before, IT2FS allows to model linguistic uncertainty, *i.e.* the uncertainty about different perceptions and concepts. Mendel [15, 16, 18, 21, 24, 25] and Melgarejo [19, 20] provided formal definitions of IT2FS. Figueroa [4, 5, 6, 7] proposed an extension of the FLP which involves linguistic uncertainty using IT2FS called Interval Type-2 Fuzzy Linear Programming (IT2FLP). Some basic definitions include the following

3.1 Basics on interval Type-2 fuzzy sets

A Type-2 fuzzy set is a collection of Type-1 fuzzy sets into a single fuzzy set. It is defined by two membership functions: a *primary membership function* defines the degree of membership over a linguistic label, and a *secondary membership function* that weights every Type-1 fuzzy set embedded into the primary function. According to Mendel [21, 22, 23], some basic definitions of Type-2 fuzzy sets include the following:

Definition 3.1 (Type-2 fuzzy set). *A Type-2 fuzzy set, $\tilde{A}$, is:*

$$\tilde{A} = \int_{x\in X} \int_{u\in J_x} f_x(u)/(x,u) = \int_{x\in X} \left[\int_{u\in J_x} f_x(u)/u\right] \Big/ x, \tag{4}$$

where x is the *primary variable*, J_x is its *primary membership function*, $J_x \subseteq [0, 1]$, u is the *secondary variable* and $\int_{u\in J_x} f_x(u)/u$ is the *secondary membership function*. Uncertainty about $\tilde{A}$ is conveyed by the union of all of the primary memberships and is called the *Footprint Of Uncertainty* of $\tilde{A}$ (FOU($\tilde{A}$)), *i.e.*

$$\text{FOU}(\tilde{A}) = \bigcup_{x\in X} J_x \tag{5}$$

Therefore, an FOU weights all the embedded J_x by using a secondary membership function $f_x(u)/u$. In a *General* Type-2 fuzzy set *(GT2FS)*, $f_x(u)/u$ is defined by a Type-1 membership function, while an *Interval Type-2* fuzzy set is a special GT2FS since its secondary membership function is 1 (one), $f_x(u) = 1$, as shown as follows

Definition 3.2 (Interval Type-2 fuzzy set). *An Interval Type-2 fuzzy set, $\tilde{A}$, is:*

$$\tilde{A} = \int_{x\in X} \int_{u\in J_x} 1/(x,u) = \int_{x\in X} \left[\int_{u\in J_x} 1/u\right] \Big/ x, \tag{6}$$

The FOU of $\tilde{A}$ is bounded by two membership functions: an *Upper* membership function *(UMF)* $\overline{\mu}_{\tilde{A}}$ and a *Lower* membership function *(LMF)* $\underline{\mu}_{\tilde{A}}$. Note that $\tilde{A}$ has embedded e sets (A_e) as well, so there is an infinite amount of A_e enclosed into the FOU of $\tilde{A}$. A graphical representation of an IT2FS, its FOU and A_e is shown in Figure 3.

Here, $\tilde{A}$ is an IT2FS defined over a set Ω, $supp(\tilde{A}) \in \Omega$, its support[2] $supp(\tilde{A})$ is the support of $\tilde{A}$, $supp(\tilde{A}) = [\check{\bar{a}}, \hat{\bar{a}}]$. $\mu_{\tilde{a}}$ is a linear Type-2 fuzzy set with parameters $\check{\bar{a}}$, $\hat{\bar{a}}$, $\underline{\check{a}}$, $\underline{\hat{a}}$ and $\overline{a}$. FOU is the *Footprint of Uncertainty* of $\tilde{A}$, and A_e is a Type-1 fuzzy set embedded on its FOU.

[2] Ω is a compact set which represents the domain of the variable *e.g.* speed, height, etcn and usually it is defined as a subset of the real numbers.

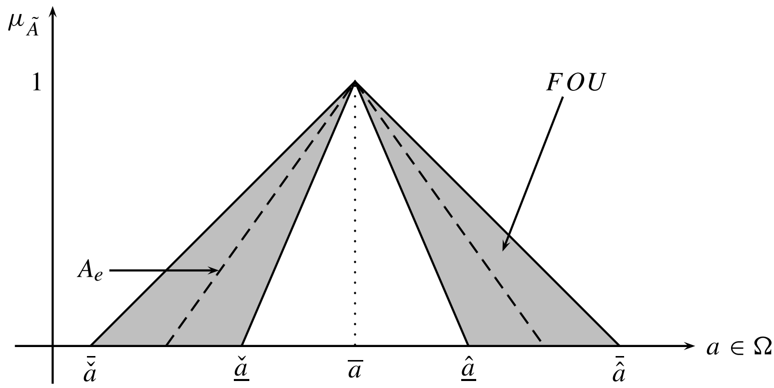

Figure 3 – Interval Type-2 Fuzzy set $\tilde{A}$.

3.2 Uncertain constraints

There are many ways to define the *"knowledgeability"* of an expert, so an infinite number of A_e fuzzy sets can be comprised into $FOU(\tilde{A})$. Each A_e is a representation of either the the knowledge of an expert about A or his perception about it. When multiple experts are defining a constraint, linguistic issues and multiple opinions about the same word A do appear, which is an uncertainty source itself.

Now we have defined what a Type-2 fuzzy set is, then an uncertain constraint can be defined as follows

Definition 3.3 (IT2FS Constraint – Figueroa [5]). *Consider a set of constraints of an FLP problem defined as an IT2FS called $\tilde{b}$ defined on the closed interval $\tilde{b}_i \in [\,\underline{b}_i, \overline{b}_i\,]$, $\{\underline{b}_i, \overline{b}_i\} \in \mathbb{R}$ and $i \in \mathbb{N}_m$. The membership function which represents $\tilde{b}_i$ is:*

$$\tilde{b}_i = \int_{b_i \in \mathbb{R}} \left[\int_{u \in J_{b_i}} 1/u \right] \Bigg/ b_i, \quad i \in \mathbb{N}_m, \ J_{b_i} \subseteq [0, 1] \tag{7}$$

Note that $\tilde{b}$ is bounded by both *Lower* and *Upper* primary membership functions, namely $\underline{\mu}_{\tilde{b}}(x)$ with parameters $\underline{\check{b}}$ and $\underline{\hat{b}}$ and $\bar{\mu}_{\tilde{b}}(x)$ with parameters $\bar{\check{b}}$ and $\bar{\hat{b}}$. Now, the *(FOU)* of the set $\tilde{b}$ can be composed by two distances called $\triangle$ and ∇, defined as follows.

Definition 3.4 (Figueroa [5]). *Consider an Interval FLP problem* (IFLP) *with restrictions in the form $\leqslant$. Then $\triangle$ is defined as the distance between $\underline{\check{b}}$ and $\bar{\check{b}}$, $\triangle = \bar{\check{b}} - \underline{\check{b}}$ and ∇ is defined as the distance between $\underline{\hat{b}}$ and $\bar{\hat{b}}$, $\nabla = \bar{\hat{b}} - \underline{\hat{b}}$.*

A graphical representation of $\tilde{b}_i$ is shown in Figure 4.

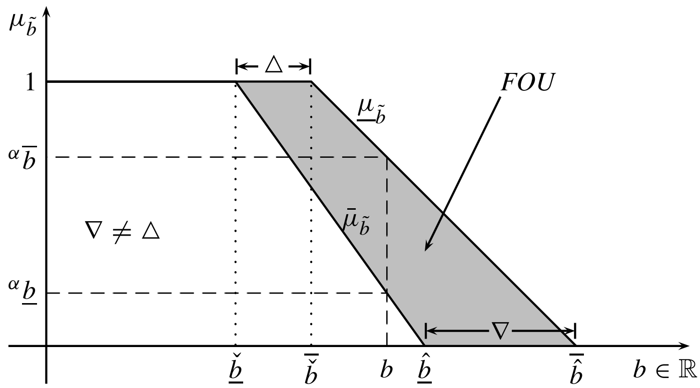

Figure 4 – IT2FS constraint with joint uncertain Δ & ∇.

In Figure 4, $\tilde{b}$ is an IT2FS with linear membership functions $\underline{\mu}_{\tilde{b}}$ and $\bar{\mu}_{\tilde{b}}$. A particular value b has an interval of infinite membership degrees $u \in J_b$, as follows

$$J_b \in \left[{}^{\alpha}\bar{\mu}_{\tilde{b}},\ {}^{\alpha}\underline{\mu}_{\tilde{b}} \right] \ \forall\ b \in \mathbb{R} \tag{8}$$

where J_b is the set of all possible membership degrees (u) associate to $b \in \mathbb{R}$. ${}^{\alpha}\bar{\mu}_{\tilde{b}}$ is the α-cut made over the upper membership function of $\tilde{b}$, and ${}^{\alpha}\underline{\mu}_{\tilde{b}}$ is the α-cut made over the upper membership function of $\tilde{b}$, where the α-cut of a fuzzy set b is defined as ${}^{\alpha}b = \{x \mid \mu_b(x) \geqslant \alpha\}$. Now, the FOU of $\tilde{b}$ can be composed by the union of all values of u, as defined as follows

Definition 3.5 (FOU of $\tilde{b}$). *As defined in* (8)*, it is possible to compose the footprint of uncertainty of $\tilde{b}$, $u \in J_b$ as follows:*

$$\text{FOU}(\tilde{b}) = \bigcup_{b \in \mathbb{R}} \left[{}^{\alpha}\bar{\mu}_{\tilde{b}},\ {}^{\alpha}\underline{\mu}_{\tilde{b}} \right] \ \forall\ b \in \tilde{b},\ u \in J_b,\ \alpha \in [0, 1] \tag{9}$$

Remark 3.1. Definition 3.2 presents an L-R Type-2 fuzzy set as the union of all possible L-R Type-1 fuzzy sets into its FOU. Definition 3.3 defines an uncertain constraint as a monotonic decreasing Type-2 fuzzy set which represents the statement "Approximately less or equal than b_i". In this way, we refer to an uncertain constraint as the IT2FS defined in Definitions 3.3 and 3.2 with a membership function as displayed in Figure 4.

The problem of having a type-2 fuzzy constrained problem cannot solved in a closed form, so there is a need for finding an appropriate solution. Some interesting ideas about the concept of an optimal solution in terms of the decision variables $x \in \mathbb{R}$ given uncertain constraints $\tilde{b}$, can be discussed. A first way would be to use Type-reduction to all IT2FS based on centroid methods, and afterwards solve the resultant interval-valued optimization problem. However, this is not recommendable because the centroid of an IT2FS constraint is usually outside its FOU. Another easy way is by using the *Center of FOU* which is simply to use the center of ∇ and Δ as extreme points of a fuzzy set embedded into the FOU of $\tilde{b}$, and then apply the Zimmermann's method. This method can be used in cases where the analyst has no a defuzzification criteria.

We have based our results in the Bellman-Zadeh fuzzy decision making principle, so the idea is to find a maximum intersection value between all constraints and Z. To do so, we need to provide some definitions of LP problems with IT2FS constraints in order to design a method for finding an optimal solution in terms of $x \in \mathbb{R}$ regarding z and $\tilde{b}$.

4 THE IT2FLP MODEL

Given the concept of an IT2FS constraint and the definition of an FLP, an uncertain constrained FLP model (IT2FLP) can be defined as follows:

$$\begin{aligned} \max_{x \in X} \; & z = c'x + c_0 \\ & s.t. \\ & Ax \precsim \tilde{b} \\ & x \geqslant 0 \end{aligned} \tag{10}$$

where $x, c \in \mathbb{R}^m$, $c_0 \in \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$. $\tilde{b}$ is an IT2FS vector defined by two primary membership functions $\underline{\mu}_b$ and $\bar{\mu}_b$. $\precsim$ is a Type-2 fuzzy partial order.

Two possible partial orders $\precsim$ and $\succsim$ can be used depending on the problem. We use only linear membership functions since we are going to use LP models (easy to optimize using classical algorithms), which means less complexity. The membership function of $\precsim$ (see Figure 4) is:

$$\underline{\mu}_{\tilde{b}}(x; \underline{\check{b}}, \underline{\hat{b}}) = \begin{cases} 1, & x \leqslant \underline{\check{b}} \\ \dfrac{\underline{\hat{b}} - x}{\underline{\hat{b}} - \underline{\check{b}}}, & \underline{\check{b}} \leqslant x \leqslant \underline{\hat{b}} \\ 0, & x \geqslant \underline{\hat{b}} \end{cases} \tag{11}$$

and its upper membership function (see Fig. 4) is:

$$\bar{\mu}_{\tilde{b}}(x; \bar{\check{b}}, \bar{\hat{b}}) = \begin{cases} 1, & x \leqslant \bar{\check{b}} \\ \dfrac{\bar{\hat{b}} - x}{\bar{\hat{b}} - \bar{\check{b}}}, & \bar{\check{b}} \leqslant x \leqslant \bar{\hat{b}} \\ 0, & x \geqslant \bar{\hat{b}} \end{cases} \tag{12}$$

A first approach for solving IT2FS problems is by reducing its complexity into a simpler form in order to use well known algorithms. In this case, we propose the following three-step methodology: 1– compute a fuzzy set of optimal solutions namely $\tilde{z}$; 2– apply a Type-reduction strategy to find a single fuzzy set Z; and 3– apply the Zimmermann's soft constraints method to find a crisp solution. This allows us to see the above problem as the problem of finding a vector of solutions $x \in \mathbb{R}^m$ such that:

$$\max_{x \in \mathbb{R}^n} \alpha \left\{ \bigcap_{i=1}^{m} \{\tilde{b}_i, \; b_i\} \bigcap \tilde{z} \right\} \tag{13}$$

where α is the α-cut made over all fuzzy constraints $\tilde{b}_i$ and $\tilde{z}$, defined as follows

$$\mu_{\tilde{z}}(\tilde{b})(z) = \sup_{z=c'x^*+c_0} \min_i \left\{ \mu_{\tilde{b}_i}(x^*) \mid x^* \in \mathbb{R}^m \right\} \tag{14}$$

Given $\mu_{\tilde{z}}$, the problem becomes in how to find the maximal intersection point between $\tilde{z}$ and $\tilde{b}$, where α is an auxiliary variable. In practice, the problem is solved by x^*, so α allows us to find x^* according to (13). The proposed methodology for using α over $\tilde{z}$ and $\tilde{b}$ to find x^* is presented in Figure 5.

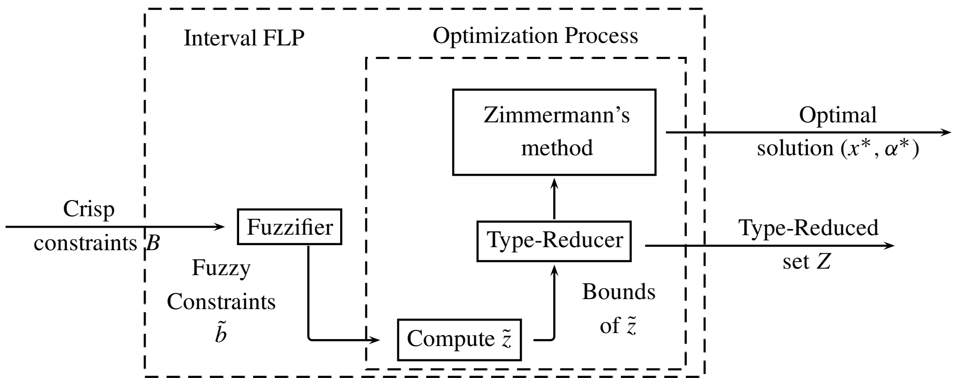

Figure 5 – IFLP proposed methodology.

Figure 5 shows the three basic steps of our proposal: fuzzification, fuzzy optimization, and defuzzification. The main idea is to compute the fuzzy set $\tilde{Z}$ before applying a Type-reduction method for Type-2 fuzzy sets, to finally obtain a crisp solution using the Zimmermann's soft constraints method. Now, there are two important conditions which ensures that an IT2FLP has an optimal solution in some point of *supp*$(\tilde{b})$: *feasibility* and *convexity* which are described next

4.1 Feasibility of an IT2FLP

An LP problem should be *feasible*. The concept of a feasible IT2FLP leads us to think that an IT2FLP should be feasible at every point of $b \in supp(\tilde{b})$. In other words:

Proposition 4.1 (Feasibility of the IT2FLP). *An IT2FLP is feasible iff the system*

$$A(x_{ij}) \leqslant \bar{\hat{b}}_i \quad \forall\ i \in \mathbb{N}_m \tag{15}$$

is feasible itself.

This means that an IT2FLP is feasible only if the broadest value of $supp\tilde{b}$ is feasible, *i.e.* the boundary provided by $\bar{\hat{b}}$. It is clear that if a solution in this point exists, then every value of $b \leqslant \bar{\hat{b}}$ is feasible as well, since they are contained into a convex hull defined by $\bar{\hat{b}}$ (see Wolsey [31], and Papadimitriou & Steiglitz [27]).

4.2 Convexity of an IT2FLP

Another important condition to be satisfied by any LP model is *convexity*. In an LP problem, convexity means that the halfspace generated by all $A(x_{ij}) \leqslant b$ should be continuous and compact. This implies that every set b should not be empty (non-null).

An IT2FLP has to guarantee two convexity conditions: a first one regarding $b \in supp(\tilde{b})$ and a second one regarding $\mu_{\tilde{b}}$. This leads us to the following proposition:

Proposition 4.2 (Convexity of an IT2FLP). *An IT2FLP is said to be convex iff*

$$A(x_{ij}) \precsim \tilde{b} \quad \forall \ i \in \mathbb{N}_m \tag{16}$$

is a non-null halfspace, and $\tilde{b}$ is composed by convex $\overline{\mu}_{\tilde{b}}$ and $\underline{\mu}_{\tilde{b}}$ membership functions.

According to Kearfott & Kreinovich [17], global optimization is only possible for convex objective functions, so the Proposition 4.2 agrees with this. As $\tilde{z}$ is a function of $\tilde{b}$, then we need to guarantee that $\mu_{\tilde{b}}$ be convex to ensure that $\tilde{z}$ be convex as well.

5 SOLUTION PROCEDURE OF AN IT2FLP

Until now our main problem is how to deal with interval fuzzy sets, since most of fuzzy optimization methods were designed for Type-1 fuzzy sets, and what we have is an interval of infinite choices. This way, our proposal is based on finding two endpoints enclosed into $\triangle$ and ∇ (see Fig. 4), and use these points as the parameters of a single fuzzy set, suitable to be optimized using the Algorithm 1.

Figueroa [4, 5, 6] proposed a method to find an optimal fuzzy set embedded into the FOU of the problem using $\triangle$, ∇ as auxiliary variables weighted by $c^{\triangle}$ and c^{∇} and the Zimmermann's method. A description of the algorithm is presented next.

Algorithm 2

1. Compute an optimal inferior boundary called *Z minimum* ($\check{z}$) by using $\check{\underline{b}} + \triangle$ as a frontier of the model, where $\triangle$ (see Definition 3.4) is an auxiliary set of variables weighted by $c^{\triangle}$ which represents the lower uncertainty interval subject to the following statement:

$$\triangle \leqslant \check{\bar{b}} - \check{\underline{b}} \tag{17}$$

To do so, $\triangle^*$ is obtained solving the following LP problem

$$\begin{aligned} \max_{x,\triangle} \ z = c'x + c_0 - c^{\triangle}{}'\triangle \\ s.t. \\ Ax - \triangle \leqslant \check{\underline{b}} \\ \triangle \leqslant \check{\bar{b}} - \check{\underline{b}} \\ x \geqslant 0 \end{aligned} \tag{18}$$

2. Compute an optimal superior boundary called *Z maximum* ($\hat{z}$) by using $\hat{\bar{b}} + \nabla$ as a frontier of the model, where ∇ (see Definition 3.4) is an auxiliary set of variables weighted by c^{∇} which represents the upper uncertainty interval subject to the following statement:

$$\nabla \leqslant \hat{\bar{b}} - \hat{\underline{b}} \tag{19}$$

To do so, ∇^* is obtained solving the following LP problem

$$\begin{aligned}
\max_{x,\nabla} \ z = c'x + c_0 - c^{\nabla\,\prime}\nabla \\
s.t. \\
Ax - \nabla \leqslant \hat{\underline{b}} \\
\nabla \leqslant \hat{\bar{b}} - \hat{\underline{b}} \\
x \geqslant 0
\end{aligned} \tag{20}$$

3. Find the final solution using the third and subsequent steps of the Algorithm 1 using the following values of $\check{b}$ and $\hat{b}$

$$\check{b} = \check{\underline{b}} + \triangle^* \tag{21}$$

$$\hat{b} = \hat{\underline{b}} + \nabla^* \tag{22}$$

Remark 5.1 (About $c^{\triangle}$ and c^{∇}). In Algorithm 2, we have defined $c^{\triangle}$ and c^{∇} as the weights of $\triangle$ and ∇. In other words, $c_i^{\triangle}$ and c_i^{∇} are the unitary cost associated to increase each resource $\check{\underline{b}}_i$ and $\hat{\underline{b}}_i$ respectively.

Remark 5.2 (max − min **objectives).** The proposed algorithm was designed for maximization problems, so equations (21) and (22) apply to a max goal. For a min goal, equations (18), (20), (21) and (22) have to be changed.

Therefore, $\triangle$ and ∇ are auxiliary variables that operate as Type-reducers[3], where $\triangle_i^*$ and ∇_i^* become $\check{b}_i$ and $\hat{b}_i$ as the inputs of the Zimmermann's method which returns $\check{z}^*$, $\hat{z}^*$ and α^* (see Section 2).

6 APPLICATION EXAMPLE

To illustrate how the proposed procedure works, we present a classical transportation problem where its demands and supplies are defined by the perception of the experts of the system in two fronts: experts of the behavior of the customer and experts of the suppliers' capabilities.

Therefore, if different experts provide opinions based on their previous knowledge, the problem is how to handle the information they have provided. Sometimes, the experts provide opinions

[3] A Type-reduction strategy regards to a method for finding a single fuzzy set embedded into the FOU of a Type-2 fuzzy set.

using words instead of numbers through sentences such as *"I think that the demand of the product X should be between* b_1 *and* b_2*"*, where b_1 and b_2 become $\check{b}_i$ and $\hat{b}_i$, as presented in Section 1.

When different experts have different opinions for the same concept, then linguistic uncertainty appears and Type-2 fuzzy sets arise as an alternative to handle this kind of uncertainty. Is in that way how we present the demands and supplies of the system defined by the experts, where the main idea is to minimize the shipping costs of the system. A general IT2FLP transportation model is as follows:

$$\min z = c_{ij}\, x_{ij} \tag{23}$$

$$s.t.$$

$$-\sum_{i=1}^{m} x_{ij} \gtrsim -\tilde{a}_j \quad \forall \ j \in \mathbb{N}_n \tag{24}$$

$$\sum_{j=1}^{n} x_{ij} \gtrsim \tilde{d}_i \quad \forall \ i \in \mathbb{N}_m \tag{25}$$

where $c_{ij}, x_{ij} \in \mathbb{R}^{n,m}$, $\tilde{a}_j$ and $\tilde{d}_i$ areIT2FS whose supports are defined over the real numbers $\mathbb{R}$, $\precsim$ and $\succsim$ are Type-2 fuzzy partial orders.

Index sets:

$\mathbb{N}_m$ is the set of all

"i'' resources, $i \in \mathbb{N}_m, \mathbb{N}_m = 1, 2, \ldots, m$.

$\mathbb{N}_n$ is the set of all "j'' products, $j \in \mathbb{N}_n, \mathbb{N}_n = 1, 2, \ldots, n$.

Decision variables:

x_{ij} = Quantity of product to be shipped from the supplier "i'' to the customer "j''.

Parameters:

a_j = Quantity of product available by the supplier "j''.

d_i = Quantity of product required by the customer "i''.

Note that $\tilde{b}$ (defined in (10)) is composed by two vectors: $\tilde{d}$ with parameters $\bar{\check{d}}, \underline{\check{d}}, \underline{\hat{d}}$ and $\bar{\check{a}}$, which are the demands of the customers, and $\tilde{a}$ with parameters $\bar{\check{a}}, \underline{\check{a}}, \underline{\hat{a}}$ and $\bar{\check{a}}$, which are the availabilities offered by suppliers.

Now, we need to compute $\tilde{z}$ and $z^* = c(x^*)$ using (14), $\tilde{d}, \tilde{a}$ and c which are provided as follows.

$$\bar{\check{d}}_i = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \underline{\check{d}}_i = \begin{bmatrix} 13 \\ 14 \\ 18 \end{bmatrix} \bar{\hat{d}}_i = \begin{bmatrix} 12 \\ 13 \\ 15 \end{bmatrix} \underline{\hat{d}}_i = \begin{bmatrix} 16 \\ 17 \\ 21 \end{bmatrix}$$

$$\bar{\check{a}}_j = \begin{bmatrix} 24 \\ 37 \\ 29 \end{bmatrix} \underline{\check{a}}_j = \begin{bmatrix} 16 \\ 30 \\ 23 \end{bmatrix} \bar{\hat{a}}_j = \begin{bmatrix} 20 \\ 32 \\ 24 \end{bmatrix} \underline{\hat{a}}_j = \begin{bmatrix} 14 \\ 25 \\ 18 \end{bmatrix}$$

$$c_{ij} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 4 \\ 1 \\ 3 \\ 2 \\ 4 \\ 2 \end{bmatrix} c^{\triangle} = \begin{bmatrix} 1 \\ 0.5 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \end{bmatrix} \triangle = \begin{bmatrix} 3 \\ 3 \\ 6 \\ 6 \\ 7 \\ 6 \end{bmatrix} c^{\nabla} = \begin{bmatrix} 1 \\ 0.5 \\ 1 \\ 1 \\ 0.5 \\ 1 \end{bmatrix} \nabla = \begin{bmatrix} 4 \\ 4 \\ 6 \\ 8 \\ 7 \\ 6 \end{bmatrix}$$

This example is composed by three suppliers and three customers whose parameters are defined by experts using IT2FS, so we apply the Algorithm 2 to find a crisp solution of the problem. The obtained fuzzy set $\tilde{Z}$ is defined by the following boundaries:

$$\begin{aligned} \check{\bar{z}} &= 76 \\ \hat{\bar{z}} &= 96 \\ \check{\underline{z}} &= 55 \\ \hat{\underline{z}} &= 67 \end{aligned}$$

6.1 Obtained results

First, we have applied the LP models shown in (18) and (20), which lead to the following results:

$$\begin{array}{rrcrr rrcrr} \triangle_1^* = & 3 & \to & \check{d}_1^* = & 10 & \nabla_1^* = & 4 & \to & \hat{d}_1^* = & 12 \\ \triangle_2^* = & 3 & \to & \check{d}_2^* = & 11 & \nabla_2^* = & 4 & \to & \hat{d}_2^* = & 13 \\ \triangle_3^* = & 6 & \to & \check{d}_3^* = & 12 & \nabla_3^* = & 6 & \to & \hat{d}_3^* = & 15 \\ \triangle_4^* = & 0 & \to & \check{a}_1^* = & -16 & \nabla_4^* = & 0 & \to & \hat{a}_1^* = & -14 \\ \triangle_5^* = & 0 & \to & \check{a}_2^* = & -30 & \nabla_5^* = & 0 & \to & \hat{a}_2^* = & -25 \\ \triangle_6^* = & 0 & \to & \check{a}_3^* = & -23 & \nabla_6^* = & 0 & \to & \hat{a}_3^* = & -18 \end{array}$$

Now, we use $\triangle^*$ and ∇^* alongside equations (21) and (22) to obtain the values of $\check{z}^* = 65.5$ and $\hat{z}^* = 79$. Then, by using $\check{d}$, $\check{a}$, $\hat{d}$, $\hat{a}$ and the Zimmermann's method we obtain the following results: $\alpha^* = 0.9412$ and $z^* = 66.29$. The shipping quantities x_{ij}^* that should be sent from suppliers to customers are shown next.

$$\begin{array}{rr rr rr} x_{11}^* = & 0 & x_{21}^* = & 0 & x_{31}^* = & 118.82 \\ x_{12}^* = & 0 & x_{22}^* = & 128.82 & x_{32}^* = & 0 \\ x_{13}^* = & 141.76 & x_{23}^* = & 0 & x_{33}^* = & 0.71 \end{array}$$

Figure 6 shows the Type-reduced fuzzy set of optimal solutions $\tilde{z}$ which is embedded into the FOU of $\tilde{Z}$, where the global satisfaction degree of $\alpha^* = 0.9412$ allows us to find a crisp solution of the problem.

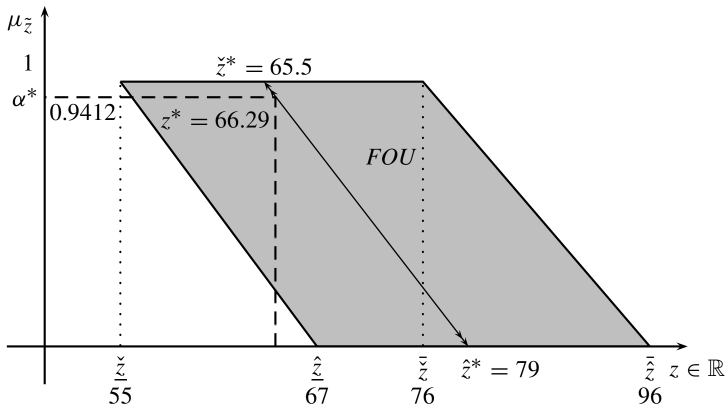

Figure 6 – Fuzzy set z embedded into the FOU of $\tilde{Z}$.

6.2 Discussion of the results

We started to solve the problem with a set of constraints $\tilde{d}$, $\tilde{a}$, in which we cannot make a shipping decision. Then we applied the Algorithm 2 to obtain the endpoints $\triangle^*$ and ∇^* which leads to $\check{d}^*$, $\check{a}^*$, $\hat{d}^*$ and $\hat{a}^*$. Finally we have applied the Algorithm 1 to find a crisp solution, which are in this case the optimal amount of demand to be satisfied, the amount of supplies to be used, and the shipping quantities x^*_{ij} to be sent through the route $i \rightarrow j$.

Note that our results depend on $c^{\triangle}$ and c^{∇}, so at a first glance, the method should not increase its delivering costs, but this does not happen as shown before. Moreover, the method decreases the satisfied demand, even by paying an additional cost $c^{\triangle}$ and c^{∇}, plus shipping costs. Also note that $\triangle^* = 0$ and $\nabla^* = 0$ when the availability of suppliers $\tilde{a}$ is increased, since this leads to increase its global shipping costs.

There is an interesting reason for: our method selects the constraints that increases the objective function, accomplishing (13) instead of the natural reasoning of treating all constraints in the same way.

Remark 6.1. Note that the presented example is feasible according to Propositions 4.1 and 4.2. This means that the problem is feasible since $\check{\bar{d}}$, $\check{\bar{a}}$ are feasible points, and the problem is convex since $\mu_{\tilde{d}}$ and $\mu_{\tilde{d}}$ are linear convex membership functions, so $\tilde{Z}$ as displayed in Figure 6 is convex as well.

Recall that our results are based on computing $\check{\bar{z}}$, $\hat{\bar{z}}$, $\check{\underline{z}}$, $\hat{\underline{z}}$, $\triangle^*$, ∇^*, $\check{z}^*$, $\hat{z}^*$ which leads to a crisp solution of α^*, z^* and x^*_{ij} through LP models which can be solved using GAMS, LINGO, MatLab or any other optimization software. This provides a well known framework for further implementations since there is no need for additional software and/or routines.

Finally, the crisp optimal solution of the problem comes from solving the Zimmermann's soft constraints method alongside $\check{d}^*$, $\check{a}^*$, $\hat{d}^*$ and $\hat{a}^*$. The results are the optimal shipping quantities to be sent x^*_{ij}, accomplishing (13). This way, what we are doing here is making a decision using the Bellman-Zadeh fuzzy decision making principle.

7 CONCLUDING REMARKS

The proposed methodology (see Section 5 and Fig. 5) deals with Type-2 fuzzy uncertainty coming from the experts by using well known fuzzy optimization techniques, achieving satisfactory results. Our proposal computes a fuzzy set $\tilde{Z}$ from $\tilde{b}$ before applying an optimization strategy, which is divided into two sub-steps: Type-reduction and defuzzification (performed by the Zimmermann's soft constraints method).

Our proposal uses the Zimmermann's method for finding a crisp solution to a Type-2 constrained problem, so many similar problems can be solved using our proposal due to its flexibility and interpretability. Note that different Type-reduction strategies may be used, each one providing supplementary information for decision making.

Finally, the proposed methodology is a guide for handling Type-2 fuzzy constraints (involving the opinions and perceptions of different experts, their knowledge, and non-probabilistic uncertainty). Other methods can be used for (see Almeida, Yamakami & Takahashi [1], Silva, Cantao & Yamakami [28], and Hernandes, Berto & Castanho [10]), so our proposal is just an approach to solve this kind of problems.

REFERENCES

[1] ALMEIDA TA, YAMAKAMI A & TAKAHASHI M. 2007. Sistema imunológico artificial para resolver o problema da árvore geradora mínima com parâmetros fuzzy. *Pesquisa Operacional*, **27**(1): 131–154.

[2] ANGELOV PP. 1997. Optimization in an intuitionistic fuzzy environment. *Fuzzy Sets and Systems*, **86**(3): 299–306.

[3] DUBEY D, CHANDRA S & MEHRA A. 2012. Fuzzy linear programming under interval uncertainty based on ifs representation. *Fuzzy Sets and Systems*, **188**(1): 68–87.

[4] FIGUEROA JC. 2008. Linear programming with interval type-2 fuzzy right hand side parameters. In: *2008 Annual Meeting of the IEEE North American Fuzzy Information Processing Society (NAFIPS).*

[5] FIGUEROA JC. 2009. Solving fuzzy linear programming problems with interval type-2 RHS. In: *2009 Conference on Systems, Man and Cybernetics*, IEEE, pp. 1–6.

[6] FIGUEROA JC. 2011. Interval type-2 fuzzy linear programming: Uncertain constraints. In:*IEEE Symposium Series on Computational Intelligence*, IEEE, pp. 1–6.

[7] FIGUEROA JC & HERNÁNDEZ G. 2012. Computing optimal solutions of a linear programming problem with interval type-2 fuzzy constraints. *Lecture Notes in Computer Science*, **7208**: 567–576.

[8] GHODOUSIANA A & KHORRAM E. 2006. Solving a linear programming problem with the convex combination of the max-min and the max-average fuzzy relation equations. *Applied Mathematics and Computation*, **180**(1): 411–418.

[9] GUU S-M & WU Y-K. 2002. Minimizing a linear objective function with fuzzy relation equation constraints. *Fuzzy Optimization and Decision Making*, **1**(4): 347.

[10] HERNANDES F, BERTON L & CASTANHO M. 2009. O problema de caminho mínimo com incertezas e restrições de tempo. *Pesquisa Operacional*, **29**(2): 471–488.

[11] INUIGUCHI M & RAMÍK J. 2000. Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy Sets and Systems*, **111**: 3–28.

[12] INUIGUCHI M & SAKAWA M. 1994. Possible and necessary optimality tests in possibilistic linear programming problems. *Fuzzy Sets and Systems*, **67**: 29–46.

[13] INUIGUCHI M & SAKAWA M. 1999. A possibilistic linear program is equivalent to a stochastic linear program in a special case. *Fuzzy Sets and Systems*, **76**(1): 309–317.

[14] KALL P & MAYER J. 2010. *Stochastic Linear Programming: Models, Theory, and Computation*. Springer-Verlag.

[15] KARNIK NN & MENDEL J. 2001. Operations on type-2 fuzzy sets. *Fuzzy Sets and Systems*, **122**: 327–348.

[16] KARNIK NN, MENDEL J & LIANG Q. 1999. Type-2 fuzzy logic systems. *Fuzzy Sets and Systems*, **17**(10): 643–658.

[17] KEARFOTT RB & KREINOVICH V. 2005. Beyond convex? global optimization is feasible only for convex objective functions: A theorem. *Journal of Global Optimization*, **33**(4): 617–624.

[18] LIANG Q & MENDEL J. 2000. Interval type-2 fuzzy logic systems: Theory and design. *IEEE Transactions on Fuzzy Systems*, **8**(5): 535–550.

[19] MELGAREJO M. 2007. A Fast Recursive Method to compute the Generalized Centroid of an Interval Type-2 Fuzzy Set. In: *Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS)*, IEEE, pp. 190–194.

[20] MELGAREJO M. 2007. Implementing Interval Type-2 Fuzzy processors. *IEEE Computational Intelligence Magazine*, **2**(1): 63–71.

[21] MENDEL J. 1994. *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Prentice Hall.

[22] MENDEL J. 2003. Fuzzy sets for words: a new beginning. In: *The IEEE International Conference on Fuzzy Systems*, pp. 37–42.

[23] MENDEL J. 2003. Type-2 Fuzzy Sets: Some Questions and Answers. *IEEE coNNectionS. A publication of the IEEE Neural Networks Society.*, 8 (2003), 10–13.

[24] MENDEL J & JOHN RI. 2002. Type-2 fuzzy sets made simple. *IEEE Transactions on Fuzzy Systems*, **10**(2): 117–127.

[25] MENDEL J, JOHN RI & LIU F. 2006. Interval type-2 fuzzy logic systems made simple. *IEEE Transactions on Fuzzy Systems*, **14**(6): 808–821.

[26] MORA HM. 2001. *Optimización no lineal y dinámica*. Universidad Nacional de Colombia.

[27] PAPADIMITRIOU CH & STEIGLITZ K. 1998. *Combinatorial Optimization: Algorithms and Complexity*. Dover Publications.

[28] SILVA RC, CANTAO LA & YAMAKAMI A. 2012. Application of an iterative method and an evolutionary algorithm in fuzzy optimization. *Pesquisa Operacional*, **32**(2): 315–329.

[29] TANAKA H & ASAI K. 1984. Fuzzy Solution in Fuzzy Linear Programming Problems. *IEEE Transactions on Systems, Man and Cybernetics*, **14**(1): 325–328.

[30] TANAKA H, ASAI K & OKUDA T. 1974. On Fuzzy Mathematical Programming. *Journal of Cybernetics*, **3**: 37–46.

[31] WOLSEY LA. 1998. *Integer Programming*. John Wiley and Sons.

[32] ZIMMERMANN HJ. 1978. Fuzzy programming and Linear Programming with several objective functions. *Fuzzy Sets and Systems*, **1**(1): 45–55.

[33] ZIMMERMANN HJ & FULLÉR R. 1993. Fuzzy Reasoning for solving fuzzy Mathematical Programming Problems. *Fuzzy Sets and Systems*, **60**(1): 121–133.