

UNIQUE SOLUTION OF INTEGRAL EQUATIONS VIA INTUITIONISTIC FUZZY B-METRIC-LIKE SPACES

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ABSTRACT. In this manuscript, our aim is to describe the concept of intuitionistic fuzzy b-metric-like spaces. We investigate some properties of this new space and give some useful examples. We also define the concepts of convergence, Cauchy sequence and completeness in intuitionistic fuzzy b-metric-like space. Furthermore, we establish some fixed point theorems in this setting and give an application to integral equations to illustrate the usability of the obtained results.

Keywords: b-metric-like spaces, intuitionistic fuzzy b-metric-like spaces, fixed point theory, contraction principle, integral equation.

1 INTRODUCTION

Fréchet (1906) and Hausdorff (1914) defined the theory of metric spaces a century ago. Banach (1922) established the Banach principle on metric spaces. Czerwik (1993) introduced the concept of b-metric space and established some generalizations of Banach's fixed point theorem in this space. The concept of metric-like space and then some fixed point results were given by Amini-Harandi (2012). Alghamdi et al. (2013) first introduced the concept of b-metric-like space which is a generalization of b-metric space and metric-like space. Then Alghamdi et al. (2013) established the existence and uniqueness of fixed points in a b-metric-like space. Moreover, an application to integral equations was given by Alghamdi et al. (2013).

The theory of fuzzy sets was introduced by Zadeh (1965). Subsequently, many authors discussed the concept of fuzzy sets in various fields, one of which is fuzzy metric spaces introduced by

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Kramosil & Michálek (1975). George & Veeramani (1994, 1995) refined the concept of fuzzy metric space, as originally proposed by Kramosil & Michálek, resulting in a stronger version of fuzzy metric space. Shukla & Abbas (2014) defined the notion of fuzzy metric-like spaces as a generalization of fuzzy metric spaces. They also studied fixed point theorems for contractive mappings in fuzzy metric-like spaces. Nadaban (2016) proposed the concept of fuzzy b-metric spaces and demonstrated that the investigation of operators on fuzzy b-metric spaces has numerous applications in Mathematics, Engineering, and Computer Science. Javed et al. (2021) introduced the concept of fuzzy b-metric-like space and discussed some related fixed point results. Also, an application for solving a first kind of Fredholm type integral equations was provided by Javed et al. (2021).

Park (2004) defined the notion of intuitionistic fuzzy metric space and Hausdorff topology on this intuitionistic fuzzy metric space. Generalizing both the concepts of intuitionistic fuzzy metric spaces and fuzzy b-metric spaces, Azam & Kanwel (2022) introduced and investigated the notion of intuitionistic fuzzy b-metric spaces. Alaca et al. (2006), in their study, generalized fixed point theorems to the setting of intuitionistic fuzzy metric spaces. Onbaşıoğlu & Pazar Varol (2023) described the concept of intuitionistic fuzzy metric-like space, which is an extension of metric-like space and intuitionistic fuzzy metric space. Moreover, Onbaşıoğlu & Pazar Varol (2023) obtained some fixed-point theorems in intuitionistic fuzzy metric-like spaces. Ahmed et al. (2023) introduced the concept of intuitionistic fuzzy b-metric-like space and discuss some fixed point results. Useful examples were given by them in this space.

In this article, we aim to redefine in an alternative way the concept of intuitionistic fuzzy b-metric-like space defined by Ahmed et al. (2023) and to prove fixed point theorems within this new structure. Our results both advance the studies in the field of intuitionistic fuzzy b-metric spaces and offer a broader perspective for future research and applications, as they encompass a wider scope than classical fixed point studies. We also present new results on the concepts of convergence, Cauchy sequences, completeness of the space, and intuitionistic fuzzy b-metric-like contraction (IFbML-contraction), which allow us to examine the structure more deeply. Moreover, we provide several useful examples and related definitions throughout the article to support our results. In addition, we apply our findings to an integral equation defined in this new setting, reinforcing the fixed point results we have obtained. For a more in-depth understanding of the concepts, we recommend reviewing the studies of Gupta et al. (2017) and Sedghi et al. (2008) on integral-type contraction mappings defined in intuitionistic fuzzy metric spaces. For applications of fuzzy fixed point theory in mathematics and engineering, we suggest the study by Younis & Abdau (2024) that generalizes the well-known results of Gregori & Sapena (2002), as well as those of Jachymski (2008) on fuzzy metric spaces.

2 PRELIMINARIES

We start this section by listing various helpful definitions and notions for readers. In this study, $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all positive integer numbers, respectively.

Definition 2.1. Let $\mathcal{U} \neq \emptyset$. A mapping $\phi : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ is called b-metric on $\mathcal{U}$ if the following three conditions hold for all $d, n, r \in \mathcal{U}$ and $b \geq 1$:

$$(bM1) \phi(d, r) = 0 \Leftrightarrow d = r,$$

$$(bM2) \phi(d, r) = \phi(r, d),$$

$$(bM3) \phi(d, r) \leq b[\phi(d, n) + \phi(n, r)].$$

The pair $(\mathcal{U}, \phi)$ is called a b-metric space (Czerwik, 1993).

Example 2.1. The space $l_p = \{(d_i) \subset \mathbb{R} : \sum_{i=1}^{\infty} |d_i|^p < \infty\}$, ($0 < p < 1$), together with the function $\phi(d, r) = (\sum_{i=1}^{\infty} |d_i - r_i|^p)^{\frac{1}{p}}$ where $d = d_i, r = r_i \in l_p$, is a b-metric space (Kir & Kiziltunc, 2013).

Example 2.2. The space L_p ($0 < p < 1$) of all real functions $h(s), s \in [0, 1]$, such that $\int_0^1 |h(s)|^p ds < \infty$ is a b-metric space with $\phi(h, g) = (\int_0^1 |h(s) - g(s)|^p ds)^{\frac{1}{p}}$ where $h, g \in L_p$ (Kir & Kiziltunc, 2013).

Definition 2.2. Let $\mathcal{U} \neq \emptyset$. A mapping $\varphi : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ is called metric-like on $\mathcal{U}$ if the following three conditions hold for all $d, n, r \in \mathcal{U}$:

$$(ML1) \varphi(d, r) = 0 \Rightarrow d = r,$$

$$(ML2) \varphi(d, r) = \varphi(r, d),$$

$$(ML3) \varphi(d, r) \leq \varphi(d, n) + \varphi(n, r).$$

The pair $(\mathcal{U}, \varphi)$ is called a metric-like space (Amini-Harandi, 2012).

Example 2.3. Let $\mathcal{U} = [0, \infty)$. Define the function $\varphi : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ by $\varphi(d, r) = \max\{d, r\}$. Then $(\mathcal{U}, \varphi)$ is a metric-like space (Amini-Harandi, 2012).

Definition 2.3. Let $\mathcal{U} \neq \emptyset$. A mapping $\wp : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ is called b-metric-like on $\mathcal{U}$ if the following three conditions hold for all $d, n, r \in \mathcal{U}$ and $b \geq 1$:

$$(bML1) \wp(d, r) = 0 \Rightarrow d = r,$$

$$(bML2) \wp(d, r) = \wp(r, d),$$

$$(bML3) \wp(d, r) \leq b[\wp(d, n) + \wp(n, r)].$$

The pair $(\mathcal{U}, \wp)$ is called a b-metric-like space (Alghamdi et al., 2013).

Example 2.4. Let $\mathcal{U} = [0, \infty)$. Define the function $\wp : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ by $\wp(d, r) = (d + r)^2$. Then $(\mathcal{U}, \wp)$ is a b-metric-like space with constant $b = 2$ (Alghamdi et al., 2013).

Example 2.5. Let $\mathcal{U} = [0, \infty)$. Define the function $\wp : \mathcal{U} \times \mathcal{U} \rightarrow [0, \infty)$ by $\wp(d, r) = (\max\{d, r\})^2$. Then $(\mathcal{U}, \wp)$ is a b-metric-like space with constant $b = 2$ (Alghamdi et al., 2013).

Definition 2.4. An intuitionistic fuzzy set $\mathfrak{R}$ is defined by $\mathfrak{R} = \{\langle d, \mu_{\mathfrak{R}}(d), \nu_{\mathfrak{R}}(d) \rangle : d \in \mathcal{U}\}$ where $\mu_{\mathfrak{R}} : \mathcal{U} \rightarrow [0, 1]$ and $\nu_{\mathfrak{R}} : \mathcal{U} \rightarrow [0, 1]$ denote membership and non-membership functions, respectively. $\mu_{\mathfrak{R}}(d)$ and $\nu_{\mathfrak{R}}(d)$ are membership and non-membership degrees of each element $d \in \mathcal{U}$ to the intuitionistic fuzzy set $\mathfrak{R}$ and $\mu_{\mathfrak{R}}(d) + \nu_{\mathfrak{R}}(d) \leq 1$ for each $d \in \mathcal{U}$ (Atanassov, 1986).

Definition 2.5. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if $*$ satisfies the following conditions for all $o, p, j, l \in [0, 1]$:

- (1) $o * 1 = o$
- (2) $o * p = p * o$ and $o * (p * j) = (o * p) * j$
- (3) If $o \leq j$ and $p \leq l$, then $o * p \leq j * l$
- (4) $*$ is continuous

(Schweizer & Sklar, 1960)

Example 2.6. The binary operations below are continuous t-norms

- (1) $o * p = op$
- (2) $o * p = \max\{0, o + p - 1\}$
- (3) $o * p = \min\{o, p\}$

(Schweizer & Sklar, 1960)

Definition 2.6. A binary operation $\diamond$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-conorm if $\diamond$ satisfies the following conditions for all $o, p, j, l \in [0, 1]$:

- (1) $o \diamond 0 = o$
- (2) $o \diamond p = p \diamond o$ and $o \diamond (p \diamond j) = (o \diamond p) \diamond j$
- (3) If $o \leq j$ and $p \leq l$, then $o \diamond p \leq j \diamond l$
- (4) $\diamond$ is continuous

(Schweizer & Sklar, 1960)

Example 2.7. The binary operations below are continuous t-conorms

- (1) $o * p = o + p - op$
- (2) $o * p = \min\{o + p, 1\}$
- (3) $o * p = \max\{o, p\}$

(Schweizer & Sklar, 1960)

Definition 2.7. A 4-tuple $(\mathcal{U}, \mathfrak{R}, *, b)$ is called fuzzy b-metric-like space if $\mathcal{U}$ is an arbitrary set, $*$ is a continuous t-norm and $\mathfrak{R}$ is a fuzzy set on $\mathcal{U}^2 \times (0, \infty)$ satisfying the following conditions, for all $d, n, r \in \mathcal{U}$, $b \geq 1$ and $v, w > 0$;

- (FbML1) $\mathfrak{R}(d, r, v) > 0$,
- (FbML2) $\mathfrak{R}(d, r, v) = 1 \Rightarrow d = r$,
- (FbML3) $\mathfrak{R}(d, r, v) = \mathfrak{R}(r, d, v)$,
- (FbML4) $\mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) \leq \mathfrak{R}(d, r, b(v + w))$,

(FbML5) $\mathfrak{R}(d, r, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous

(Javed et al., 2021).

Example 2.8. Take $\mathcal{U} = (0, \infty)$. Consider the t-norm defined by $o * p = op$, then

$\mathfrak{R}(d, r, v) = e^{-\frac{(d+r)^2}{v}}$ ($\forall d, r \in \mathcal{U}$ and $v > 0$) is a fuzzy b-metric-like (Javed et al., 2021).

Definition 2.8. A 5-tuple $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond)$ is called intuitionistic fuzzy metric space if $\mathcal{U}$ is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mathfrak{R}, \mathfrak{S}$ are fuzzy sets on $\mathcal{U}^2 \times (0, \infty)$ satisfying the following conditions, for all $d, n, r \in \mathcal{U}$ and $v, w > 0$;

(IFM1) $\mathfrak{R}(d, r, v) + \mathfrak{S}(d, r, v) \leq 1$,

(IFM2) $\mathfrak{R}(d, r, v) > 0$,

(IFM3) $\mathfrak{R}(d, r, v) = 1 \Leftrightarrow d = r$,

(IFM4) $\mathfrak{R}(d, r, v) = \mathfrak{R}(r, d, v)$,

(IFM5) $\mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) \leq \mathfrak{R}(d, r, v + w)$,

(IFM6) $\mathfrak{R}(d, r, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,

(IFM7) $\mathfrak{S}(d, r, v) < 1$,

(IFM8) $\mathfrak{S}(d, r, v) = 0 \Leftrightarrow d = r$,

(IFM9) $\mathfrak{S}(d, r, v) = \mathfrak{S}(r, d, v)$,

(IFM10) $\mathfrak{S}(d, n, v) \diamond \mathfrak{S}(n, r, w) \geq \mathfrak{S}(d, r, v + w)$,

(IFM11) $\mathfrak{S}(d, r, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Then $(\mathfrak{R}, \mathfrak{S})$ is called an intuitionistic fuzzy metric on $\mathcal{U}$ (Park, 2004).

Example 2.9. Let $(\mathcal{U}, \wp)$ be a metric space. Let $o * p = op$ and $o \diamond p = \min\{o + p, 1\}$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{S}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = \frac{hv^n}{hv^n + m\wp(d, r)} \text{ and } \mathfrak{S}(d, r, v) = \frac{\wp(d, r)}{kv^n + m\wp(d, r)}$$

for all $d, r \in \mathcal{U}$, $v > 0$ and all $h, k, m, n \in \mathbb{R}^+$. Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond)$ is an intuitionistic fuzzy metric space (Park, 2004).

Remark 2.1. Note that the above example holds even with the t-norm $o * p = \min\{o, p\}$ and the t-conorm $o \diamond p = \max\{o, p\}$ and hence $(\mathfrak{R}, \mathfrak{S})$ is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = k = m = n = 1$ we get

$$\mathfrak{R}(d, r, v) = \frac{v}{v + \wp(d, r)} \text{ and } \mathfrak{S}(d, r, v) = \frac{\wp(d, r)}{v + \wp(d, r)}.$$

This intuitionistic fuzzy metric induced by a metric $\wp$ is called standard intuitionistic fuzzy metric (Park, 2004).

Definition 2.9. A 6-tuple $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ is called intuitionistic fuzzy b-metric space if $\mathcal{U}$ is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mathfrak{R}, \mathfrak{S}$ are fuzzy sets on $\mathcal{U}^2 \times (0, \infty)$ satisfying the following conditions, for all $d, n, r \in \mathcal{U}$, $b \geq 1$ and $v, w > 0$;

$$(IFbM1) \mathfrak{R}(d, r, v) + \mathfrak{S}(d, r, v) \leq 1,$$

$$(IFbM2) \mathfrak{R}(d, r, 0) = 0,$$

$$(IFbM3) \mathfrak{R}(d, r, v) = 1 \Leftrightarrow d = r,$$

$$(IFbM4) \mathfrak{R}(d, r, v) = \mathfrak{R}(r, d, v),$$

$$(IFbM5) \mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) \leq \mathfrak{R}(d, r, b(v + w)),$$

$$(IFbM6) \mathfrak{R}(d, r, \cdot) : [0, \infty) \rightarrow [0, 1] \text{ is left continuous and } \lim_{v \rightarrow \infty} \mathfrak{R}(d, r, v) = 1,$$

$$(IFbM7) \mathfrak{S}(d, r, 0) = 1,$$

$$(IFbM8) \mathfrak{S}(d, r, v) = 0 \Leftrightarrow d = r,$$

$$(IFbM9) \mathfrak{S}(d, r, v) = \mathfrak{S}(r, d, v),$$

$$(IFbM10) \mathfrak{S}(d, n, v) \diamond \mathfrak{S}(n, r, w) \geq \mathfrak{S}(d, r, b(v + w)),$$

$$(IFbM11) \mathfrak{S}(d, r, \cdot) : [0, \infty) \rightarrow [0, 1] \text{ is right continuous and } \lim_{v \rightarrow \infty} \mathfrak{S}(d, r, v) = 0$$

(Azam & Kanwel, 2022).

Example 2.10. Let $(\mathcal{U}, \phi, b)$ be a b-metric space. Let $o * p = \min\{o, p\}$ and $o \diamond p = \max\{o, p\}$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{S}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows: $\mathfrak{R}(d, r, v) = \begin{cases} \frac{v}{v + \phi(d, r)}, & v > 0 \\ 0, & v = 0, \end{cases}$ and $\mathfrak{S}(d, r, v) = \begin{cases} \frac{\phi(d, r)}{v + \phi(d, r)}, & v > 0 \\ 1, & v = 0. \end{cases}$

Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ is intuitionistic fuzzy b-metric space (Azam & Kanwel, 2022).

Definition 2.10. A 5-tuple $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond)$ is called intuitionistic fuzzy metric-like space if $\mathcal{U}$ is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mathfrak{R}, \mathfrak{S}$ are fuzzy sets on $\mathcal{U}^2 \times (0, \infty)$ satisfying the following conditions, for all $d, n, r \in \mathcal{U}$ and $v, w > 0$;

$$(IFML1) \mathfrak{R}(d, r, v) + \mathfrak{S}(d, r, v) \leq 1,$$

$$(IFML2) \mathfrak{R}(d, r, v) > 0,$$

$$(IFML3) \mathfrak{R}(d, r, v) = 1 \Rightarrow d = r,$$

$$(IFML4) \mathfrak{R}(d, r, v) = \mathfrak{R}(r, d, v),$$

$$(IFML5) \mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) \leq \mathfrak{R}(d, r, v + w),$$

$$(IFML6) \mathfrak{R}(d, r, \cdot) : (0, \infty) \rightarrow (0, 1] \text{ is continuous,}$$

$$(IFML7) \mathfrak{S}(d, r, v) < 1,$$

$$(IFML8) \mathfrak{S}(d, r, v) = 0 \Rightarrow d = r,$$

$$(IFML9) \mathfrak{S}(d, r, v) = \mathfrak{S}(r, d, v),$$

$$(IFML10) \mathfrak{I}(d, n, v) \diamond \mathfrak{I}(n, r, w) \geq \mathfrak{I}(d, r, v + w),$$

$$(IFML11) \mathfrak{I}(d, r, \cdot) : (0, \infty) \rightarrow [0, 1] \text{ is continuous.}$$

$(\mathfrak{R}, \mathfrak{I})$ are called an intuitionistic fuzzy metric-like on $\mathcal{U}$ with $*$ and $\diamond$ (Onbaşıoğlu & Pazar Varol, 2023).

Example 2.11. Let $\mathcal{U} = IR^+$, $h \in IR^+$ and $m > 0$. Let $o * p = op$ and $o \diamond p = \min\{o + p, 1\}$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{I}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = \frac{hv}{hv+m(\max\{d, r\})} \text{ and } \mathfrak{I}(d, r, v) = \frac{\max\{d, r\}}{hv+m(\max\{d, r\})}$$

for all $d, r \in \mathcal{U}$, $v > 0$. Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond)$ is an intuitionistic fuzzy metric-like space (Onbaşıoğlu & Pazar Varol, 2023).

Example 2.12. Let $\mathcal{U} = IR^+$. Let $o * p = op$ and $o \diamond p = \min\{o + p, 1\}$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{I}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = \frac{1}{e^{\frac{\max\{d, r\}}{v}}} \text{ and } \mathfrak{I}(d, r, v) = \frac{e^{\frac{\max\{d, r\}}{v}} - 1}{e^{\frac{\max\{d, r\}}{v}}}$$

for all $d, r \in \mathcal{U}$, $v > 0$. Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond)$ is an intuitionistic fuzzy metric-like space (Onbaşıoğlu & Pazar Varol, 2023).

3 INTUITIONISTIC FUZZY B-METRIC-LIKE SPACE

We start this section with the definition of intuitionistic fuzzy b-metric-like space and study its properties to support the structure. We give examples and also define convergence, Cauchy sequence, completeness in intuitionistic fuzzy b-metric-like space. After that, we prove fixed point theorems in this new structure.

Definition 3.1. A six tuple $(\mathcal{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, b)$ is called intuitionistic fuzzy b-metric-like space (shortly, IFbMLS) if $\mathcal{U}$ is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mathfrak{R}, \mathfrak{I}$ are fuzzy sets on $\mathcal{U}^2 \times (0, \infty)$ satisfying the following conditions, for all $d, n, r \in \mathcal{U}$, $b \geq 1$ and $v, w > 0$;

$$(IFbML1) \mathfrak{R}(d, r, v) + \mathfrak{I}(d, r, v) \leq 1,$$

$$(IFbML2) \mathfrak{R}(d, r, v) > 0,$$

$$(IFbML3) \mathfrak{R}(d, r, v) = 1 \Rightarrow d = r,$$

$$(IFbML4) \mathfrak{R}(d, r, v) = \mathfrak{R}(r, d, v),$$

$$(IFbML5) \mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) \leq \mathfrak{R}(d, r, b(v + w)),$$

$$(IFbML6) \mathfrak{R}(d, r, \cdot) : (0, \infty) \rightarrow (0, 1] \text{ is continuous,}$$

$$(IFbML7) \mathfrak{I}(d, r, v) < 1,$$

$$(IFbML8) \mathfrak{I}(d, r, v) = 0 \Rightarrow d = r,$$

$$(IFbML9) \mathfrak{I}(d, r, v) = \mathfrak{I}(r, d, v),$$

$$(IFbML10) \mathfrak{I}(d, n, v) \diamond \mathfrak{I}(n, r, w) \geq \mathfrak{I}(d, r, b(v+w)),$$

$$(IFbML11) \mathfrak{I}(d, r, \cdot) : (0, \infty) \rightarrow [0, 1] \text{ is continuous.}$$

Then $(\mathfrak{R}, \mathfrak{I})$ is called an intuitionistic fuzzy b-metric-like on $\mathcal{U}$.

Example 3.1. Let $\mathcal{U} = [0, \infty)$. Let $o * p = op$ and $o \diamond p = o + p - op$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{I}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = \frac{v}{v+(d+r)^2}, \mathfrak{I}(d, r, v) = 1 - \mathfrak{R}(d, r, v) = \frac{(d+r)^2}{v+(d+r)^2}$$

for all $d, r \in \mathcal{U}$, $v > 0$. Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, 2)$ is an IFbMLS.

(IFbML1)-(IFbML4), (IFbML6)-(IFbML9) and (IFbML11) are obvious. Now, we prove (IFbML5) and (IFbML10). Since $(d+r)^2 \leq 2[(d+n)^2 + (n+r)^2]$ for all $d, n, r \in \mathcal{U}$, we have

$$\begin{aligned} (d+r)^2 &\leq 2\left[\frac{v+w}{v}(d+n)^2 + \frac{v+w}{w}(n+r)^2\right] \\ \Rightarrow \frac{(d+r)^2}{2(v+w)} &\leq \frac{(d+n)^2}{v} + \frac{(n+r)^2}{w} \\ &= \frac{w(d+n)^2 + v(n+r)^2}{vw} \\ \Rightarrow 1 + \frac{(d+r)^2}{2(v+w)} &\leq 1 + \frac{w(d+n)^2 + v(n+r)^2}{vw} \\ \Rightarrow \frac{2(v+w) + (d+r)^2}{2(v+w)} &\leq \frac{vw + w(d+n)^2 + v(n+r)^2}{vw} \\ &\leq \frac{vw + w(d+n)^2 + v(n+r)^2 + (d+n)^2(n+r)^2}{vw} \\ \Rightarrow \frac{vw}{vw + w(d+n)^2 + v(n+r)^2 + (d+n)^2(n+r)^2} &\leq \frac{2(v+w)}{2(v+w) + (d+r)^2} \\ \Rightarrow \frac{v}{v+(d+n)^2} \cdot \frac{w}{w+(n+r)^2} &\leq \frac{2(v+w)}{2(v+w) + (d+r)^2} \\ \Rightarrow \mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) &\leq \mathfrak{R}(d, r, 2(v+w)) \end{aligned}$$

for all $d, n, r \in \mathcal{U}$ and $v, w > 0$. Hence (IFbML5) holds.

We know that $\frac{v}{v+(d+n)^2} \cdot \frac{w}{w+(n+r)^2} \leq \frac{2(v+w)}{2(v+w) + (d+r)^2}$ for all $d, n, r \in \mathcal{U}$ and $v, w > 0$.

$$\begin{aligned} \Rightarrow 1 - \frac{2(v+w)}{2(v+w) + (d+r)^2} &\leq 1 - \frac{v}{v+(d+n)^2} \cdot \frac{w}{w+(n+r)^2} \\ &= 1 - \frac{v}{v+(d+n)^2} \cdot \frac{w}{w+(n+r)^2} + 1 - 1 + \frac{v}{v+(d+n)^2} - \frac{v}{v+(d+n)^2} + \frac{w}{w+(n+r)^2} - \frac{w}{w+(n+r)^2} \\ &= \left(1 - \frac{v}{v+(d+n)^2}\right) + \left(1 - \frac{w}{w+(n+r)^2}\right) - \left[1 - \frac{v}{v+(d+n)^2} - \frac{w}{w+(n+r)^2} + \frac{v}{v+(d+n)^2} \cdot \frac{w}{w+(n+r)^2}\right] \\ &= \left(1 - \frac{v}{v+(d+n)^2}\right) + \left(1 - \frac{w}{w+(n+r)^2}\right) - \left[\left(1 - \frac{v}{v+(d+n)^2}\right) \cdot \left(1 - \frac{w}{w+(n+r)^2}\right)\right] \\ \Rightarrow \mathfrak{I}(d, n, v) \diamond \mathfrak{I}(n, r, w) &\geq \mathfrak{I}(d, r, 2(v+w)) \end{aligned}$$

for all $d, n, r \in \mathcal{U}$ and $v, w > 0$. Hence (IFbML10) holds.

Remark 3.1. Note that the above example also holds for $o * p = op$ and $o \diamond p = \min\{1, o + p\}$.

Example 3.2. Let $\mathcal{U} = [0, \infty)$. Let $o * p = op$ and $o \diamond p = o + p - op$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{I}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = \frac{v}{v+(\max\{d, r\})^2}, \mathfrak{I}(d, r, v) = \frac{(\max\{d, r\})^2}{v+(\max\{d, r\})^2} \text{ for all } d, r \in \mathcal{U}, v > 0.$$

Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, 2)$ is an IFbMLS.

(IFbML1)-(IFbML4),(IFbML6)-(IFbML9) and (IFbML11) are obvious. Now, we prove (IFbML5) and (IFbML10). Since $(\max\{d, r\})^2 \leq 2[(\max\{d, n\})^2 + (\max\{n, r\})^2]$ for all $d, n, r \in \mathcal{U}$, we have

$$\begin{aligned}
 (\max\{d, r\})^2 &\leq 2\left[\frac{v+w}{v}(\max\{d, n\})^2 + \frac{v+w}{w}(\max\{n, r\})^2\right] \\
 \Rightarrow \frac{(\max\{d, r\})^2}{2(v+w)} &\leq \frac{(\max\{d, n\})^2}{v} + \frac{(\max\{n, r\})^2}{w} \\
 &= \frac{w(\max\{d, n\})^2 + v(\max\{n, r\})^2}{vw} \\
 \Rightarrow 1 + \frac{(\max\{d, r\})^2}{2(v+w)} &\leq 1 + \frac{w(\max\{d, n\})^2 + v(\max\{n, r\})^2}{vw} \\
 \Rightarrow \frac{2(v+w) + (\max\{d, r\})^2}{2(v+w)} &\leq \frac{vw + w(\max\{d, n\})^2 + v(\max\{n, r\})^2}{vw} \\
 &\leq \frac{vw + w(\max\{d, n\})^2 + v(\max\{n, r\})^2 + (\max\{d, n\})^2(\max\{n, r\})^2}{vw} \\
 \Rightarrow \frac{vw}{vw + w(\max\{d, n\})^2 + v(\max\{n, r\})^2 + (\max\{d, n\})^2(\max\{n, r\})^2} &\leq \frac{2(v+w)}{2(v+w) + (\max\{d, r\})^2} \\
 \Rightarrow \frac{v}{v + (\max\{d, n\})^2} \cdot \frac{w}{w + (\max\{n, r\})^2} &\leq \frac{2(v+w)}{2(v+w) + (\max\{d, r\})^2} \\
 \Rightarrow \mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) &\leq \mathfrak{R}(d, r, 2(v+w))
 \end{aligned}$$

for all $d, n, r \in \mathcal{U}$ and $v, w > 0$. Hence (IFbML5) holds.

We know that $\frac{v}{v + (\max\{d, n\})^2} \cdot \frac{w}{w + (\max\{n, r\})^2} \leq \frac{2(v+w)}{2(v+w) + (\max\{d, r\})^2}$ for all $d, n, r \in \mathcal{U}$ and $v, w > 0$.

$$\begin{aligned}
 \Rightarrow 1 - \frac{2(v+w)}{2(v+w) + (\max\{d, r\})^2} &\leq 1 - \frac{v}{v + (\max\{d, n\})^2} \cdot \frac{w}{w + (\max\{n, r\})^2} \\
 &= 1 - \frac{v}{v + (\max\{d, n\})^2} \cdot \frac{w}{w + (\max\{n, r\})^2} + 1 - 1 \\
 &+ \frac{v}{v + (\max\{d, n\})^2} - \frac{v}{v + (\max\{d, n\})^2} + \frac{w}{w + (\max\{n, r\})^2} - \frac{w}{w + (\max\{n, r\})^2} \\
 &= \left(1 - \frac{v}{v + (\max\{d, n\})^2}\right) + \left(1 - \frac{w}{w + (\max\{n, r\})^2}\right) \\
 &- \left[1 - \frac{v}{v + (\max\{d, n\})^2} - \frac{w}{w + (\max\{n, r\})^2} + \frac{v}{v + (\max\{d, n\})^2} \cdot \frac{w}{w + (\max\{n, r\})^2}\right] \\
 &= \left(1 - \frac{v}{v + (\max\{d, n\})^2}\right) + \left(1 - \frac{w}{w + (\max\{n, r\})^2}\right) \\
 &- \left[\left(1 - \frac{v}{v + (\max\{d, n\})^2}\right) \cdot \left(1 - \frac{w}{w + (\max\{n, r\})^2}\right)\right] \\
 \Rightarrow \mathfrak{S}(d, n, v) \diamond \mathfrak{S}(n, r, w) &\geq \mathfrak{S}(d, r, 2(v+w))
 \end{aligned}$$

for all $d, n, r \in \mathcal{U}$ and $v, w > 0$. Hence (IFbML10) holds.

Remark 3.2. Note that the above example also holds for $o * p = op$ and $o \diamond p = \min\{1, o + p\}$.

Proposition 3.1. Let $(\mathcal{U}, \wp)$ be a b-metric-like space with constant $b \geq 1$. Let $o * p = op$ and $o \diamond p = o + p - op$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{S}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = \frac{v}{v + \wp(d, r)}, \quad \mathfrak{S}(d, r, v) = \frac{\wp(d, r)}{v + \wp(d, r)}$$

for all $d, r \in \mathcal{U}$, $v > 0$. Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ is an IFbMLS.

Proof. (IFbML1)-(IFbML4),(IFbML6)-(IFbML9) and (IFbML11) are obvious. Now, we prove (IFbML5) and (IFbML10). Since $\wp(d, r) \leq b[\wp(d, n) + \wp(d, r)]$ for all $d, n, r \in \mathcal{U}$, $b \geq 1$, we have

$$\wp(d, r) \leq b\left[\frac{v+w}{v}\wp(d, n) + \frac{v+w}{w}\wp(n, r)\right].$$

$$\begin{aligned}
 \Rightarrow \frac{\wp(d,r)}{b(v+w)} &\leq \frac{\wp(d,n)}{v} + \frac{\wp(n,r)}{w} \\
 &= \frac{w\wp(d,n) + v\wp(n,r)}{vw} \\
 \Rightarrow 1 + \frac{\wp(d,r)}{b(v+w)} &\leq 1 + \frac{w\wp(d,n) + v\wp(n,r)}{vw} \\
 \Rightarrow \frac{b(v+w) + \wp(d,r)}{b(v+w)} &\leq \frac{vw + w\wp(d,n) + v\wp(n,r)}{vw} \\
 &\leq \frac{vw + w\wp(d,n) + v\wp(n,r) + \wp(d,n)\wp(n,r)}{vw} \\
 \Rightarrow \frac{vw}{vw + w\wp(d,n) + v\wp(n,r) + \wp(d,n)\wp(n,r)} &\leq \frac{b(v+w)}{b(v+w) + \wp(d,r)} \\
 \Rightarrow \frac{v}{v + \wp(d,n)} \cdot \frac{w}{w + \wp(n,r)} &\leq \frac{b(v+w)}{b(v+w) + \wp(d,r)} \\
 \Rightarrow \mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) &\leq \mathfrak{R}(d, r, b(v+w))
 \end{aligned}$$

for all $d, n, r \in \mathcal{U}$ and $v, w > 0$. Hence (IFbML5) holds.

We know that $\frac{v}{v + \wp(d,n)} \cdot \frac{w}{w + \wp(n,r)} \leq \frac{b(v+w)}{b(v+w) + \wp(d,r)}$ for all $d, n, r \in \mathcal{U}$ and $v, w > 0$.

$$\begin{aligned}
 \Rightarrow 1 - \frac{b(v+w)}{b(v+w) + \wp(d,r)} &\leq 1 - \frac{v}{v + \wp(d,n)} \cdot \frac{w}{w + \wp(n,r)} \\
 &= 1 - \frac{v}{v + \wp(d,n)} \cdot \frac{w}{w + \wp(n,r)} + 1 - 1 + \frac{v}{v + \wp(d,n)} - \frac{v}{v + \wp(d,n)} + \frac{w}{w + \wp(n,r)} - \frac{w}{w + \wp(n,r)} \\
 &= \left(1 - \frac{v}{v + \wp(d,n)}\right) + \left(1 - \frac{w}{w + \wp(n,r)}\right) - \left[1 - \frac{v}{v + \wp(d,n)} - \frac{w}{w + \wp(n,r)} + \frac{v}{v + \wp(d,n)} \cdot \frac{w}{w + \wp(n,r)}\right] \\
 &= \left(1 - \frac{v}{v + \wp(d,n)}\right) + \left(1 - \frac{w}{w + \wp(n,r)}\right) - \left[\left(1 - \frac{v}{v + \wp(d,n)}\right) \cdot \left(1 - \frac{w}{w + \wp(n,r)}\right)\right] \\
 \Rightarrow \mathfrak{S}(d, n, v) \diamond \mathfrak{S}(n, r, w) &\geq \mathfrak{S}(d, r, b(v+w))
 \end{aligned}$$

for all $d, n, r \in \mathcal{U}$ and $v, w > 0$. Hence (IFbML10) holds. $\square$

Remark 3.3. Note that the above proposition also holds for $o * p = op$ and $o \diamond p = \min\{1, o + p\}$.

Proposition 3.2. Let $(\mathcal{U}, \wp)$ be a b-metric-like space with constant $b \geq 1$. Let $o * p = op$ and $o \diamond p = o + p - op$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{S}$ on $\mathcal{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = e^{-\frac{\wp(d,r)}{v^m}}, \mathfrak{S}(d, r, v) = 1 - e^{-\frac{\wp(d,r)}{v^m}}$$

for all $d, r \in \mathcal{U}$, $v > 0$ where $m \in \mathbb{N}$. Then $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ is an IFbMLS.

Proof. (IFbML1)-(IFbML4), (IFbML6)-(IFbML9) and (IFbML11) are obvious. Now, we prove (IFbML5) and (IFbML10). Since $\wp(d, r) \leq b[\wp(d, n) + \wp(d, r)]$ for all $d, n, r \in \mathcal{U}$, $b \geq 1$, we have

$$\begin{aligned}
 \frac{\wp(d,r)}{(v+w)^m} &\leq \frac{b[\wp(d,n) + \wp(n,r)]}{(v+w)^m} \\
 \Rightarrow \frac{\wp(d,r)}{(b(v+w))^m} &\leq \frac{\wp(d,r)}{b(v+w)^m} \leq \frac{\wp(d,n) + \wp(n,r)}{(v+w)^m} \\
 &\leq \frac{\wp(d,n)}{v^m} + \frac{\wp(n,r)}{w^m} \\
 \Rightarrow e^{-\frac{\wp(d,r)}{(b(v+w))^m}} &\geq e^{-\frac{\wp(d,r)}{b(v+w)^m}} \geq e^{-\frac{\wp(d,n)}{v^m}} \cdot e^{-\frac{\wp(n,r)}{w^m}} \\
 \Rightarrow \mathfrak{R}(d, n, v) * \mathfrak{R}(n, r, w) &\leq \mathfrak{R}(d, r, b(v+w))
 \end{aligned}$$

for all $d, n, r \in \mathcal{U}$ and $v, w > 0$. Hence (IFbML5) holds.

We know that $e^{-\frac{\wp(d,r)}{(b(v+w))^m}} \geq e^{-\frac{\wp(d,n)}{v^m}} \cdot e^{-\frac{\wp(n,r)}{w^m}}$ for all $d, n, r \in \mathcal{U}$ and $v, w > 0$.

$$\begin{aligned}
 \Rightarrow 1 - e^{\frac{-\wp(d,r)}{(b(v+w))^m}} &\leq 1 - \left(e^{\frac{-\wp(d,n)}{v^m}} \cdot e^{\frac{-\wp(n,r)}{w^m}} \right) \\
 &= 1 - \left(e^{\frac{-\wp(d,n)}{v^m}} \cdot e^{\frac{-\wp(n,r)}{w^m}} \right) + 1 - 1 + e^{\frac{-\wp(d,n)}{v^m}} - e^{\frac{-\wp(d,n)}{v^m}} + e^{\frac{-\wp(n,r)}{w^m}} - e^{\frac{-\wp(n,r)}{w^m}} \\
 &= \left(1 - e^{\frac{-\wp(d,n)}{v^m}} \right) + \left(1 - e^{\frac{-\wp(n,r)}{w^m}} \right) - \left[1 - e^{\frac{-\wp(d,n)}{v^m}} - e^{\frac{-\wp(n,r)}{w^m}} + e^{\frac{-\wp(d,n)}{v^m}} \cdot e^{\frac{-\wp(n,r)}{w^m}} \right] \\
 &= \left(1 - e^{\frac{-\wp(d,n)}{v^m}} \right) + \left(1 - e^{\frac{-\wp(n,r)}{w^m}} \right) - \left[\left(1 - e^{\frac{-\wp(d,n)}{v^m}} \right) \cdot \left(1 - e^{\frac{-\wp(n,r)}{w^m}} \right) \right]
 \end{aligned}$$

$$\Rightarrow \mathfrak{I}(d, n, v) \diamond \mathfrak{I}(n, r, w) \geq \mathfrak{I}(d, r, b(v+w))$$

for all $d, n, r \in \mathfrak{U}$ and $v, w > 0$. Hence (IFbML10) holds. $\square$

Remark 3.4. Note that the above proposition also holds for $o * p = op$ and $o \diamond p = \min\{1, o + p\}$.

Proposition 3.3. Let $(\mathfrak{U}, \mathfrak{R}, *, b)$ be a fuzzy b-metric-like space with constant $b \geq 1$.

Let $o \diamond p = 1 - [(1 - o) * (1 - p)]$ for all $o, p \in [0, 1]$. Then $(\mathfrak{U}, \mathfrak{R}, \mathfrak{I} = 1 - \mathfrak{R}, *, \diamond, b)$ is an IFbMLS.

Definition 3.2. Let $(\mathfrak{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, b)$ be an IFbMLS.

(a) A sequence (d_i) in $\mathfrak{U}$ is called convergent to $d \in \mathfrak{U}$ if $\lim_{i \rightarrow \infty} \mathfrak{R}(d_i, d, v) = \mathfrak{R}(d, d, v)$ and $\lim_{i \rightarrow \infty} \mathfrak{I}(d_i, d, v) = \mathfrak{I}(d, d, v)$ for all $v > 0$.

(b) A sequence (d_i) in $\mathfrak{U}$ is called Cauchy sequence if $\lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v)$ and $\lim_{i \rightarrow \infty} \mathfrak{I}(d_{i+e}, d_i, v)$ exist and finite for all $v > 0, e \geq 1$.

(c) $(\mathfrak{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, b)$ is called complete if every Cauchy sequence (d_i) in $\mathfrak{U}$ converges to some $d \in \mathfrak{U}$ such that

$$\lim_{i \rightarrow \infty} \mathfrak{R}(d_i, d, v) = \mathfrak{R}(d, d, v) = \lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v) \text{ and}$$

$$\lim_{i \rightarrow \infty} \mathfrak{I}(d_i, d, v) = \mathfrak{I}(d, d, v) = \lim_{i \rightarrow \infty} \mathfrak{I}(d_{i+e}, d_i, v) \text{ for all } v > 0, e \geq 1.$$

Remark 3.5. In an IFbMLS, the limit of a convergent sequence may not be unique. Consider Example 3.2. and define a sequence (d_i) in $\mathfrak{U}$ by $(d_i) = 1 + \frac{1}{i}$ for all $i \in \mathbb{N}$. If $d \geq 2$, then

$$\lim_{i \rightarrow \infty} \mathfrak{R}(d_i, d, v) = \lim_{i \rightarrow \infty} \frac{v}{v + (\max\{d_i, d\})^2} = \lim_{i \rightarrow \infty} \frac{v}{v + d^2} = \frac{v}{v + (\max\{d, d\})^2} = \mathfrak{R}(d, d, v)$$

and

$$\lim_{i \rightarrow \infty} \mathfrak{I}(d_i, d, v) = \lim_{i \rightarrow \infty} \frac{(\max\{d_i, d\})^2}{v + (\max\{d_i, d\})^2} = \lim_{i \rightarrow \infty} \frac{d^2}{v + d^2} = \frac{(\max\{d, d\})^2}{v + (\max\{d, d\})^2} = \mathfrak{I}(d, d, v)$$

for all $v > 0$. Hence, the sequence (d_i) convergence to all $d \in \mathfrak{U}$ with $d \geq 2$.

Definition 3.3. Let $(\mathfrak{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, b)$ be an IFbMLS. A mapping $F : \mathfrak{U} \rightarrow \mathfrak{U}$ is called intuitionistic fuzzy b-metric-like contraction (shortly IFbML-contraction) if there exists $\mu \in (0, 1)$ such that $\frac{1}{\mathfrak{R}(F(d), F(r), v)} - 1 \leq \mu \cdot \left[\frac{1}{\mathfrak{R}(d, r, v)} - 1 \right]$ and $\mathfrak{I}(F(d), F(r), v) \leq \mu \cdot \mathfrak{I}(d, r, v)$ for all $d, r \in \mathfrak{U}$ and $v > 0$. Here μ is called the IFbML-contraction constant of F .

Theorem 3.1. Let $(\mathfrak{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, b)$ be an complete IFbMLS and $F : \mathfrak{U} \rightarrow \mathfrak{U}$ be an IFbML-contraction with IFbML-contraction constant μ , then F has a unique fixed point $c \in \mathfrak{U}$ and $\mathfrak{R}(c, c, v) = 1, \mathfrak{I}(c, c, v) = 0$ for all $v > 0$.

Proof. Let $(\mathfrak{U}, \mathfrak{R}, \mathfrak{I}, *, \diamond, b)$ be a complete IFbMLS. For an arbitrary $d_0 \in \mathfrak{U}$, let define a sequence $(d_i) \subset \mathfrak{U}$ by $d_1 = F(d_0), d_2 = F(d_1), \dots, d_i = F(d_{i-1})$ for all $i \in \mathbb{N}$. If $d_i = d_{i-1}$ for some $i \in \mathbb{N}$,

then d_i is a fixed point of F . Now, let assume that $d_i \neq d_{i-1}$ for all $i \in \mathbb{N}$. For $v > 0$ and $i \in \mathbb{N}$, we get following from Definition 3.3;

$$\frac{1}{\mathfrak{R}(d_i, d_{i+1}, v)} - 1 = \frac{1}{\mathfrak{R}(F(d_{i-1}), F(d_i), v)} - 1 \leq \mu \left[\frac{1}{\mathfrak{R}(d_{i-1}, d_i, v)} - 1 \right] = \frac{\mu}{\mathfrak{R}(d_{i-1}, d_i, v)} - \mu.$$

Let take $\mathfrak{R}(d_i, d_{i+1}, v) = \mathfrak{R}_i(v)$ and $1 - \mu = k$, then we have that $\frac{1}{\mathfrak{R}_i(v)} \leq \frac{\mu}{\mathfrak{R}_{i-1}(v)} + k$ for all $v > 0$.

Continuing in the above inequality, we get

$$\frac{1}{\mathfrak{R}_i(v)} \leq \frac{\mu^i}{\mathfrak{R}_0(v)} + \mu^{i-1}k + \mu^{i-2}k + \dots + k = \frac{\mu^i}{\mathfrak{R}_0(v)} + (\mu^{i-1} + \mu^{i-2} + \dots + 1)k = \frac{\mu^i}{\mathfrak{R}_0(v)} + 1 - \mu^i,$$

then, we obtain

$$\frac{1}{\frac{\mu^i}{\mathfrak{R}_0(v)} + 1 - \mu^i} \leq \mathfrak{R}_i(v) \quad \text{for all } v > 0, i \in \mathbb{N}. \quad (1)$$

Now, for $e \geq 1$ and $i \in \mathbb{N}$, we get

$$\begin{aligned} \mathfrak{R}(d_{i+e}, d_i, v) &\geq \mathfrak{R}(d_i, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, d_{i+2}, \frac{v}{2b}) \\ &\geq \mathfrak{R}(d_i, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, d_{i+2}, \frac{v}{2^2b^2}) * \mathfrak{R}(d_{i+2}, d_{i+3}, \frac{v}{2^2b^2}) \\ &\geq \mathfrak{R}(d_i, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, d_{i+2}, \frac{v}{2^2b^2}) * \dots * \mathfrak{R}(d_{i+e-2}, d_{i+e-1}, \frac{v}{2^{e-1}b^{e-1}}) \\ &\quad * \mathfrak{R}(d_{i+e-1}, d_{i+e}, \frac{v}{2^{e-1}b^{e-1}}) \\ &= \mathfrak{R}_i(\frac{v}{2b}) * \mathfrak{R}_{i+1}(\frac{v}{2^2b^2}) * \dots * \mathfrak{R}_{i+e-2}(\frac{v}{2^{e-1}b^{e-1}}) * \mathfrak{R}_{i+e-1}(\frac{v}{2^{e-1}b^{e-1}}) \end{aligned}$$

By using (1) in the above inequality, we obtain

$$\begin{aligned} \mathfrak{R}(d_{i+e}, d_i, v) &\geq \frac{1}{\frac{\mu^i}{\mathfrak{R}_0(\frac{v}{2b})} + 1 - \mu^i} * \frac{1}{\frac{\mu^{i+1}}{\mathfrak{R}_0(\frac{v}{2^2b^2})} + 1 - \mu^{i+1}} * \dots * \frac{1}{\frac{\mu^{i+e-1}}{\mathfrak{R}_0(\frac{v}{2^{e-1}b^{e-1}})} + 1 - \mu^{i+e-1}} \\ &\geq \frac{1}{\frac{\mu^i}{\mathfrak{R}_0(\frac{v}{2b})} + 1} * \frac{1}{\frac{\mu^{i+1}}{\mathfrak{R}_0(\frac{v}{2^2b^2})} + 1} * \dots * \frac{1}{\frac{\mu^{i+e-1}}{\mathfrak{R}_0(\frac{v}{2^{e-1}b^{e-1}})} + 1}. \end{aligned}$$

Here, $\mu \in (0, 1)$, using the properties of continuous t-norm we have from the above expression that $\lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v) = 1$ for all $v > 0$, $e \geq 1$.

For any $i \in \mathbb{N}$ and $v > 0$, similarly, from Definition 3.3. we obtain that,

$$\mathfrak{I}(F(d_i), F(d_{i+1}), v) \leq \mu \mathfrak{I}(d_i, d_{i+1}, v). \quad \text{Then, } \mathfrak{I}(d_{i+1}, d_{i+2}, v) = \mathfrak{I}(F(d_i), F(d_{i+1}), v) \leq \mu \mathfrak{I}(d_i, d_{i+1}, v).$$

Setting, $\mathfrak{I}(d_{i+1}, d_{i+2}, v) = \mathfrak{I}_i(v)$ and $1 - \mu = k$, it follows from the above inequality that

$$\mathfrak{I}_i(v) \leq \mu \mathfrak{I}_{i-1}(v) = (1 - k) \mathfrak{I}_{i-1}(v) = \mathfrak{I}_{i-1}(v) - k \mathfrak{I}_{i-1}(v).$$

From the above inequality we have

$$\begin{aligned} \mathfrak{I}_i(v) &\leq \mathfrak{I}_{i-1}(v) - k \mathfrak{I}_{i-1}(v) \\ &\leq (1 - k) \mathfrak{I}_{i-2}(v) - (1 - k) k \mathfrak{I}_{i-2}(v). \\ &\leq (1 - k)^2 \mathfrak{I}_{i-3}(v) - (1 - k)^2 k \mathfrak{I}_{i-3}(v). \\ &\leq (1 - k)^{i-1} \mathfrak{I}_0(v) - (1 - k)^{i-1} k \mathfrak{I}_0(v). \\ &= (1 - k)^{i-1} [\mathfrak{I}_0(v) - k \mathfrak{I}_0(v)]. \\ &= \mu^{i-1} [\mathfrak{I}_0(v) - k \mathfrak{I}_0(v)]. \end{aligned}$$

$$\text{Then we get } \mathfrak{I}_i(v) \leq \mu^{i-1} [\mathfrak{I}_0(v) - k \mathfrak{I}_0(v)] \quad \text{for all } v > 0, i \in \mathbb{N}. \quad (2)$$

Now, for $e \geq 1$ and $i \in \mathbb{N}$, we get

$$\begin{aligned} \mathfrak{S}(d_{i+e}, d_i, v) &\leq \mathfrak{S}(d_i, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(d_{i+1}, d_{i+e}, \frac{v}{2b}) \\ &\leq \mathfrak{S}(d_i, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(d_{i+1}, d_{i+2}, \frac{v}{2^2 b^2}) \diamond \mathfrak{S}(d_{i+2}, d_{i+e}, \frac{v}{2^2 b^2}) \\ &\leq \mathfrak{S}(d_i, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(d_{i+1}, d_{i+2}, \frac{v}{2^2 b^2}) \diamond \dots \diamond \mathfrak{S}(d_{i+e-2}, d_{i+e-1}, \frac{v}{2^{e-1} b^{e-1}}) \diamond \mathfrak{S}(d_{i+e-1}, d_{i+e}, \frac{v}{2^{e-1} b^{e-1}}). \end{aligned}$$

Using (2) in the above inequality, we have

$$\begin{aligned} \mathfrak{S}(d_{i+e}, d_i, v) &\leq \mathfrak{S}_{i-1}(\frac{v}{2b}) \diamond \mathfrak{S}_i(\frac{v}{2^2 b^2}) \diamond \dots \diamond \mathfrak{S}_{i+e-3}(\frac{v}{2^{e-1} b^{e-1}}) \diamond \mathfrak{S}_{i+e-2}(\frac{v}{2^{e-1} b^{e-1}}) \\ &\leq \mu^{i-2} [\mathfrak{S}_0(\frac{v}{2b}) - k \mathfrak{S}_0(\frac{v}{2b})] \diamond \mu^{i-1} [\mathfrak{S}_0(\frac{v}{2^2 b^2}) - k \mathfrak{S}_0(\frac{v}{2^2 b^2})] \diamond \dots \\ &\diamond \mu^{i+e-4} [\mathfrak{S}_0(\frac{v}{2^{e-1} b^{e-1}}) - k \mathfrak{S}_0(\frac{v}{2^{e-1} b^{e-1}})] \diamond \mu^{i+e-3} [\mathfrak{S}_0(\frac{v}{2^{e-1} b^{e-1}}) - k \mathfrak{S}_0(\frac{v}{2^{e-1} b^{e-1}})]. \end{aligned}$$

Here, $\mu \in (0, 1)$, using the properties of continuous t-conorm we obtain from the above expression that $\lim_{i \rightarrow \infty} \mathfrak{S}(d_{i+e}, d_i, v) = 0$ for all $v > 0$, $e \geq 1$.

Therefore, since $\lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v) = 1$ and $\lim_{i \rightarrow \infty} \mathfrak{S}(d_{i+e}, d_i, v) = 0$ for all $v > 0$, $e \geq 1$, (d_i) is Cauchy sequence in $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$.

Since $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ is a complete IFbMLS, there exists $c \in \mathcal{U}$ such that

$$\lim_{i \rightarrow \infty} \mathfrak{R}(d_i, c, v) = \lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v) = \mathfrak{R}(c, c, v) = 1. \quad (3)$$

and

$$\lim_{i \rightarrow \infty} \mathfrak{S}(d_i, c, v) = \lim_{i \rightarrow \infty} \mathfrak{S}(d_{i+e}, d_i, v) = \mathfrak{S}(c, c, v) = 0 \text{ for all } v > 0, e \geq 1. \quad (4)$$

Now, we prove that c is a fixed point for F . For this, we obtain from Definition 3.3. that

$$\begin{aligned} \frac{1}{\mathfrak{R}(F(d_i), F(c), v)} - 1 &\leq \mu \cdot [\frac{1}{\mathfrak{R}(d_i, c, v)} - 1] = \frac{\mu}{\mathfrak{R}(d_i, c, v)} - \mu, \\ \frac{1}{\frac{\mu}{\mathfrak{R}(d_i, c, v)} + 1 - \mu} &\leq \mathfrak{R}(F(d_i), F(c), v) \text{ and } \mathfrak{S}(F(d_i), F(c), v) \leq \mu \mathfrak{S}(d_i, c, v). \end{aligned}$$

Using the above inequalities, we obtain

$$\begin{aligned} \mathfrak{R}(c, F(c), v) &\geq \mathfrak{R}(c, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, F(c), \frac{v}{2b}) \\ &= \mathfrak{R}(c, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(F(d_i), F(c), \frac{v}{2b}) \\ &\geq \mathfrak{R}(c, d_{i+1}, \frac{v}{2b}) * \frac{1}{\frac{\mu}{\mathfrak{R}(d_i, c, \frac{v}{2b})} + 1 - \mu} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{S}(c, F(c), v) &\leq \mathfrak{S}(c, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(d_{i+1}, F(c), \frac{v}{2b}) \\ &= \mathfrak{S}(c, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(F(d_i), F(c), \frac{v}{2b}) \\ &\leq \mathfrak{S}(c, d_{i+1}, \frac{v}{2b}) \diamond \mu \mathfrak{S}(d_i, c, \frac{v}{2b}). \end{aligned}$$

Taking limit as $i \rightarrow \infty$ and using (3)-(4) in the above inequalities we get $\mathfrak{R}(c, F(c), v) = 1$ and $\mathfrak{S}(c, F(c), v) = 0$, that is $F(c) = c$. Hence, c is a fixed point of F and $\mathfrak{R}(c, c, v) = 1$ and $\mathfrak{S}(c, c, v) = 0$ for all $v > 0$.

We investigate the uniqueness of the fixed point c of F . Let z be another fixed point of F such that $\mathfrak{R}(c, z, v) < 1$ and $\mathfrak{S}(c, z, v) > 0$ for some $v > 0$, it follows that from the Definition 3.3.

$$\begin{aligned} \frac{1}{\mathfrak{R}(c, z, v)} - 1 &= \frac{1}{\mathfrak{R}(F(c), F(z), v)} - 1 \leq \mu [\frac{1}{\mathfrak{R}(c, z, v)} - 1] < \frac{1}{\mathfrak{R}(c, z, v)} - 1 \text{ and} \\ \mathfrak{S}(c, z, v) &= \mathfrak{S}(F(c), F(z), v) \leq \mu \mathfrak{S}(c, z, v) < \mathfrak{S}(c, z, v), \text{ a contradiction.} \end{aligned}$$

Hence, we must have $\mathfrak{R}(c, z, v) = 1$ and $\mathfrak{S}(c, z, v) = 0$ for all $v > 0$ and therefore $c = z$. $\square$

Example 3.3. Let $\mathfrak{U} = [0, 2]$. Let $o * p = op$ and $o \diamond p = \max\{o, p\}$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{S}$ on $\mathfrak{U}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(d, r, v) = e^{-\frac{(\max\{d, r\})^2}{v}}, \mathfrak{S}(d, r, v) = 1 - e^{-\frac{(\max\{d, r\})^2}{v}} \text{ for all } d, r \in \mathfrak{U}, v > 0.$$

Then $(\mathfrak{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ is a complete IFbMLS. Define $F : \mathfrak{U} \rightarrow \mathfrak{U}$ as follows:

$$F(d) = \begin{cases} 0, & d = 1; \\ \frac{d}{2}, & d \in [0, 1), \\ \frac{d}{4}, & d \in (1, 2] \end{cases}$$

Then, we have nine cases:

Case 1: If $d = r = 1$, then $F(d) = F(r) = 0$.

Case 2: If $d = 1$ and $r \in [0, 1)$, then $F(d) = 0, F(r) = \frac{r}{2}$.

Case 3: If $d = 1$ and $r \in (1, 2]$, then $F(d) = 0, F(r) = \frac{r}{4}$.

Case 4: If $d \in [0, 1)$ and $r \in (1, 2]$, then $F(d) = \frac{d}{2}, F(r) = \frac{r}{4}$.

Case 5: If $d \in [0, 1)$ and $r \in [0, 1)$, then $F(d) = \frac{d}{2}, F(r) = \frac{r}{2}$.

Case 6: If $d \in [0, 1)$ and $r = 1$, then $F(d) = \frac{d}{2}, F(r) = 0$.

Case 7: If $d \in (1, 2]$ and $r = 1$, then $F(d) = \frac{d}{4}, F(r) = 0$.

Case 8: If $d \in (1, 2]$ and $r \in (1, 2]$, then $F(d) = \frac{d}{4}, F(r) = \frac{r}{4}$.

Case 9: If $d \in (1, 2]$ and $r \in [0, 1)$, then $F(d) = \frac{d}{4}, F(r) = \frac{r}{2}$.

All the above cases hold the IFbML-contraction:

$$\frac{1}{\mathfrak{R}(F(d), F(r), v)} - 1 \leq \mu \cdot \left[\frac{1}{\mathfrak{R}(d, r, v)} - 1 \right]$$

and

$$\mathfrak{S}(F(d), F(r), v) \leq \mu \cdot \mathfrak{S}(d, r, v)$$

with the IFbML-contraction constant $\mu \in [\frac{1}{2}, 1)$. Hence F is an IFbML-contraction with $\mu \in [\frac{1}{2}, 1)$. All the conditions of Theorem 3.1. are satisfied. Also, 0 is the unique fixed point of F and $\mathfrak{R}(0, 0, v) = 1, \mathfrak{S}(0, 0, v) = 0$ for all $v > 0$.

Theorem 3.2. Let $(\mathfrak{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ be a complete IFbMLS such that $\lim_{v \rightarrow \infty} \mathfrak{R}(d, r, v) = 1$ and $\lim_{v \rightarrow \infty} \mathfrak{S}(d, r, v) = 0$ for all $d, r \in \mathfrak{U}, v > 0$ and $F : \mathfrak{U} \rightarrow \mathfrak{U}$ be an mapping satisfying the condition:

$\mathfrak{R}(F(d), F(r), \alpha v) \geq \mathfrak{R}(d, r, v)$ and $\mathfrak{S}(F(d), F(r), \alpha v) \leq \mathfrak{S}(d, r, v)$ for all $d, r \in \mathfrak{U} v > 0$ where $\alpha \in (0, 1)$. Then F has a unique fixed point $c \in \mathfrak{U}$ and $\mathfrak{R}(c, c, v) = 1, \mathfrak{S}(c, c, v) = 0$ for all $v > 0$.

Proof. Let $(\mathfrak{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ be a complete IFbMLS. For an arbitrary $d_0 \in \mathfrak{U}$, define a sequence (d_i) in $\mathfrak{U}$ by $d_1 = F(d_0), d_2 = F^2(d_0) = F(d_1), \dots, d_i = F^i(d_0) = F(d_{i-1})$ for all $i \in \mathbb{N}$.

If $d_i = d_{i-1}$ for some $i \in \mathbb{N}$, then d_i is a fixed point of F . We suppose that $d_i \neq d_{i-1}$ for all $i \in \mathbb{N}$. For $v > 0$ and $i \in \mathbb{N}$ we have from conditions in the hypothesis that

$$\mathfrak{R}(d_i, d_{i+1}, v) = \mathfrak{R}(F(d_{i-1}), F(d_i), v) \geq \mathfrak{R}(d_{i-1}, d_i, \frac{v}{\alpha}) \text{ and}$$

$$\mathfrak{S}(d_i, d_{i+1}, v) = \mathfrak{S}(F(d_{i-1}), F(d_i), v) \leq \mathfrak{S}(d_{i-1}, d_i, \frac{v}{\alpha}) \text{ for all } i \in \mathbb{N} \text{ and } v > 0.$$

Let $\mathfrak{R}(d_i, d_{i+1}, v) = \mathfrak{R}_i(v)$ and $\mathfrak{S}(d_i, d_{i+1}, v) = \mathfrak{S}_i(v)$ and apply the above expression repeatedly, then we deduce that

$$\begin{aligned} \mathfrak{R}_i(v) &= \mathfrak{R}(d_i, d_{i+1}, v) = \mathfrak{R}(F(d_{i-1}), F(d_i), v) \geq \mathfrak{R}(d_{i-1}, d_i, \frac{v}{\alpha}) = \mathfrak{R}(F(d_{i-2}), F(d_{i-1}), \frac{v}{\alpha}) \geq \\ &\mathfrak{R}(d_{i-2}, d_{i-1}, \frac{v}{\alpha^2}) \geq \dots \geq \mathfrak{R}(d_0, d_1, \frac{v}{\alpha^i}). \end{aligned}$$

So

$$\mathfrak{R}_i(v) \geq \mathfrak{R}_0(\frac{v}{\alpha^i}) \quad (5)$$

and similarly

$$\mathfrak{S}_i(v) \leq \mathfrak{S}_0(\frac{v}{\alpha^i}) \text{ for all } i \in \mathbb{N} \text{ and } v > 0. \quad (6)$$

If $i \in \mathbb{N}$ and $e \geq 1$, then we get

$$\begin{aligned} \mathfrak{R}(d_{i+e}, d_i, v) &\geq \mathfrak{R}(d_i, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, d_{i+2}, \frac{v}{2b}) \\ &\geq \mathfrak{R}(d_i, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, d_{i+2}, \frac{v}{2^2 b^2}) * \mathfrak{R}(d_{i+2}, d_{i+3}, \frac{v}{2^2 b^2}) \\ &\geq \mathfrak{R}(d_i, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, d_{i+2}, \frac{v}{2^2 b^2}) * \dots * \mathfrak{R}(d_{i+e-2}, d_{i+e-1}, \frac{v}{2^{e-1} b^{e-1}}) \\ &* \mathfrak{R}(d_{i+e-1}, d_{i+e}, \frac{v}{2^{e-1} b^{e-1}}) \\ &= \mathfrak{R}_i(\frac{v}{2b}) * \mathfrak{R}_{i+1}(\frac{v}{2^2 b^2}) * \dots * \mathfrak{R}_{i+e-2}(\frac{v}{2^{e-1} b^{e-1}}) * \mathfrak{R}_{i+e-1}(\frac{v}{2^{e-1} b^{e-1}}). \end{aligned}$$

and

$$\begin{aligned} \mathfrak{S}(d_{i+e}, d_i, v) &\leq \mathfrak{S}(d_i, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(d_{i+1}, d_{i+2}, \frac{v}{2b}) \\ &\leq \mathfrak{S}(d_i, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(d_{i+1}, d_{i+2}, \frac{v}{2^2 b^2}) \diamond \mathfrak{S}(d_{i+2}, d_{i+3}, \frac{v}{2^2 b^2}) \\ &\leq \mathfrak{S}(d_i, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{S}(d_{i+1}, d_{i+2}, \frac{v}{2^2 b^2}) \diamond \dots \diamond \mathfrak{S}(d_{i+e-2}, d_{i+e-1}, \frac{v}{2^{e-1} b^{e-1}}) \\ &\diamond \mathfrak{S}(d_{i+e-1}, d_{i+e}, \frac{v}{2^{e-1} b^{e-1}}) \\ &= \mathfrak{S}_i(\frac{v}{2b}) \diamond \mathfrak{S}_{i+1}(\frac{v}{2^2 b^2}) \diamond \dots \diamond \mathfrak{S}_{i+e-2}(\frac{v}{2^{e-1} b^{e-1}}) \diamond \mathfrak{S}_{i+e-1}(\frac{v}{2^{e-1} b^{e-1}}). \end{aligned}$$

By using (5)-(6) in the above inequalities, we get

$$\mathfrak{R}(d_{i+e}, d_i, v) \geq \mathfrak{R}_0(\frac{v}{2b\alpha^i}) * \mathfrak{R}_0(\frac{v}{2^2 b^2 \alpha^{i+1}}) * \dots * \mathfrak{R}_0(\frac{v}{2^{e-1} b^{e-1} \alpha^{i+e-1}}) \text{ and}$$

$$\mathfrak{S}(d_{i+e}, d_i, v) \leq \mathfrak{S}_0(\frac{v}{2b\alpha^i}) \diamond \mathfrak{S}_0(\frac{v}{2^2 b^2 \alpha^{i+1}}) \diamond \dots \diamond \mathfrak{S}_0(\frac{v}{2^{e-1} b^{e-1} \alpha^{i+e-1}}).$$

Since $0 < \alpha < 1$, $\lim_{v \rightarrow \infty} \mathfrak{R}(d, r, v) = 1$ and $\lim_{v \rightarrow \infty} \mathfrak{S}(d, r, v) = 0$ for all $d, r \in \mathcal{U}$ and by the properties of continuous t-norm and t-conorm we obtain from the above expression that

$$\lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v) \geq 1 * 1 * \dots * 1 = 1 \text{ and } \lim_{i \rightarrow \infty} \mathfrak{S}(d_{i+e}, d_i, v) \leq 0 \diamond 0 \diamond \dots \diamond 0 = 0 \text{ for all } v > 0, e \geq 1.$$

From that, we have $\lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v) = 1$ and $\lim_{i \rightarrow \infty} \mathfrak{S}(d_{i+e}, d_i, v) = 0$.

Hence, (d_i) is a Cauchy sequence in $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$. Since, $(\mathcal{U}, \mathfrak{R}, \mathfrak{S}, *, \diamond, b)$ is complete IFbMLS, there exists $c \in \mathcal{U}$ such that

$$\lim_{i \rightarrow \infty} \mathfrak{R}(d_i, c, v) = \lim_{i \rightarrow \infty} \mathfrak{R}(d_{i+e}, d_i, v) = \mathfrak{R}(c, c, v) = 1 \quad (7)$$

$$\lim_{i \rightarrow \infty} \mathfrak{Z}(d_i, c, v) = \lim_{i \rightarrow \infty} \mathfrak{Z}(d_{i+e}, d_i, v) = \mathfrak{Z}(c, c, v) = 0 \text{ for all } v > 0, e \geq 1. \quad (8)$$

Now, we derive that $c \in \mathfrak{U}$ is a fixed point of F . To demonstrate this we continue like below for all $i \in \mathbb{N}$ and $v > 0$, we obtain from the hypothesis that

$$\begin{aligned} \mathfrak{R}(c, F(c), v) &\geq \mathfrak{R}(c, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_{i+1}, F(c), \frac{v}{2b}) \\ &= \mathfrak{R}(c, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(F(d_i), F(c), \frac{v}{2b}) \\ &\geq \mathfrak{R}(c, d_{i+1}, \frac{v}{2b}) * \mathfrak{R}(d_i, c, \frac{v}{2b\alpha}) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{Z}(c, F(c), v) &\leq \mathfrak{Z}(c, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{Z}(d_{i+1}, F(c), \frac{v}{2b}) \\ &= \mathfrak{Z}(c, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{Z}(F(d_i), F(c), \frac{v}{2b}) \\ &\leq \mathfrak{Z}(c, d_{i+1}, \frac{v}{2b}) \diamond \mathfrak{Z}(d_i, c, \frac{v}{2b\alpha}). \end{aligned}$$

Taking limit as $i \rightarrow \infty$ and using (7)-(8) in the above inequalities, we get $\mathfrak{R}(c, F(c), v) = 1$ and $\mathfrak{Z}(c, F(c), v) = 0$. Hence, c is a fixed point of F and $\mathfrak{R}(c, c, v) = 1$ and $\mathfrak{Z}(c, c, v) = 0$, for all $v > 0$.

To show the uniqueness of fixed point, let z be another fixed point of F . Using the conditions of the hypothesis, we get

$$\mathfrak{R}(c, z, v) = \mathfrak{R}(F(c), F(z), v) \geq \mathfrak{R}(c, z, \frac{v}{\alpha}),$$

$$\mathfrak{Z}(c, z, v) = \mathfrak{Z}(F(c), F(z), v) \leq \mathfrak{Z}(c, z, \frac{v}{\alpha}).$$

That is $\mathfrak{R}(c, z, v) \geq \mathfrak{R}(c, z, \frac{v}{\alpha})$ and $\mathfrak{Z}(c, z, v) \leq \mathfrak{Z}(c, z, \frac{v}{\alpha})$, for all $v > 0$.

Since the above inequality holds for all $v > 0$, we get $\mathfrak{R}(c, z, v) \geq \mathfrak{R}(c, z, \frac{v}{\alpha^i})$ and $\mathfrak{Z}(c, z, v) \leq \mathfrak{Z}(c, z, \frac{v}{\alpha^i})$, for all $i \in \mathbb{N}$.

Now, let us take limit as $i \rightarrow \infty$ and use $\lim_{v \rightarrow \infty} \mathfrak{R}(d, r, v) = 1$, $\lim_{v \rightarrow \infty} \mathfrak{Z}(d, r, v) = 0$ for all $d, r \in \mathfrak{U}$ and $v > 0$, we obtain $\mathfrak{R}(c, z, v) = 1$, $\mathfrak{Z}(c, z, v) = 0$ and so $c = z$. Hence, the fixed point is unique.

□

4 AN APPLICATION TO INTEGRAL EQUATION

In this part, we examine the existence of solution for integral equations.

Consider the following integral equation:

$$\sigma(t) = \int_0^H \delta(t, z, \sigma(z)) dz \quad (9)$$

where $H > 0$ and $\delta : [0, H] \times [0, H] \times IR \rightarrow IR$.

Let $\mathcal{N} = C[0, H]$ be the set of all continuous real valued functions defined on $[0, H]$. Let $o * p = op$ and $o \diamond p = o + p - op$ for all $o, p \in [0, 1]$. Define fuzzy sets $\mathfrak{R}$ and $\mathfrak{Z}$ on $\mathcal{N}^2 \times (0, \infty)$ as follows:

$$\mathfrak{R}(\sigma(t), \tau(t), v) = \max_{t \in [0, H]} e^{\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}$$

$$\mathfrak{Z}(\sigma(t), \tau(t), v) = 1 - \max_{t \in [0, H]} e^{\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}$$

for all $\sigma, \tau \in \mathcal{N} = C[0, H]$, $v > 0$. Clearly $(\mathcal{N}, \Re, \Im, *, \diamond, b)$ is a complete IFbMLS.

Let $\Phi\sigma(t) = \int_0^H \delta(t, z, \sigma(z)) dz$ for all $\sigma \in \mathcal{N}$ and $t \in [0, H]$. Observe that the existence of a solution of (9) is equivalent to the existence of a fixed point of Φ .

Theorem 4.1. Assume that the following conditions hold.

- (1) $\delta : [0, H] \times [0, H] \times IR \rightarrow IR$ is continuous
- (2) There is a continuous function $\kappa : [0, H] \times [0, H] \rightarrow IR^+$ for all $t, z \in [0, H]$ such that $|\delta(t, z, \sigma(z))| + |\delta(t, z, \tau(z))| \leq \alpha^{\frac{1}{2}} \kappa(t, z)(|\sigma(t)| + |\tau(t)|)$ where $\alpha \in (0, 1)$
- (3) $\lim_{v \rightarrow \infty} \Re(\sigma(t), \tau(t), v) = 1$ and $\sup_{t \in [0, H]} \int_0^H \kappa(t, z) dz \leq 1$
 $\lim_{v \rightarrow \infty} \Im(\sigma(t), \tau(t), v) = 0$ and $\inf_{t \in [0, H]} \int_0^H \kappa(t, z) dz \geq 0$

Then the integral equation (9) has a unique solution.

Proof. For all $t \in [0, H]$, we have

$$\begin{aligned}
 e^{\frac{-(|\Phi\sigma(t)| + |\Phi\tau(t)|)^2}{\alpha v}} &= e^{\frac{-(\int_0^H \delta(t, z, \sigma(z)) dz + \int_0^H \delta(t, z, \tau(z)) dz)^2}{\alpha v}} \\
 &\geq e^{\frac{-(\int_0^H |\delta(t, z, \sigma(z))| dz + \int_0^H |\delta(t, z, \tau(z))| dz)^2}{\alpha v}} \\
 &= e^{\frac{-(\int_0^H (|\delta(t, z, \sigma(z))| + |\delta(t, z, \tau(z))|) dz)^2}{\alpha v}} \\
 &\geq e^{\frac{-(\int_0^H \alpha^{\frac{1}{2}} \kappa(t, z)(|\sigma(t)| + |\tau(t)|) dz)^2}{\alpha v}} \\
 &= e^{\frac{-(\int_0^H \alpha^{\frac{1}{2}} \kappa(t, z)((|\sigma(t)| + |\tau(t)|)^2)^{\frac{1}{2}} dz)^2}{\alpha v}} \\
 &= e^{\frac{-\alpha(|\sigma(t)| + |\tau(t)|)^2 (\int_0^H \kappa(t, z) dz)^2}{\alpha v}} \\
 &\geq e^{\frac{-\alpha(|\sigma(t)| + |\tau(t)|)^2 (\sup_{t \in [0, H]} \int_0^H \kappa(t, z) dz)^2}{\alpha v}} \\
 &\geq e^{\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}
 \end{aligned}$$

Then, for all $t \in [0, H]$, we have

$$\max_{t \in [0, H]} e^{\frac{-(|\Phi\sigma(t)| + |\Phi\tau(t)|)^2}{\alpha v}} \geq e^{\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}.$$

It follows that

$$\max_{t \in [0, H]} e^{\frac{-(|\Phi\sigma(t)| + |\Phi\tau(t)|)^2}{\alpha v}} \geq \max_{t \in [0, H]} e^{\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}.$$

Thus we have

$$\Re(\Phi\sigma(t), \Phi\tau(t), \alpha v) \geq \Re(\sigma(t), \tau(t), v).$$

We know that for all $t \in [0, H]$,

$$e^{\frac{-(|\Phi\sigma(t)| + |\Phi\tau(t)|)^2}{\alpha v}} \geq e^{\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}.$$

Hence we have for all $t \in [0, H]$,

$$1 - e^{\frac{-(|\Phi\sigma(t)| + |\Phi\tau(t)|)^2}{\alpha v}} \leq 1 - e^{\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}.$$

It follows that,

$$1 - \max_{t \in [0, H]} e^{-\frac{-(|\Phi\sigma(t)| + |\Phi\tau(t)|)^2}{\alpha v}} \leq 1 - \max_{t \in [0, H]} e^{-\frac{-(|\sigma(t)| + |\tau(t)|)^2}{v}}.$$

Thus we have $\mathfrak{I}(\Phi\sigma(t), \Phi\tau(t), \alpha v) \leq \mathfrak{I}(\sigma(t), \tau(t), v)$.

Also, observe that all conditions of Theorem 3.2. are satisfied. Therefore, the operator Φ has a unique fixed point. This means that the integral equation (9) has a unique solution. $\square$

5 CONCLUSION

In this manuscript, we introduced the concept of intuitionistic fuzzy b-metric-like space and established some properties. Also, we gave several fixed point results which is an important issue in applications. This study is the extended form of fuzzy b-metric-like spaces (Javed et al., 2021). The obtained results support the approaches in the literature. As is well-known, in recent years, the study of fixed point theory has been extensively researched due to its fundamental role in various fields of mathematics, science, and engineering. By proving fixed point theorems based on new contractive mappings in intuitionistic fuzzy b-metric-like space, researchers can give new applications of these theorems such as navigation and equilibrium Point in control systems.

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