Portfolio optimization using Mean Absolute Deviation (MAD) and Conditional Value-at-Risk (CVaR)

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Abstract

This paper investigates the efficiency of traditional portfolio optimization models when the returns of financial assets are highly volatile, e.g., in financial crises periods. We also develop alternative optimization models that combine the mean absolute deviation (MAD) and the conditional value at risk (CVaR), attempting to mitigate inefficient, low return and/or high-risk, portfolios. Three methodologies for estimating the probability of the asset's historical returns are also compared. By using historical data on the Brazilian stock market between 2004 and 2013, we analyze the efficiency of the proposed approaches. Our results show that the traditional models provide portfolios with higher returns, but our propose model are able to generate lower risk portfolios, which might be more attractive in volatile markets. In addition, we find that models that do not use equiprobable scenarios produce better results in terms of return and risk.

Keywords

Portfolio optimization. Mean Absolute Deviation. Conditional Value-at-risk. Brazilian stock market.

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1. Introduction

In the financial market, one of the most important issues relates to the composition of a stock portfolio that fits the investor's desire, and investor satisfaction is directly related to the risk and return that the portfolio or the share offers. The theory of portfolio selection published by Harry Markowitz in 1952 was the major breakthrough in financial decision making (Kolm et al., 2014). Many researchers developed other models based on Markowitz's model, which is a quadratic programming model, and attempted to linearize it (Sharpe, 1971). Konno & Yamazaki (1991) suggested a linear programming model in which portfolio risk is measured with the mean absolute deviation (MAD) instead of variance. The linearization of Markowitz's model reduced the computational time of determining the optimal portfolio. Real features were included in the models to make them more applicable in the real world. Mansini & Speranza (2005) incorporated real features such as transaction costs, minimum transaction lots, cardinality constraints, and thresholds on maximum or minimum investments. Kolm et al. (2014) also highlighted some of the new trends in portfolio optimization, such as diversification methods, risk-parity portfolios, the mixing of different sources of alpha, and practical multi-period portfolio optimization.

The motivation for this study is that we understand that traditional optimization models might fail to provide efficient portfolios when the returns of financial assets are highly volatile, e.g., in financial crises periods. Therefore, the main question we want to answer is: Can alternative risk models mitigate the probability of generating inefficient portfolios (low return and/or high risk) during such crises? To answer this question, we propose extensions to the "Beta" portfolio mathematical model to incorporate the so-called *conditional value at risk* (*CVaR*) as an additional fashion to quantify and mitigate risk. We propose to combine both MAD and CVaR in three different ways to analyze the tradeoffs between risk and return provided by these two risk measures. As most models found in the literature considered equiprobable scenarios for the returns, we also test three methodologies of weighting those scenarios. The portfolio models are tested according to assets traded in the Brazilian market between 2004 and 2013.

The remainder of this paper is organized as follows. Section 2 presents the literature review of the portfolio optimization. Section 3 presents the proposed Beta-CVaR model. Section 4 presents the methodology used to collect data and to evaluate the key parameters of the model. Section 5 discusses the computational results. Conclusions and research directions are depicted in Section 6.

2. Literature review

Markowitz's (1952) pioneering work showed that in addition to the proportion of the assets, the risk of an investment portfolio depends on the covariance between the assets. The author revealed that the return of a portfolio is determined by the weighted average return on assets, whereas risk depends on the covariance between the assets comprising the portfolio. Markowitz's (1952) model may be written as the following non-linear quadratic programming model:

("Markowitz" model)

$$\min \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} x_i x_j \sigma_{ij} \tag{1}$$

$$\text{s.t.:} \sum_{j\in\mathcal{N}} x_j r_j \geq \omega_0 M_0 \tag{2}$$

$$\sum_{j\in\mathcal{N}} x_j = M_0 \tag{3}$$

$$x_j \geq 0,\ j\in\mathcal{N}, \tag{4}$$

where $\mathcal{N}=\{1,\ldots,N\}$ is the set of assets; σ_{ij} is the covariance between r eturns i and j; r_j is the average expected return of asset j; ω_0 is the minimum return required by the investor; M_0 is the capital available for investment; and decision variable x_j is the proportion of capital allocated to invest in asset j. Objective function (1) minimizes the portfolio risk, i.e., the covariance between returns. Constraint (2) states that the portfolio return is equal to or greater than the return required by the investor, where such a return is obtained by multiplying the capital available for investment, M_0, and the rate of return that an investor would like to obtain, ω_0. Constraint (3) guarantees that the sum of the capital proportions invested in each asset j is equal to the capital available. The set of constraints (4) establishes the domain of the decision variables.

To overcome the potential computational intractability of Markowitz's nonlinear model (1952), some authors have proposed to use alternative, more tractable, risk measures. Sharpe (1971), for instance, assumed that assets are correlated with a single index representing the return of all market assets (Zanini & Figueiredo, 2005). Konno & Yamazaki (1991) introduced the so-called mean absolute deviation (MAD) to measure the risk of a portfolio. MAD results in a linear programming model, which is proved to be equivalent to Markowitz's model and computationally more tractable. The MAD linear programming model, proposed by Konno & Yamazaki (1991), can be posed as follows:

("MAD" model)

$$\min \sum_{t\in\mathcal{T}} p_t y_t, \tag{5}$$

s.t.: "Markowitz" constraints (2)-(3)

$$y_t + \sum_{j\in\mathcal{N}} \left(r_{jt} - r_j\right) x_j \geq 0,\ t\in\mathcal{T} \tag{6}$$

$$y_t - \sum_{j\in\mathcal{N}} \left(r_{jt} - r_j\right) x_j \geq 0,\ t\in\mathcal{T} \tag{7}$$

$$0 \leq x_j \leq u_j,\ j\in\mathcal{N} \tag{8}$$

$$y_t \geq 0, t \in \mathcal{T}, \tag{9}$$

where *rjt* is the average expected return of asset *j* in scenario *t*, $\mathcal{T} = \{1, \ldots, T\}$ is the set of scenarios (or periods) *t* relative to the asset realizations, and variable p_t is the probability of scenario *t*. The objective function (5) minimizes the expected risk, given by the mean absolute deviation. Constraints (6) and (7) account for the deviation of the values below and above the expected value of the portfolio, respectively. Constraint (8) ensures that the proportion of the amount invested in each asset *j* must be less than or equal to the limit u_j in an attempt to ensure a greater portfolio diversification. Constraint (9) is the domain of the new variable y_t.

Mansini & Speranza (2005) proposed extensions for Konno's & Yamazaki's (1991) model to consider other real characteristics in the composition of optimal portfolios, such as the purchase of shares by batch and the incorporation of direct and indirect transaction costs. In addition, the authors weighed the expected return and portfolio risk in the objective function using a *mean-risk* model. With these changes, the mixed-integer linear programming model proposed by Mansini & Speranza (2005) may be formulated as follows:

("MS" model)

$$\max \sum_{j \in \mathcal{N}} \left[(1-g) r_j s_j x_j - c_j z_j \right] - \sum_{t \in \mathcal{T}} p_t y_t \tag{10}$$

$$\text{s.t.}: y_t + \sum_{j \in \mathcal{N}} \left(r_{jt} - r_j \right) s_j x_j \geq 0, \ \ t \in \mathcal{T} \tag{11}$$

$$\sum_{j \in \mathcal{N}} \left[(1-g) r_j s_j x_j - c_j z_j \right] \geq \omega_0 \sum_{j \in \mathcal{N}} s_j x_j \tag{12}$$

$$\sum_{j \in \mathcal{N}} s_j x_j \leq M_0 \tag{13}$$

$$x_j \leq u_j z_j, \ \ j \in \mathcal{N} \tag{14}$$

$$y_t \geq 0, \ \ t \in \mathcal{T} \tag{15}$$

$$x_j \in \mathbb{Z}_+, \ \ j \in \mathcal{N} \tag{16}$$

$$z_j \in \{0,1\}, \ \ j \in \mathcal{N}, \tag{17}$$

where *g* is the tax paid over the obtained return r_j; c_j is the fixed cost incurred only if there is investment in asset *j*; S_j is the price quote of asset *j*; and z_j is a binary decision variable that takes value 1 if asset *j* is selected and 0 otherwise. The mean-risk type objective function (10) maximizes the difference between the expected return of the portfolio and the risk measured by the mean absolute semi-deviation. Constraint (11) considers only the semi-deviation below the mean. Constraint (12) ensures that the expected return on invested capital is equal to or greater than the return specified by the investor. Constraint (13) represents the total capital available for investment. Constraint (14) requires a maximum investment limit for each selected asset *j*. Constraints (15)-(17) represent the domain of the decision variables. Note that unlike the previous cases, Mansini's & Speranza's model (2005) requires that decision variables x_j be integer values to represent the purchase of shares in batches.

Albuquerque (2009) studied the problem of choosing an optimal portfolio applied to the Brazilian capital market. The model proposed by the author can be considered an extension of MS model proposed by Mansini & Speranza (2005) and features the inclusion of diversifiable and non-diversifiable risks in the linear model studied. Diversified risk was considered by imposing a minimum number of assets in the composition of the optimal portfolio, whereas non-diversifiable risk was considered using the beta coefficient of the portfolio. Since most studies in the literature address the scenarios as being equiprobable, another aspect that differentiates Albuquerque's work (2009) from the others was the allocation of different probabilities for different scenarios. Thus, the Beta model proposed by Albuquerque (2009) can be considered an adaptation of MS model and is written as:

("Beta" model)

$$\max \sum_{j\in\mathcal{N}} \left[(1-g) r_j s_j x_j - c_j z_j\right] - \sum_{t\in\mathcal{T}} p_t y_t, \tag{18}$$

s.t.: "MS" constraints (11)-(13) and (15)-(17)

$$\sum_{j\in\mathcal{N}} \left(\beta_{max} - \beta_j\right) s_j x_j \geq 0, \tag{19}$$

$$\sum_{j\in\mathcal{N}} \left(\beta_j - \beta_{min}\right) s_j x_j \geq 0, \tag{20}$$

$$\sum_{j\in\mathcal{N}} z_j \geq k, \tag{21}$$

$$l_j z_j \leq x_j \leq u_j z_j, \ j \in \mathcal{N}, \tag{22}$$

where l_j is the lower bound for the number of acquired asset *j*; *k* is the minimum number of assets that should make up the portfolio; β_{max} is the maximum value the beta of the assets can take; β_{min} is the minimum value the beta of the assets can take; and β_j is the beta of asset *j*. New constraints (19) and (20) limit the portfolios to the range [β_{min}, β_{max}]. Note that it is possible that the β's of some assets are not within the range [β_{min}, β_{max}]. This possibility implies that the sum of the β's of the assets included in the portfolio is greater than or equal to zero. Constraint (21) establishes that the portfolio has at least *k* different assets. The difference between constraints (22) and (14) in model MS is the incorporation of a minimum threshold for variables x_j.

In addition to the previously presented models, other (deterministic) formulations to the problem of optimal portfolio are based on the minimax criterion (Young, 1998) and goal programming (Charnes et al., 1955; Charnes & Cooper, 1957; Lee, 1972; Aouni et al., 2014). Other extensions include cardinality constraints, multi-period horizons and approaches that allow the quantification of uncertainty and risk in the estimation of expected returns through Bayesian techniques, stochastic programming and robust optimization (Kolm et al., 2014).

Mansini et al. (2014) reviewed the variety of LP solvable portfolio optimization models presented in the literature, the real features that have been modeled, and the solution approaches to the resulting models. They surveyed the main algorithms proposed in the literature for portfolio problems and classified them according to their nature in heuristic and exact solution approaches. Aouni et al. (2014) also reviewed papers that studied the portfolio optimization problem, but they presented the different variants of the goal programming (GP) model that have been applied to the financial portfolio selection problem from the 1970s to the present.

Although many papers have already discussed different solutions to solve the portfolio optimization problem, questions regarding portfolio management techniques during crises remain unclear. Sandoval Junior & Franca (2012) showed that high market volatility is strongly correlated with great crashes. The authors investigated some of the largest financial market crises that had occurred in recent years. After the crisis, many researchers started to study the portfolio optimization problem through a multi-objective approach.

This research and most aforementioned works considered one objective function, but recent papers have studied the portfolio optimization problem through multi-objective functions. Anagnostopoulos & Mamanis (2010) presented a portfolio model with three objectives and discrete variables. The first two objectives were to determine the variance (minimize risk) and the expected return (maximize return) of the portfolio, as commonly utilized in portfolio selection problems, and the third objective was to measure the number of assets held in the portfolio and minimize risk. The authors used multi-objective optimization techniques to solve the mixed-integer multi-objective optimization problem. Pouya et al. (2016) added two attributes to the primary Markowitz mean-variance model as two objectives. The first attribute was the P/E criterion, which captures the current expectations of market activists regarding different companies, and the other attribute was experts' recommendations on the market sector, which captures experts' predictions about the future of the stock market. The authors also used three methods to solve the model: the proposed Invasive Weed Optimization (IWO), the particle swarm optimization algorithm (PSO), and the reduced gradient method (RGM). Babaei et al. (2015) formulated the portfolio optimization problem as multi-objective mixed integer programming. The authors developed a two-step procedure to investigate the portfolio optimization problem in a more realistic manner. In the first step, they concentrated on modelling financial data by adopting stable distributions as the margins of the portfolio return. In the second step, the extension of Markowitz's model was taken into account, and the risk was measured by value at risk. Two multi-objective algorithms based on the particle swarm concept were proposed to solve the model and were compared against the non-dominated sorting genetic algorithm (NSGAII) and Strength Pareto Evolutionary Algorithm 2 (SPEA2).

In addition to using the multi-objective function in the portfolio optimization problem, some researchers have used fuzzy numbers to represent the uncertainty of future returns. Mashayekhi & Omrani (2016) proposed a multi-objective model that incorporates DEA cross-efficiency into Markowitz's mean-variance model and considers the return, risk and efficiency of the portfolio. Because the computationally tractability of the proposed model, the authors applied a second version of the non-dominated sorting genetic algorithm (NSGAII). Saborido et al. (2016) also used fuzzy numbers to consider the uncertainty of future return. The authors considered the Mean-Downside Risk-Skewness (MDRS) model, which takes into account both the multidimensional nature of the portfolio selection problem and the requirements imposed by the investor. The quantification of uncertain future return on a given portfolio is approximated by means of LR-fuzzy numbers, while the moments of its return are evaluated using possibility theory. The authors proposed an algorithm to solve the MDRS model, incorporating the operators suggested into the NSGAII, MOEA/D and GWASF-GA algorithms. Chen (2015) discussed the portfolio optimization problem with real-world constraints under the assumption that the returns of risky assets are fuzzy numbers. The author proposed a possibilistic mean semi-absolute deviation model, in which transaction costs, cardinality and quantity constraints are considered. The model becomes a mixed integer nonlinear programming problem, and the author developed a modified artificial bee colony (MABC) algorithm to solve the optimization problem.

The accurate estimation of financial asset returns is a major challenge in portfolio optimization. A common practice is to use historical data to estimate future returns. However, this practice can lead to inaccurate and/or unrealistic values, especially in unstable and volatile markets, such as the Brazilian financial market. After the Brazilian economic stabilization in 1994, there was optimism about the financial market, but it was accompanied by a fear of volatility in the country. Verma & Soydemir (2006) argued that the prize obtained by investments in Latin American markets is large compared to developed economies. According to the same authors, Latin American countries practice high interest rates, providing attractiveness to investors. However, despite offering attractive returns, these markets are unstable environments, where the prices of financial assets show great variability compared to stable markets.

3. Beta-CVaR models with different weights

A limitation of *value at risk* (VaR) based-models is that it does not provide information regarding the value of the loss that exceeds the VaR, which occurs with probability $(1 - \alpha)$. This limitation can result in inefficient portfolios, especially when the assets present long tail distributions or high tail dependence. Consequently, determining the expected conditional value of the losses of a portfolio, known as CVaR, may provide relevant information to the composition of a portfolio as it takes into account large losses with low probabilities. Mathematically, the conditional expectation can be represented as $\text{CVaR} = \text{E}(\xi|\xi \geq \text{VaR}$, where ξ is a random variable. Therefore, the Beta-CVaR model includes the risk measure CVaR in the Beta model.

We proposed in this paper three different objective functions that use both CVaR and MAD to model risk to provide an alternative, less risky, tractable portfolio optimization problem. The Beta-CVaR models can be represented as follows:

("Beta-CVaR" model)

$$\max \ \lambda\left[\sum_{j=1}^{n}\left[(1-g)r_j s_j x_j - c_j z_j\right] - \sum_{t=1}^{T} p_t y_t\right] + (1-\lambda)\left[\eta - \frac{1}{1-\alpha}\sum_{t=1}^{T} p_t d_t\right] \tag{23}$$

$$\max \ \lambda\sum_{j=1}^{n}\left[(1-g)r_j s_j x_j - c_j z_j\right] + (1-\lambda)\left[\eta - \frac{1}{1-\alpha}\sum_{t=1}^{T} p_t d_t - \sum_{t=1}^{T} p_t y_t\right] \tag{24}$$

$$\max \ (1-\lambda)\left[\eta - \frac{1}{1-\alpha}\sum_{t=1}^{T} p_t d_t\right] - \lambda\sum_{t=1}^{T} p_t y_t \tag{25}$$

s.t.: Constraints (11)-(13), (15)-(17), (20)-(21)

$$\eta - \sum_{j=1}^{n}\left[(1-g)r_{jt}s_jx_j - c_jz_j\right] \le d_t,\ t = 1,\ldots,T; \tag{26}$$

$$d_t \ge 0,\ t = 1,\ldots,T; \tag{27}$$

where η is a free and independent variable that represents the *β-quantil* value in the optimization or the (VaR) value; $d_t = \max\{0, \eta - h_t\}$ measures the deviation of the realization in portfolio h_t when $h_t < \eta$; λ is the so-called risk parameter; and (1 – α) is the confidence level.

The objective function (23) determines the tradeoff between Beta and CVaR models. The first term $\left[\sum_{j=1}^{n}\left[(1-g)r_js_jx_j - c_jz_j\right] - \sum_{t=1}^{T}p_ty_t\right]$ represents the objective function of the Beta model, whereas the second term $\left[\eta - \frac{1}{1-\alpha}\sum_{t=1}^{T}p_td_t\right]$ represents the CVaR model. Thus, $\lambda = 1$ results in a model equivalent to Beta, whereas $\lambda = 0$ corresponds to a model equivalent to CVaR. For $\lambda \in [0,1]$, there are solutions that consider the importance of both models.

The objective function (24) provides the tradeoff between return and risk when the risk is represented by the linear combination of the MAD and the CVaR. When $\lambda = 1$, the return maximization is preferred, regardless of the risk associated with the assets. When $\lambda = 0$, the solution to the maximization problem leads to a more conservative model in which only the risk is minimized, regardless of how low the returns of the selected assets are.

Finally, the objective function (25) represents the tradeoff between the two risk measures, i.e., CVaR and MAD. For λ – 1, only the risk defined by MAD is considered, whereas $\lambda = 0$ implies minimizing only the CVaR risk measure. Note that it is expected that the solutions of the problems defined by objective functions (23) and (25) are similar. The set of constraints (10)-(13) and (15)-(17) are those used in the MS model, while constraints (19)-(22) are the ones defined in the Beta model. Finally, constraints (26) and (27) define variable *dt* as the deviation of the realization in portfolio *ht* to η when $ht < \eta$ and zero otherwise, i.e., $dt = \max\{0, \eta - ht\}$, where $h_t = \sum_{j=1}^{n}\left[(1-g)r_js_jx_j - c_jz_j\right]$ is the realization of the portfolio in scenario *t* when the fixed and proportional costs are taken into account.

4. Data description and parameters estimation

The data used in this study were collected from the Economatica® system and consist of monthly returns of stocks included in the *Ibovespa* index traded in at least 90% of the analyzed period. After collecting the data, two samples with different time horizons were built. The first dataset included the monthly returns between 2004 and 2013 (34 assets with approximately 64% of the market share in the *Ibovespa* index), and the second included the monthly returns between 2007 and 2013 (48 assets with approximately 75% of the share in *Ibovespa*).

4.1. Evaluating the probability of scenarios

Several studies consider the scenarios to be equiprobable (Markowitz, 1952; Konno & Yamazaki, 1991; Mansini & Speranza, 2005), and only a few studies have employed alternative methods to estimate non-equiprobable probabilities for the scenarios of the asset's returns (Albuquerque, 2009). In this paper, besides the equiprobable scenarios' method, we also examine two methods. The first, following Albuquerque (2009), attributed higher probabilities to the most stable scenarios. The second is histogram-based, and the different classes of the histogram are considered to be representative of the different scenarios.

4.1.1. Method 1: Prioritizing more stable scenarios

The idea is to assign a greater probability to more stable scenarios, which might imply a better representation of the behaviour of the Brazilian market (Albuquerque, 2009). Therefore, the monthly *Ibovespa* returns from 2004 to 2013 were used. Twelve scenarios were generated, one for each month of the year. The probability of the scenarios was obtained by the dispersion of the monthly returns of the respective periods relative to the average of the scenario. For instance, only January returns from 2004 to 2013 were grouped, and then, the average and dispersion of the January returns were determined. Three different dispersion measures were used to evaluate which one performed better: the standard deviation, the variance and the mean absolute deviation. Therefore, the probability of each scenario was monthly based and dependent on the average and dispersion of each month's returns. Next, the steps for evaluating the probabilities of the scenarios are presented.

To assign higher probabilities to the scenarios with a lower dispersion, the scenario probabilities must be inversely proportional to the measure of dispersion of the scenario, denoted by λ_t, i.e.,

$$p_t = K/\lambda_t, t = 1, \ldots, T, \tag{28}$$

where $K = \frac{1}{\sum_{t=1}^{T_c} \frac{1}{\lambda_t}}$ is the proportionality constant needed to ensure that p_t is a probability measure.

The probability values can be used to determine expected returns on assets j, expressed by $r_j = E\left[R_j\right] = \sum_{t=1}^{T} p_t r_{jt}$. In the case of the MAD, MS and Beta models, these weights are also used in the objective function.

Figure 1 presents the probabilities assigned to the scenarios using the dispersion measures. These probabilities were calculated taking into account the data of the *Ibovespa* monthly returns over the 2004-2013 period.

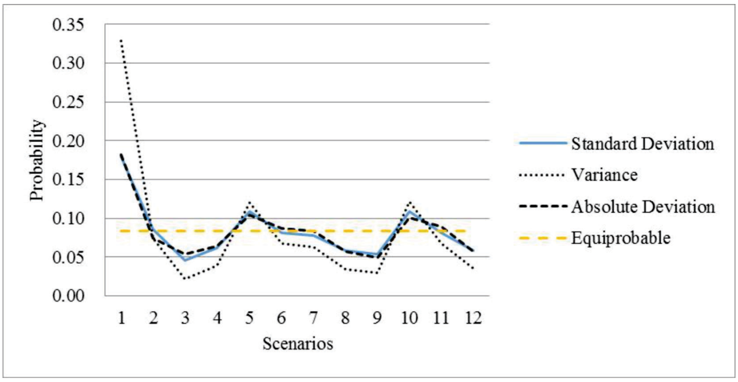

Figure 1. Probabilities of the scenarios through the stable scenario. Source: Authors.

4.1.2. Method 2: Histogram based method

For the histogram method, the *Ibovespa* monthly returns were used, and 119 months were considered, from February 2004 to December 2013. MINITAB® 14 software was used. Figure 2 shows the histogram of the monthly returns of the *Ibovespa* index.

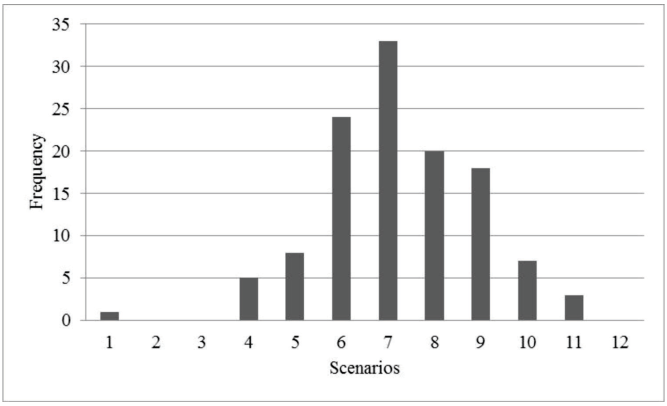

Figure 2. Histogram of the *Ibovespa* monthly returns from 02/2004 to 12/2013. Source: Authors.

The data were classified into 12 scenarios to be compared with the previous method. The results of the histogram revealed that values with greater probabilities were found between scenarios 6 and 9, and the probabilities of scenarios 2, 3 and 12 were zero. In this method, the probability of the occurrence of each scenario represents the probability of the occurrence of the month of the year.

4.2. Determining the betas of the assets

The present study used daily *Ibovespa* returns as the market indicator. The betas of each candidate asset to compose the portfolio were obtained through the linear regression between the return of the specific asset (dependent variable) and the *Ibovespa* index (explanatory variable). It is noteworthy that for the evaluation of the asset betas, the daily returns of the assets were used, but to solve the portfolios, the monthly returns were used.

5. Computational results

In this section, we present the computational tests to illustrate the performance of the traditional models, i.e., "Markowitz," "MAD", "MS", "Beta", and our proposed "Beta-CVaR" approaches, in generating efficient portfolios. The comparison of these models aims to identify which is the most suitable model for the Brazilian market and to present the advantages and disadvantages of each model. We also compared the proposed three methods for evaluating the probability of the scenarios regarding the asset's returns. For this purpose, we generate six test groups by varying the number of assets, the period, and the probability criteria, as depicted in Table 1. In the end, we assess the impact of these characteristics on the optimal composition of the portfolio.

All the models were coded in Algebraic Modeling System GAMS 24.0.2, and the tests were run on an HP Pavilion dv6 Notebook PC machine with Intel ® Core ™ i7-2670QM CPU @ 2.20GHz, 8.0 GB of RAM under Windows 7 operating system.

Table 1. Summary of the computational tests carried out.

Test Groups	Number of Assets	Period	Type of weighting
1	34	2004-2013	Equiprobable
2	34	2004-2013	Stable Scenarios
3	34	2004-2013	Histogram
4	48	2007-2013	Equiprobable
5	48	2007-2013	Stable Scenarios
6	48	2007-2013	Histogram

Source: Authors.

5.1. Analysis of the proposed Beta-CVaR models

To analyze the three objective functions of the Beta-CVaR model and its behavior when a variation occurs in parameter (λ), computational tests were performed in the Beta-CVaR model in test group 3. This test group was chosen because it generated the best results for the models (Markowitz, MAD, MS and Beta), as shown in section 5.1.

This computational test consisted of a variation of parameter (λ) in the Beta-CVaR model. The parameter varies from 0 to 1. Figures 3, 4, and 5 show the tradeoff between the effective return and the risk of the optimal portfolios generated by the Beta-CVaR model, when the values of parameter λ vary from 0 to 1 for the three objective functions used in the proposed Beta-CVaR model.

Figure 3 shows the tradeoff between the Beta model and the CVaR model for the optimal portfolios generated by the Beta-CVaR model considering objective function (23). We can observe in Figure 3 that the results of effective return remain constant with the increase in the values of λ. However, the risk increases when λ increases, except for $\lambda = 0.2$ and $\lambda = 1$.

The optimal portfolios generated with $\lambda = 1$ provided the same result as the Beta model within the same test group, i.e., with value $\lambda = 1$, the proportion of the use of risk measures CVaR is zero. Therefore, only the MAD risk measure was used to solve the model. Because the risk measure used was the same, it presented the same result as the Beta model. For the Beta-CVaR model considering objective function (23), the value of $\lambda = 0.2$ was chosen to perform the computational test because among the values of λ that considered both risk measurements (MAD and CVaR), it presented the lowest risk.

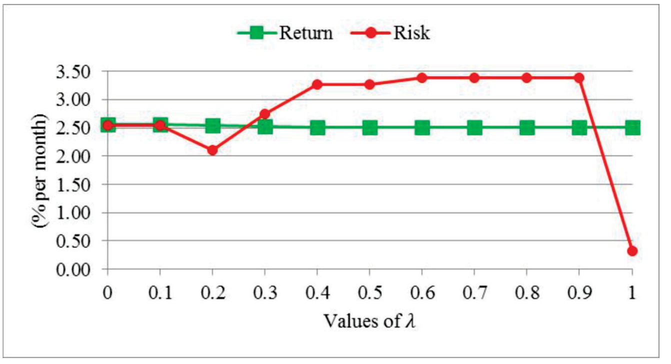

Figure 3. Tradeoff return x risk chart generated by Beta-CVaR model with objective function (23). Source: Authors.

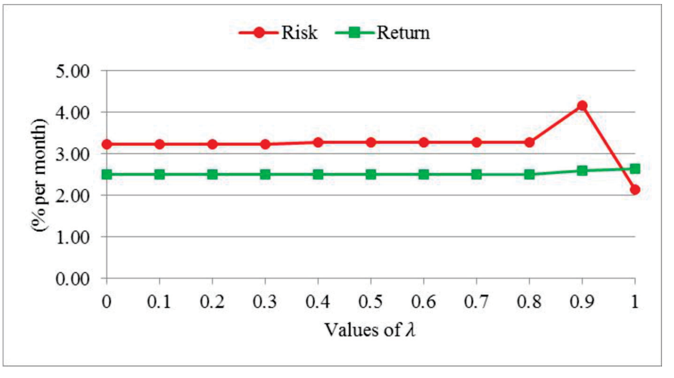

Figure 4. Tradeoff return x risk chart generated by the Beta-CVaR model with objective function (24). Source: Authors.

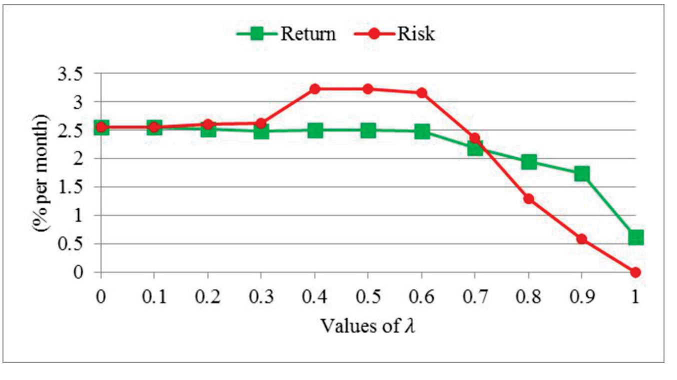

Figure 5. Tradeoff return x risk chart generated by the Beta-CVaR model with objective function (25). Source: Authors.

Figure 4 shows the tradeoff between the effective return and risk (MAD and CVaR) for the optimal portfolios generated by the Beta-CVaR model considering objective function (24). When the value of $\lambda = [0,0.8]$, the optimal portfolio generated obtained similar results, creating portfolios with capital invested in the same assets. With the value of $\lambda = 1$, the optimal portfolio generated seeks to maximize returns without considering investment risk, and in this case, the model will consist of assets with the highest returns. For the Beta-CVaR model considering objective function (24), the value $\lambda = 1$ was chosen because at this point, the percentage of the effective return is greater than the percentage of risk.

Figure 5 shows the tradeoff between the MAD and the CVaR risks for the optimal portfolios generated by the Beta-CVaR model considering objective function (25). When $\lambda = 1$, the optimal portfolio generated seeks to minimize the MAD risk and generates the same results as the optimal portfolio formed by the MAD model (8 – 12). This was expected because the proportion of CVaR was zero. As λ ratio increases, the weighting that aims to minimize the risk MAD increases, and the weighting of CVaR decreases. For the Beta-CVaR model considering objective function (25), the value of $\lambda = 1$ was selected for the computational tests because at this point, both risk measurements are considered, and it is the part that the percentage of effective return becomes greater than the percentage of risk.

The three objective functions showed good results. Objective function (23) establishes the weighting of Beta x CVaR model. In this case, when the value was $\lambda - 1$, the optimal portfolio obtained the same results of the Beta model because when $\lambda = 1$ the Beta-CVaR model considered only the proportion of the Beta model in the objective function. Objective function (24) weights the return x risk. In this case, the objective function has the advantage of seeking to maximize returns without regard to risk, and it may thus be used for risk-seeking investors. Objective function (25) performs the weighting between the risks (MAD and CVaR), and it has the advantage of using risk minimization as an objective; thus, it may be used for more conservative investors. It is noteworthy that for the value of $\lambda = 1$, the generated optimal portfolio achieves the same results of the MAD model because when the value of $\lambda = 1$, the model aims to minimize the risk MAD. Therefore, it is up to investors to analyze what their preferences are in order to choose the best weighting for each objective function.

5.2. Computational tests of the portfolio models

For the six test groups analyzed in this work, the return required by the investor was set at zero ($\omega_0 = 0.00$), and the capital available for each portfolio was 20,000 (currency). Table 2 shows the test results. From now on, consider that "O.F." means objective function. We set the risk parameter λ differently for each proposed O.F.: $\lambda = 0.2$ in the first objective (23); $\lambda = 1$ in the second objective (24); and $\lambda = 0.8$ in the third objective (25).

As shown in Table 2, Markowitz's mathematical model generated effective returns and equal numbers of risks and assets, regardless of the test group. This can be explained by the model's prioritization of the minimization of the portfolio variance. Regarding the number of assets that make up the optimal portfolio, the models generated investment portfolios with a high number of assets. This was expected, as Markowitz's model seeks to diversify the portfolio when it gives preference to assets with lower covariance.

The analysis of the results was divided into three parts. First, the differences among the models were analyzed in the same group test. Second, the differences between the groups were analyzed considering 34 assets (data collection period: 2004-2013) and 48 assets (data collection period: 2007-2013). In other words, the analysis compared group test 1 with group test 4, group test 2 with group test 5, and group test 3 with group test 6. Finally, to identify the differences between the types of scenario weighting, the analysis compared group test 1 with the group tests 2 and 3 and compared group test 4 with group tests 5 and 6.

For the first analysis, the proposed Beta-CVaR model obtained the highest values of effective return of all models, especially the portfolio that was generated with objective function (24), which obtained a higher effective return than objective functions (23) and (25). The portfolio considering objective function (24) in test group 2 obtained the highest effective return (3.28%) of all models. Regarding risk, the optimal portfolio generated by the Beta-CVaR model considering the three objective functions resulted in the lowest values of risk in test groups 3 and 6. In this first analysis, it was concluded that the portfolio generated with the proposed Beta-CVaR model showed higher values of effective return; thus, this model could be used for investors who aim to obtain higher returns.

In the second analysis, the effective return for the models generated in group test 1 was higher than that for the models generated in group test 4. A comparison of group test 2 with group test 5 and of group test 3 with group test 6 revealed the same results. In these cases, the group test that considered 34 assets showed higher values of effective return. It was thus concluded that in the groups with 34 and 48 assets, the optimal portfolios generated using mathematical models that used the group with 34 assets showed higher results in terms of effective returns. This can be considered an indication that considering the database of the assets with the largest collection period (in this case, the data of the assets were from 2004 to 2013) may generate better estimates.

Considering the three criteria for estimating the probability of the scenarios in the last analysis, of the MAD, MS, Beta, Beta-CVaR models (notice that the Markowitz model does not present a probability parameter), histogram-based probabilities generally showed lower values in terms of risk for the optimal portfolios generated by the models. A comparison of group test 1 with group tests 2 and 3 revealed that the models obtained the lowest risk in group test 3 among the group tests within the same models. A comparison of group test 4 with group tests 5 and 6 revealed the models obtained the lowest risk in group test 6 among the groups tests within the same models. In turn, the model that obtained the highest effective return among the models generated in all group tests was the Beta-CVaR model through stable scenarios as scenario weighting. However, it is not

Table 2. Summary of results of the optimal portfolios generated in the six test groups.

Model	Effective Return (%)	Return ($)	Risk ($)	Risk (%)	No. of assets	O.F.	Computational Time (s)
Test Group 1							
Markowitz	1.1501	230.02	20.00	0.1000	17	0.001	0.062
MAD	1.1940	238.79	67.20	0.3360	9	67.20	0.016
MS	2.4407	488.13	129.06	0.6453	4	359.07	0.327
Beta	2.4404	488.06	129.05	0.6452	5	359.02	0.310
Beta-CVaR O.F. (23)	2.5638	512.76	1031.74	5.1587	5	443.49	0.005
Beta-CVaR O.F. (24)	2.5638	512.76	170.47	0.8524	5	512.75	0.005
Beta-CVaR O.F. (25)	1.3657	273.14	293.55	1.4677	7	20.39	0.005
Test Group 2							
Markowitz	1.1501	230.02	20.00	0.1000	17	0.001	0.062
MAD	1.2309	246.18	43.43	0.2171	7	43.43	0.031
MS	2.3430	468.60	116.60	0.5830	2	351.99	0.109
Beta	2.2260	445.21	93.35	0.4668	5	351.86	0.093
Beta-CVaR O.F. (23)	3.2547	651.81	1553.80	7.7690	5	517.49	0.005
Beta-CVaR O.F. (24)	3.2811	656.23	639.79	3.1989	5	656.04	0.005
Beta-CVaR O.F. (25)	1.6190	323.79	361.46	1.8073	7	25.57	0.005
Test Group 3							
Markowitz	1.1501	230.02	20.00	0.1000	17	0.001	0.062
MAD	0.6139	122.77	0.776	0.0004	7	0.776	1.482
MS	2.5061	501.19	63.65	0.3182	3	437.54	0.125
Beta	2.5055	501.08	63.70	0.3185	5	437.38	0.046
Beta-CVaR O.F. (23)	2.5514	510.22	421.44	2.1072	5	467.57	0.005
Beta-CVaR O.F. (24)	2.6384	527.53	427.45	2.1372	5	527.65	0.005
Beta-CVaR O.F. (25)	1.9359	387.17	258.95	1.2948	5	57.52	0.125
Test Group 4							
Markowitz	0.8362	167.24	20.00	0.1000	18	0.001	0.015
MAD	0.9567	191.33	65.36	0.3268	8	65.36	0.031
MS	1.7334	346.69	73.47	0.3674	6	273.22	0.063
Beta	1.7334	346.69	73.47	0.3674	6	273.22	0.172
Beta-CVaR O.F. (23)	1.8505	370.10	970.12	4.8506	5	427.53	0.005
Beta-CVaR O.F. (24)	2.0167	403.34	671.07	3.3553	5	403.33	0.005
Beta-CVaR O.F. (25)	0.6893	137.86	575.19	2.8759	5	82.585	
Test Group 5							
Markowitz	0.8362	167.24	20.00	0.1000	18	0.001	0.015
MAD	0.9257	185.15	39.65	0.1983	8	39.65	0.031
MS	1.6871	337.42	70.88	0.3544	5	266.54	0.141
Beta	1.6871	337.42	70.88	0.3544	5	266.54	0.093
Beta-CVaR O.F. (23)	0.2532	50.64	1419.50	7.0975	5	447.95	0.005
Beta-CVaR O.F. (24)	2.9375	587.50	587.43	2.9371	5	587.35	0.005
Beta-CVaR O.F. (25)	0.0100	0.11	669.38	3.3469	5	92.842	0.005
Test Group 6							
Markowitz	0.8362	167.24	20.00	0.1000	18	0.001	0.015
MAD	0.0001	0.02	0.03	0.0001	10	0.03	0.047
MS	1.7765	355.29	52.28	0.2614	3	303.02	0.109
Beta	1.7762	355.24	52.22	0.2611	5	303.02	0.109
Beta-CVaR O.F. (23)	1.7850	357.00	657.43	3.2871	5	431.11	0.005
Beta-CVaR O.F. (24)	1.7873	357.46	658.93	3.2946	5	395.865	0.005
Beta-CVaR O.F. (25)	0.7333	146.67	391.75	1.9587	5	97.513	0.005

O.F. means objective function. Source: Authors.

possible to say that this method will always generate better results because the histogram-based method showed better results for the MS and Beta models.

To compare the types of risk measures, Table 3 presents the results of the investment portfolios and the dispersion measures. Next, only the standard deviation, the mean absolute deviation, the VaR and the CVaR were analyzed, as they have the same unit of measurement. The portfolios generated using Markowitz's model showed lower values in terms of risks. The values of the standard deviations, mean absolute deviations, VaR and CVaR were lower for test groups 4, 5, and 6. Moreover, the standard deviation and mean absolute deviation values were lower for test group 1, and VaR and CVaR were lower for test group 3. Comparing the results of the portfolios generated using Markowitz's model among the test groups, the models generated by groups that considered 48 assets showed less dispersion.

With the MAD model, the optimal portfolio with the lowest risk values relative to the standard deviation and mean absolute deviation were the investment portfolio generated in test group 1. Regarding the VaR and the CVaR values, the portfolio generated by group test 2 showed the lowest value. Considering the four risk types,

Table 3. Dispersion measurements of the models of test groups 1 to 6.

Model	Return ($)	Effective Return (%)	No. of assets	Standard Deviation	Absolute Deviation	VaR (C.I. of 95%)	CVaR
Test Group 1							
Markowitz	230.02	1.1501	17	1570.11	1221.44	1155.85	1540.72
MAD	238.79	1.1940	9	1487.65	1143.83	1357.23	1728.24
MS	488.13	2.4406	4	1812.96	1343.93	2044.86	3042.61
Beta	488.06	2.4403	5	1812.79	1343.80	2044.64	3042.55
Beta-CVaR O.F. (23)	512.76	2.5638	5	1680.11	1221.93	2244.24	2797.25
Beta-CVaR O.F. (24)	512.77	2.5638	5	1679.52	1221.45	2244.29	2797.44
Beta-CVaR O.F. (25)	273.14	1.3657	7	1501.68	1146.11	1329.03	1605.07
Test Group 2							
Markowitz	230.02	1.1501	17	1570.11	1221.44	1155.85	1540.72
MAD	246.18	1.2309	7	1504.44	1156.64	983.09	1303.52
MS	468.60	2.3430	2	1768.38	1305.42	2108.96	2968.81
Beta	445.21	2.2260	5	1768.74	1305.68	2107.34	2965.73
Beta-CVaR O.F. (23)	651.81	3.2547	5	2447.19	1809.27	3380.43	4192.03
Beta-CVaR O.F. (24)	656.23	3.2811	5	2441.54	1795.59	3570.41	4541.05
Beta-CVaR O.F. (25)	323.79	1.6190	7	1694.19	1282.96	1538.99	2065.14
Test Group 3							
Markowitz	230.02	1.1501	17	1570.11	1221.44	1155.85	1540.72
MAD	122.77	0.6139	7	1670.14	1299.14	1058.40	1369.66
MS	501.19	2.5059	3	1785.98	1319.23	2074.57	3041.18
Beta	501.08	2.5054	5	1787.36	1320.53	2072.28	3038.36
Beta-CVaR O.F. (23)	510.22	2.5514	5	1995.52	1515.69	2129.91	3427.96
Beta-CVaR O.F. (24)	527.53	2.6384	5	2215.24	1713.61	3217.79	4590.95
Beta-CVaR O.F. (25)	387.17	1.9359	5	1665.15	1257.15	1622.21	1972.42
Test Group 4							
Markowitz	167.24	0.8362	17	1574.57	1198.43	1142.68	1438.74
MAD	191.33	0.9567	8	1937.17	1451.55	1675.05	2100.29
MS	346.69	1.7335	6	1661.08	1326.32	1446.53	2294.18
Beta	346.69	1.7335	6	1661.08	1326.32	1446.53	2294.18
Beta-CVaR O.F. (23)	370.10	1.8505	5	1602.65	1300.98	2264.52	3242.14
Beta-CVaR O.F. (24)	403.34	2.0167	5	1442.25	1145.30	1965.79	2402.22
Beta-CVaR O.F. (25)	137.86	0.6893	5	2451.44	1821.79	3039.84	3805.15
Test Group 5							
Markowitz	167.24	0.8362	17	1574.57	1198.43	1142.68	1438.74
MAD	185.15	0.9257	8	1909.96	1430.94	1759.84	2300.77
MS	337.42	1.6871	5	1881.60	1471.95	1794.41	2740.17
Beta	337.42	1.6871	5	1881.60	1471.95	1794.41	2740.17
Beta-CVaR O.F. (23)	50.64	0.2532	5	1751.09	1392.02	2597.36	3296.71
Beta-CVaR O.F. (24)	587.50	2.9375	5	2224.80	1677.27	3482.22	4461.47
Beta-CVaR O.F. (25)	0.11	0.0000	5	1870.44	1443.68	1981.44	2557.12
Test Group 6							
Markowitz	167.24	0.8362	17	1574.57	1198.43	1142.68	1438.74
MAD	0.02	0.0001	10	2064.52	1547.71	1765.47	2343.90
MS	355.29	1.7765	3	1848.77	1447.08	2009.42	2302.25
Beta	355.24	1.7762	5	1848.80	1447.06	2009.08	2302.23
Beta-CVaR O.F. (23)	357.00	1.7850	5	1678.48	1343.51	2143.89	2986.65
Beta-CVaR O.F. (24)	357.46	1.7873	5	1678.03	1343.24	2144.88	2987.01
Beta-CVaR O.F. (25)	146.67	0.7333	5	1795.24	1431.16	1981.23	2741.40

The variance of each optimal portfolio is a very high value because it is calculated as expressed in Markowitz's equation (1952): $\boldsymbol{\sigma_x^2 = \sum_{j=1}^{n} y_j^2 \sigma_j^2 + \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} y_j y_k \sigma_{jk}}$, where σ_x^2 = portfolio variance, σ_{jk} = covariance between assets j and k, σ_j^2 = asset variance j, y_j = proportion invested in asset j and y_k = proportion invested in asset k and O.F. means objective function. Source: Authors.

the test group that showed low values in terms of risk was the portfolio generated by test group 1, where the lowest values were for the standard deviation and mean absolute deviation risk measures, and the third lowest value was for VaR and CVaR. That is, for the portfolios generated using the MAD model, the equiprobable scenario weighting showed better values in general.

The portfolios generated using the MS and Beta models showed investment portfolios with results very close in value. In this case, the lower results for the standard deviation risk measures, VaR and CVaR, were found in the optimal portfolios generated by test group 4, and the third smallest value for the mean absolute deviation risk measure was found in this test group. The lowest value for the mean absolute deviation was found in the portfolio generated in test group 2. For investment portfolios generated using the MS and Beta models, the lowest values for the risk measures were found using equiprobable scenario and the test group that considered 48 assets.

The portfolios generated by the Beta-CVaR models considering the three objective functions, in general, resulted in lower values of risk measurements for standard deviation and mean absolute deviation compared to portfolios generated using the Beta and MS models. However, for VaR and CVaR, the portfolios generated by the MS and Beta models obtained lower values in almost all test groups when compared with the portfolios generated by the Beta-CVaR considering the three objective functions.

Finally, it is possible to note in Table 3 that lower results in terms of risk measures were found in the portfolios generated using Markowitz's model, which can be explained by the fact that this model's main goal is to minimize risk. Regarding the number of assets, the portfolio groups that considered 48 assets showed better results because they generated a more diversified portfolio than the groups that considered 34 assets.

6. Conclusion

In this study, different types of portfolio optimization models were analyzed. Four models in the literature (Markowitz, MAD, MS, and Beta) were compared with the proposed Beta-CVaR model, which was divided into three different models, each with a different objective function. The proposed extensions for the Beta portfolio optimization model aimed at considering the CVaR as a risk measure to produce more efficient portfolios for the volatile Brazilian market. The proposed models (called Beta-CVaR) combine the CVaR with the MAD in the objective function in three distinct ways to emphasize different tradeoffs. The results obtained with the proposed models were satisfactory, especially with objective function (24), which generated an optimal portfolio with a higher effective return in comparison with the other models in all test groups. In addition, the Beta-CVaR model can be flexible in terms of the usage of parameter λ. This parameter can be used in the comparison with different tradeoffs. In objective function (23), the parameter makes the tradeoff between the Beta model and the CVaR model. Parameter λ was used in objective function (24) to make the tradeoff between the effective return and risk (MAD and CVaR), and it made the tradeoff between the MAD and the CVaR risk in objective function (25). The second aim of this study was to compare three methods for evaluating the probability of the scenarios for the asset's returns. The results indicated that the weights of different scenarios assigning different probabilities to the scenarios perform better than the traditional method of using equiprobable scenarios. In this case, the histogram-based method presented the best results.

A comparison of the types of mathematical models should take into account the investor's preference because according to the results, the Markowitz and MAD models generated diversified portfolios with lower risk. The MS, Beta, and Beta-CVaR models, in turn, showed good results in terms of effective return.

As further work, we suggest considering a broader set of assets, including other market indicators and using fixed-income securities as an investment option with low risk. In addition, we suggest that the calculation of the expected return of each asset be based on the capital asset pricing model (CAPM). Moreover, a multi-objective function can be used in the Beta-CVaR model. For instance, considering two objectives in the model, the first function could be the maximization of the expected return, and the second could be the minimization of the MAD and CVaR. The multi-objective approach is used due to the financial crisis that changed the view of the modern portfolio theory.

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