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Beyond the massless approximation: The rotation of a heavy tourniquet with a load on one end

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Using dynamical approach, we consider the strain distribution in the rotating heavy tourniquet with a load on one end for the case of its small deformations (small angular velocities of the rotation). It appears that the tension force is maximum at the endpoint, where the tourniquet connects to the vertical axis of rotation, and decreases quadratically along the tourniquet length. We conclude that if the tourniquet mass m is not negligible against the load mass M, its effect may be expressed by adding to the mass M the fraction 1/3 of the mass m.
Keywords: Rotating heavy tourniquet, small deformations, tension force.

1. Introduction

The scientific ideal physics models are one of the most common classes of concepts, which are considered in the high school course of physics [1]. Among them the limit transition abstractions play a key role, i.e. the models constructed by limit transitions for the selected number of characteristics to their maximum, minimum or constant values [2]. For example, in the model of a rigid body one approaches its hardness to infinity. Within the model of an ideal gas the molecule sizes and the values of interaction forces between them are tended to zero. For the model of uniform vector field its vector-valued function is assumed to be independent of the spatial coordinates, i.e. it is constant in direction and modulus. Such idealizations describe the behaviour of real physical objects not only qualitatively but also (most importantly) quantitatively.

The correctness of the applicability of a particular model depends on the specific physical situation itself and, strictly speaking, should be determined by means of a physical experiment. However, for education purposes the accuracy of a model, i.e. the errors arising as a consequence of neglecting several factors considered to be insignificant in the construction of the model can be estimated theoretically within the extended model taking into account these neglected factors.

One of the most important limit transition models is the model of massless object, in which its mass is equal to zero. There are many illustrations of using massless approximation in physics education. For example, the study of an Atwood's machine usually simplify the problem considering the pulley and string as massless objects. Tarnopolski [3] has taken into account their finite masses, while analysing the motion of the two weights. He concluded that a non-zero pulley mass leaves this motion uniformly accelerated. At the same time, a non-zero string mass causes an increase in acceleration with time.

Another important example have been considered by Galloni and Kohen [4]. They have investigated the influence of the spring mass on its static and dynamic effects. In this paper, we consider the similar issue, namely, we explore the strain distribution in the rotating heavy tourniquet with a load on one end. The study of strain distribution along some non-rigid bodies has already been carried out earlier in a number of education research papers [5–7]. The consideration of this problem will be useful for the undergraduates studying the basics of elasticity theory.

2. The Problem

Let us consider the homogeneous tourniquet of mass m and natural (initial) length l_0 with stiffness (elastic constant) $k = ES_0/l_0$ [8], where E is the Young's modulus, S_0 is the initial cross-sectional area of the tourniquet. One of its end is fixed on a smooth (frictionless) table, and point mass M is attached to the other end. We assume that the tourniquet uniformly rotates in horizontal plane with an angular velocity of ω. If the resulting horizontal force acting on each element of the tourniquet much greater than the gravity force then the tourniquet can be represented by a horizontal line, whose deformation has only horizontal displacement, neglecting any vertical deformation (this is achieved in the case of sufficient angular speed). Let x be the horizontal coordinate of some cross-section before stretching (Figure 1). In this case, according to

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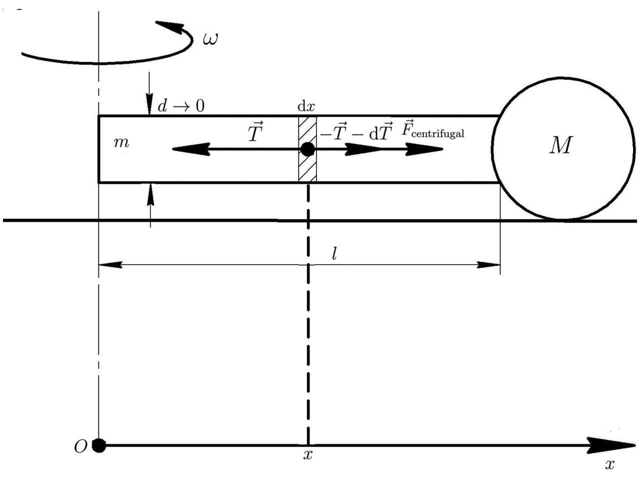

Figure 1: Geometry of the problem.

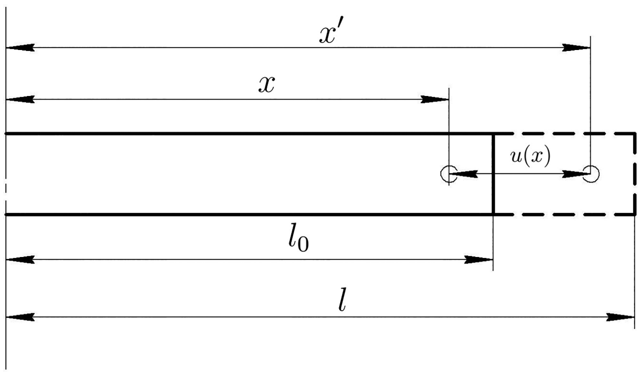

Figure 2: The geometric characteristics of the elongation deformation.

the Hooke's law, the absolute value of tension force $\vec{T}$ at the cross-section with coordinate x is equal to [4]:

$$T(x) = ES_0\frac{\mathrm{d}u}{\mathrm{d}x} = kl_0\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{1}$$

where $u(x) = x'(x) - x > 0$ is the elongation of the tourniquet element of length x (Figure 2). Equation (1) (the linear theory) is valid for small deformations, where $u(x) \ll x$ and $\mathrm{d}u \ll \mathrm{d}x$.

In the reference frame associated with the rotating tourniquet, the equilibrium equation of its infinitesimal element $\mathrm{d}x$, according to Figure 1 (in this figure, the width d of the tourniquet is greatly exaggerated ($d \to 0$), so that the reader can clearly discern all the x components of the forces acting on the element $\mathrm{d}x$) has the following form:

$$T + \mathrm{d}T + F_{\text{centrifugal}} - T = 0. \tag{2}$$

During rotation, the element $\mathrm{d}x$ is located from the axis of rotation by the distance $x + u(x)$. Then, the centrifugal force will be:

$$F_{\text{centrifugal}} = \mathrm{d}m\omega^2[x + u(x)] \approx \mathrm{d}m\omega^2 x, \tag{3}$$

and

$$\mathrm{d}T(x) \approx -\mathrm{d}m\omega^2 x, \tag{4}$$

where $\mathrm{d}m = (m/l)\mathrm{d}x \approx (m/l_0)\mathrm{d}x$ is the mass of the infinitesimal element $\mathrm{d}x$ of the tourniquet. Here we use the approximate equality $l \approx l_0$, which is valid under the linear elasticity theory (l is the total length of the rotating tourniquet (the final length)).

Using equations (1) and (2), we get:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -\frac{m\omega^2}{kl_0^2}x. \tag{5}$$

Hence

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{m\omega^2}{2kl_0^2}x^2 + C_1. \tag{6}$$

Since $T(l_0 + u(l_0)) = M\omega^2(l_0 + u(l_0)) \approx T(l_0) \approx M\omega^2 l_0$, then using equations (1) and (6), we obtain:

$$\frac{M\omega^2}{k} = -\frac{m\omega^2}{2k} + C_1. \tag{7}$$

Then, the integration constant:

$$C_1 = \left(M + \frac{m}{2}\right)\frac{\omega^2}{k}. \tag{8}$$

Therefore, in the case of $m \neq 0$ the deformation is inhomogeneous ($\mathrm{d}u/\mathrm{d}x \neq \text{const}$). Considering equations (1), (6), and (8), we get:

$$T(x) = \left(M + \frac{m}{2}\right)\omega^2 l_0 - \frac{m\omega^2}{2l_0}x^2. \tag{9}$$

The equation (9) tells us that the tension force is maximum at the endpoint, where the tourniquet connects to the vertical axis of rotation, and decreases quadratically with increasing x-coordinate.

Considering again equations (6) and (8), we derive:

$$u(x) = -\frac{m\omega^2}{6kl_0^2}x^3 + \left(M + \frac{m}{2}\right)\frac{\omega^2}{k}x + C_2. \tag{10}$$

At $x = 0$ $u = 0$. Then $C_2 = 0$. Applying equation (10), we find the total elongation:

$$\Delta l - l - l_0 = u(l_0) = \left(M + \frac{m}{3}\right)\frac{\omega^2 l_0}{k}. \tag{11}$$

3. Conclusions

Galloni and Kohen [4] have concluded that if the spring mass m is not negligible against the mass M suspended at its end, its effect for the static case may be expressed by adding to the mass M the fraction $1/2$ of the spring mass. In the dynamic case of the oscillating load, this fraction is approximately equal to $1/3$. Our calculations show that for the case of rotating heavy tourniquet with a load on one end, such a fraction makes exactly

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third part of the tourniquet mass, when determining the total elongation. This circumstance can be physically explained by the presence of the centrifugal force acting on each element dx of the tourniquet due to its non-zero mass. This leads to the fact that the resulting elongation is somewhat increased compared to the ideal case $m = 0$.

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