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Green's function for the lossy wave equation

(Função de Green para a equação da onda dispersiva)

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Using an integral representation for the first kind Hankel (Hankel-Bessel Integral Representation) function we obtain the so-called Basset formula, an integral representation for the second kind modified Bessel function. Using the Sonine-Bessel integral representation we obtain the Fourier cosine integral transform of the zero order Bessel function. As an application we present the calculation of the Green's function associated with a second-order partial differential equation, particularly a wave equation for a lossy two-dimensional medium. This application is associated with the transient electromagnetic field radiated by a pulsed source in the presence of dispersive media, which is of great importance in the theory of geophysical prospecting, lightning studies and development of pulsed antenna systems.

Keywords: Sonine-Bessel, integral representation, dissipative wave equation.

Usando uma representação integral para a função de Hankel de primeira espécie (representação integral de Hankel-Bessel) obtemos a chamada fórmula de Basset, uma representação integral para a função de Bessel modificada de segunda espécie. A partir de uma representação integral de Sonine-Bessel obtemos a transformada de Fourier em co-senos da função de Bessel de ordem zero. Como uma aplicação, apresentamos o cálculo da função de Green associada a uma equação diferencial parcial de segunda ordem, a saber, a equação da onda em um meio dissipativo de dimensão dois. Esta aplicação está associada ao campo eletromagnético transiente irradiado por uma fonte tipo pulso na presença de meios dispersivos, o qual é de grande importância na teoria de prospecção geofísica, estudos sobre luz e desenvolvimento de sistemas de antenas tipo pulso.

Palavras-chave: Sonine-Bessel, representação integral, equação da onda dissipativa.

1. Introduction

In the study of classical special functions, *e.g.*, Bessel functions and Legendre polynomials, two fundamental methods must be mentioned: Rodrigues type formula where the particular special function is presented in terms of derivatives, and integral representations where the particular special function is given by an integral in the complex plane. We mention in passing that all classical special function can be presented by a Frobenius-type series.

Several authors prefer to work with Rodrigues formula [1-3]. This method is convenient to study several properties of special functions as, for example, their recurrence relations. Other authors prefer to work with a suitable integral representation in the complex plane [4]. This method is best suited when one uses integral transforms (Laplace, Fourier, Hankel and Mellin integral transforms) to solve an ordinary or partial differential equation.

Here we follow the second method because we are interested in calculating the Green's function associated with the wave equation. Moreover, we use the Laplace and Hankel integral transforms (the joint transform method) to solve a non-homogeneous second order partial differential equation.

As possible applications we may cite the radiation problem in a curved homogeneous dielectric slab-waveguide, which was investigated by Chang-Barnes [5], and the transient electromagnetic field radiated by a pulsed source in the presence of dispersive media, which is of great importance in the theory of geophysical prospecting, lightning studies and in the development of pulsed antenna systems. We remark that for this last problem Kuester [6] has obtained an exact integral representation for the transient field of a pulsed line source above a plane reflecting surface which can be expressed as a finite integral over the transient plane-wave solution for complex angles of incidence.

This paper is organized as follows: in section 2 we present some applications as a motivation in section 3 we discuss an integral representation for the first kind

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Hankel function, in order to obtain an integral representation for the Bessel function, the so-called Basset formula. In section 4, using the Sonine-Bessel integral representation we calculate an integral involving a Bessel function, which can be interpreted as a Fourier cosine transform of the zero order Bessel function. In section 5, using Laplace and Hankel integral transforms, the so-called joint transform method, we discuss a non-homogeneous second order partial differential equation with constant coefficients and as an application we obtain, in a closed form, the Green's function, associated with the wave equation for a lossy two-dimensional medium. Finally we present our concluding remarks.

2. Some applications

In this paper we derived the Green's function of a non-homogeneous second order partial differential equation in three independent variables with constant coefficients, *i.e.*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-a\frac{\partial^2}{\partial t^2}-b\frac{\partial}{\partial t}\right)u(x,y,t)=f(x,y),$$

where a and b are positive constants and $f(x,y)$ is a continuous function of the form $f(x,y)=f(\sqrt{x^2+y^2})$, a radial function. The equation above is the wave equation.

Waves are important phenomena nowadays. Wave equations appear in many applications of mathematics in physical sciences, like geophysical prospecting, lightning studies and development of pulsed antenna systems. For example, we know that homogeneous Maxwell's equations satisfy the wave equation. Using this fact Maxwell discovered that the solutions of his homogeneous system were electromagnetic waves that propagate in the empty space (vacuum) [7].

As an application on geophysical prospecting we cite the methods of migration. Bleistein [8] says that migration is "the dominant method for reflector imaging from seismic data in geophysics today. The objective of this method might be viewed as moving the reflectors from their time location to their spatial location."So, from seismic data, that are in the time domain, we use this method to recover its shape in the depth domain. One of them is the wavefield migration. This method uses the solution of wave equation to recover the shape of the subsurface. For lightning studies we can find an application using the wave equation to study the transient electromagnetic field of a pulsed line source.

So, as we can see that waves are important in several problems in many different areas. Sezginer and Chew [9] solved the problem of finding the Green's funcion associated with equation above. We used another mathematical technique to solve this problem.

3. An integral representation

In Watson's book [4] it was shown that several contour integrals can be obtained as generalisations of Poisson's integral and several integral representations for Bessel and Hankel (Bessel function of the third kind) functions were obtained with convenient modifications of Hankel's contour integrals. A particular such representation is

$$H_\nu^{(1)}(z)=\frac{2\Gamma(1/2-\nu)\,(z/2)^\nu}{\Gamma(1/2)}\times \frac{1}{2\pi i}\int_{\infty i}^{(1+)}\mathrm{e}^{izt}(t^2-1)^{\nu-1/2}dt, \tag{1}$$

with $|\arg(z)|<\pi/2$ and $\Re(\frac{1}{2}-\nu)>0$, where

$$H_\nu^{(1)}(xz\,\mathrm{e}^{i\pi/2})=\frac{2}{\pi}\mathrm{e}^{-\nu\pi i/2}K_{-\nu}(xz) \tag{2}$$

and $K_\mu(x)$ is a second kind modified Bessel function.

Considering in Eq. (2) x as a positive number and z as a complex number with $|\arg(z)|<\pi/2$ we can write Eq. (1) as

$$H_{-\nu}^{(1)}(xz\,\mathrm{e}^{i\pi/2})=\frac{2\Gamma(\nu+1/2)}{\Gamma(1/2)}\left(\frac{ixz}{2}\right)^{-\nu}\times \frac{1}{2\pi i}\int_{+\infty}^{(1+)}\frac{\mathrm{e}^{-xzt}}{(t^2-1)^{\nu+1/2}}dt, \tag{3}$$

with $\Re(\nu+1/2)>0$.

Now, when $\Re(\nu+1/2)\geq 0$, the integral, taken on arcs of a circle from ρ to $\rho\,\mathrm{e}^{\pm\pi i/2-\theta}$, where $\theta=\arg(z)$, tends to zero as $\rho\to\infty$, by Jordan's lemma [10]. Hence, by Cauchy's theorem, the path of integration may be opened out until it becomes the line on which $\Re(zt)=0$. If we write $zt=iu$, the phase of $[-(u^2/z^2)-1]$ is $-\pi$ at the origin in the u-plane [4]. Using Eq. (2) and the parity of the integral we can write Eq. (3) in the following form

$$K_\nu(xz)=\frac{\Gamma(\nu+1/2)}{\Gamma(1/2)}\left(\frac{2z}{x}\right)^\nu\times \int_0^\infty\frac{\cos(xu)}{(u^2+z^2)^{\nu+1/2}}du, \tag{4}$$

with $\Re(\nu)\geq -1/2$, $x>0$ and $|\arg(z)|<\pi/2$. This expression, an integral representation for second kind modified Bessel function, is known by the name of Basset formula [4].

As a particular case of Eq. (4) we consider $\nu=0$ and then we obtain

$$K_0(xz)=\int_0^\infty\frac{\cos(xu)}{\sqrt{u^2+z^2}}du\,, \tag{5}$$

with $x>0$ and $|\arg(z)|<\pi/2$. We refer that another way to calculate this integral representation is to consider the following convenient limit

$$\lim_{n\to\infty}Q_n[\cosh(z/n)]=K_0(z)\,,$$

where $Q_n(z)$ is the second kind Legendre function, given by the integral representation

$$Q_n(z) = \int_0^\infty \frac{d\theta}{(z+\sqrt{z^2-1}\cosh\theta)^{n+1}}\,,$$

with $|z|>1$. This integral representation can be obtained by means of an integral representation for a hypergeometric function [11] and the expression

$$Q_n(z) = \frac{2^{2n+1}(n!)^2}{(2n+1)!}\xi^{-n-1}\,{}_2F_1\left(\frac{1}{2},1+n;\frac{3}{2}+n;\frac{1}{\xi^2}\right),$$

where $\xi = z+\sqrt{z^2-1}$ and one must take the positive square root for $|z|>1$.

4. Fourier cosine transform of Bessel function

In this section we calculate an integral involving a Bessel function, an integral that can be interpreted as a Fourier cosine transform of the zero order Bessel function by using Sonine's integral representation.

We are interested in calculating the integral

$$\Omega(t,r,\epsilon) = \int_0^\infty \cos(ru)\,J_0(t\sqrt{u^2-\epsilon^2})\,du, \quad (6)$$

where $r>0$, $t>0$ and ϵ is a positive real parameter.

To perform this integral we begin with the integral representation, in the complex plane, for the Bessel function

$$J_\nu(z) = \frac{(z/2)^\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty}\xi^{-\nu-1}\exp(\xi - z^2/4\xi)\,d\xi\,, \quad (7)$$

with $c>0$, which is known as Sonine's integral [4]. We note that in this expression the contour is the so-called Bromwich contour, the same contour used in calculations of inverse Laplace integral transform [11].

Here we are interested in the case $\nu=0$, only. Then, taking $\nu=0$ and $z=+t\sqrt{u^2-\epsilon^2}$ we can write for the Bessel function

$$J_0(t\sqrt{u^2-\epsilon^2}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{d\xi}{\xi}\exp(\xi-\beta^2/4\xi), \quad (8)$$

where $\beta^2 = t^2(u^2-\epsilon^2)$ and $c>0$.

Introducing Eq. (8) in Eq. (6) and changing the order of integration (both integrals are uniformly convergent) we get

$$\Omega(t,r,\epsilon) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{d\xi}{\xi}\exp(\xi+\epsilon^2t^2/4\xi)\times \int_0^\infty du\cos(ru)\,e^{-\gamma u^2},$$

where $\gamma = t^2/4\xi$.

To perform the integral in variable u we complete the square and obtain

$$\Omega(t,r,\epsilon) = \sqrt{\frac{\pi}{\gamma}}\frac{1}{t}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}du\,u^{-1/2}\,e^{(u-\mu^2/4u)}, \quad (9)$$

where $\mu^2 = -\epsilon^2(t^2-r^2)$.

The above equation can be identified with Eq. (7), for $\nu=-1/2$ and we get

$$\Omega(t,r,\epsilon) = \frac{\sqrt{\pi}}{\sqrt{t^2-r^2}}\left(\frac{i\epsilon}{2}\sqrt{t^2-r^2}\right)^{1/2}\times J_{-1/2}(i\epsilon\sqrt{t^2-r^2})\,.$$

Using the following relation between Bessel and modified Bessel functions

$$J_\nu(iz) = e^{i\pi\nu/2}\,I_\nu(z),$$

and, for the particular value $\nu=-1/2$, the relation

$$I_{-1/2}(z) = \left(\frac{\pi z}{2}\right)^{-1/2}\cosh z$$

we finally obtain for our initial integral, Eq. (7)

$$\Omega(t,r,\epsilon) \equiv \int_0^\infty du\cos(ru)\,J_0(t\sqrt{u^2-\epsilon^2}) = \frac{\cosh(\epsilon\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}, \quad (10)$$

for $0<r<t$ and zero otherwise.

5. Green's function

In this section we introduce Laplace and Hankel integral transforms (the joint transform method) to solve a non-homogeneous second order partial differential equation in three independent variables with constant coefficients, *i.e.*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-a\frac{\partial^2}{\partial t^2}-b\frac{\partial}{\partial t}\right)u(x,y,t) = f(x,y),$$

where a and b are positive constants and $f(x,y)$ is a continuous function of the form $f(x,y)=f(\sqrt{x^2+y^2})$, a radial function.

To solve this partial differential equation it is sufficient to look for the associated Green's function, which is the solution of the non-homogeneous partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-a\frac{\partial^2}{\partial t^2}-b\frac{\partial}{\partial t}\right)g(\vec{r}-\vec{r'},t-t') = -2\pi\,\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (11)$$

where we consider the causality condition

$$g(|\mathbf{r}-\mathbf{r}'|,t-t') = 0 \quad \text{for} \quad c(t-t') < |\mathbf{r}-\mathbf{r}'|,$$

where $a=1/c^2$ and c is a constant (velocity of light).

Introducing polar coordinates (translational invariance) $x = r\cos\theta$ and $y = r\sin\theta$ we can rewrite Eq. (11) as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - a\frac{\partial^2}{\partial \tau^2} - b\frac{\partial}{\partial \tau}\right) g(r,\tau) = -\frac{1}{r}\delta(r)\delta(\tau), \tag{12}$$

where $r = |\mathbf{r} - \mathbf{r}'|$ and $\tau = t - t'$.

This equation is the same equation discussed by Sezginer and Chew [9] in a paper where they obtain a closed form expression of the Green's function for the time-domain wave equation for a lossy two-dimensional medium, using Fourier transform. This equation is satisfied for the electric field due to a line current source parallel to the z axis in a conductive medium.[2]

To calculate the Green's function we must introduce the boundary and initial conditions. For the initial conditions we take (homogeneous conditions)

$$g(r,0) = 0 = \frac{d}{d\tau}g(r,\tau)|_{\tau=0},$$

and use as boundary conditions

$$|g(r,\tau)| < \infty \qquad \text{and} \qquad \lim_{r\to\infty} g'(r,\tau) \to 0\,,$$

which guarantee the existence of Laplace and Hankel integral transforms. We note that in Ref. [9] the authors obtain an integral representation for the Green's function in terms of Hankel function.

We observe that the importance in solving this problem by means of Hankel transform resides in the fact that the integral representations for second kind modified Bessel functions have a well known representation as we have discussed in section 3, *i.e.*, our Eq. (4).

Firstly we introduce the Laplace integral transform, $g(r,s)$, in the time variable τ,

$$g(r,s) = \int_0^\infty g(r,\tau)\,\mathrm{e}^{-s\tau} d\tau\,,$$

with $\Re(s) > 0$; using the initial conditions we get the non-homogeneous ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s^2}{c^2} - \frac{s}{c^2\zeta}\right) g(r,s) = -\frac{1}{r}\delta(r),$$

where c and ζ were introduced in footnote 2.

Next, we introduce the Hankel integral transform $g(k,s)$ in the radial variable r,

$$\int_0^\infty g(r,s)\, r\, J_0(kr)\, dr = g(k,s)\,,$$

where $J_0(x)$ is a Bessel function, and using boundary conditions we obtain an algebraic equation for $g(k,s)$ which has the solution

$$g(k,s) = \frac{1}{k^2 + s(s+\beta)},$$

where $\beta^{-1} = c\,\zeta$ and $s/c \to s$.

Now, our procedure is to evaluate the respective inverse transforms. The inverse Hankel integral transform is given by [12]

$$\begin{aligned} g(r,s) &= \int_0^\infty g(k,s)\, k\, J_0(kr)\, dk \\ &= \int_0^\infty \frac{k\, J_0(kr)}{k^2 + s(s+\beta)} dk \\ &= K_0\left(r\sqrt{s^2 + s\beta}\right), \end{aligned}$$

where $K_0(x)$ is a second kind modified Bessel function. Another way to calculate this integral is by means of the residue theorem [10].

To recover our Green's function we must calculate the inverse Laplace integral transform, *i.e.*

$$\begin{aligned} &\mathcal{L}^{-1}[g(r,s)] \equiv g(r,\tau) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(r,s)\,\mathrm{e}^{s\tau}\, ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_0\left(r\sqrt{s^2 + s\beta}\right)\,\mathrm{e}^{s\tau} ds\,, \end{aligned} \tag{13}$$

where $c > 0$.

To perform this integral we can use a suitable contour in the complex plane (modified Bromwich contour) or use the integral representation obtained in section 3. Firstly, we can write the above equation as

$$g(r,\tau) = \mathrm{e}^{-\epsilon\tau}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_0\left(r\sqrt{s^2 - \epsilon^2}\right)\,\mathrm{e}^{s\tau} ds,$$

and using Eq. (5) we get

$$g(r,\tau) = \mathrm{e}^{-\epsilon\tau}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\int_0^\infty \frac{\cos(ru)}{\sqrt{u^2 + s^2 - \epsilon^2}} du\right\} \times \mathrm{e}^{s\tau} ds.$$

Both integrals are convergent and we can write

$$\begin{aligned} g(r,\tau) = \mathrm{e}^{-\epsilon\tau}\int_0^\infty \cos(ru)\, du\, \times \\ \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{e}^{s\tau}}{\sqrt{u^2 + s^2 - \epsilon^2}} ds\right) = \\ \mathrm{e}^{-\epsilon\tau}\int_0^\infty \cos(ru)\, J_0(\tau\sqrt{u^2 - \epsilon^2})\, du, \end{aligned}$$

[2] As for a physical problem we must introduce the coefficients σ=conductivity; μ=permittivity and ϵ=dielectric constant, related by the expressions $c = (\epsilon\mu)^{-1/2}$ and $\zeta = \epsilon/\sigma$, where c=speed of light and ζ=relaxation time.

because, Laplace, $\mathcal{L}[f(\tau)]$, and Laplace inverse, $\mathcal{L}^{-1}[F(s)]$, transforms are related by means of [11]

$$\mathcal{L}[J_0(\tau\sqrt{u^2-\epsilon^2})] = \frac{1}{\sqrt{u^2+s^2-\epsilon^2}} \Longleftrightarrow$$
$$\mathcal{L}^{-1}\left[\frac{1}{\sqrt{u^2+s^2-\epsilon^2}}\right] = J_0(\tau\sqrt{u^2-\epsilon^2}),$$

with $\Re(s) > 0$.

The remaining integral is performed with the help of Eq. (10) and we finally get

$$g(r,\tau) = \mathrm{e}^{-\epsilon\tau}\frac{\cosh(\epsilon\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}},$$

for $0 < r < t$ and zero otherwise. In order to simplify this expression we introduce Heaviside theta function and then we obtain

$$g(r,\tau) = \mathrm{e}^{-\epsilon\tau}\frac{\cosh(\epsilon\sqrt{\tau^2-r^2})}{\sqrt{\tau^2-r^2}}\,\Theta(\tau-r),$$

which is the same expression obtained in Ref. [9] by using another procedure.

Taking the limit $\epsilon \to 0$ in the equation above we get

$$\lim_{\epsilon\to 0} g(r,\tau) \equiv g_0(r,\tau) = \frac{\Theta(\tau-r)}{\sqrt{\tau^2-r^2}},$$

which is the Green's function associated with the lossless case.

We conclude this section calling that, in general, our inverse Laplace transform given by Eq. (13) can be written as

$$\frac{1}{2\pi i}\int_\Gamma K_0\left(\mu\sqrt{p}\sqrt{p+d}\right)\mathrm{e}^{pt}dp =$$
$$\begin{cases} \dfrac{\mathrm{e}^{-td/2}}{\sqrt{t^2-\mu^2}}\cosh\left(\dfrac{d}{2}\sqrt{t^2-\mu^2}\right) & t > \mu \\ 0 & t < \mu, \end{cases}$$

where Γ is the modified Bromwich contour [10] and $d \geq 0$. Taking the same limit as above we can get

$$\frac{1}{2\pi i}\int_\Gamma K_0(\mu p)\,\mathrm{e}^{pt}dp = \begin{cases} \dfrac{1}{\sqrt{t^2-\mu^2}} & t > \mu \\ 0 & t < \mu, \end{cases}$$

which can be interpreted as the inverse Laplace transform of the second kind modified Bessel function.

Concluding remarks

In this paper we point out the importance of Hankel integral transform in the calculation of a Green's function associated with a problem involving propagation in two independent variables.

Using an integral representation for the second kind modified Bessel function we calculate an integral involving a zero order Bessel function which can be interpretated as a Fourier cosine transform of Bessel function and then we obtain the Green's function associated with the wave equation.

We note that our result can be used in the calculation of the Green's function associated with the wave equation for a damped oscillator and the telegraph equations. This will be done in forthcoming paper.

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