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About the Teaching of the Inertial Field as Maxwell like-type

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This paper has a didactic aim. The Einstein General Theory of Relativity is very difficult for undergraduates students and also for graduates who have not followed a course of study in gravitational physics. For example, the calculation of some of its known consequences, such as the gravitational time dilation, requires familiarity with space-time metrics. In this paper, starting with the analogy between the electromagnetic field and the inertial one, we want to analyze, through the Einstein Equivalence Principle (EEP), some simple effect in a fictious gravitational field by using the inertial potentials in analogy with the electromagnetic ones.

Keywords: Inertial potential, Equivalence Principle, Gravitational time dilation.

1. Introduction

In the classical mechanics, it is well known that for the holonomic systems is valid the following Lagrange equation of motion [1–3]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \tag{1}$$

where $T = T(t, q, \dot{q})$ is the kinetic energy, $\dot{q}_i = \frac{dq_i}{dt}$ are called generalized velocities while Q_i are the generalized components of the forces defined by

$$Q_i = \sum_{j=1}^{n} F_j \cdot \frac{\partial P_j}{\partial q_i}, \tag{2}$$

where F_j is the total force applied to the point at P_j assumed to be a function of time. Moreover, if we have a conservative system it is possible to write

$$Q_i = -\frac{\partial U}{\partial q_i}(q), \tag{3}$$

where $U(q)$ is the potential energy.

By introducing the Lagrangian $L = T - U$, the equation (1) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{4}$$

Now we consider a velocity-dependent force as, for example, the Lorentz force

$$\mathbf{F} = e\mathbf{v} \wedge \mathbf{B}, \tag{5}$$

where $\wedge$ is the usual cross product, e is the particle charge, $\mathbf{v}$ the particle velocity and $\mathbf{B}$ the magnetic field. Let us remember that exists a vectorial field $\mathbf{A}$ with $\mathbf{B} = \nabla \wedge \mathbf{A}$ where the symbol ∇ denotes the usual vector operator nabla [4–6]. Then, the scalar function

$$U = e\mathbf{v} \cdot \mathbf{A}, \tag{6}$$

is called generalized potential energy for the Lorentz force and we have

$$F_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right). \tag{7}$$

The scalar function (6) depends on more than just the particle position but, despite this fact, whenever the relation (7) is valid, the Lagrange equation (4) still holds for $L = T - U$. In the following sections, we review the formal analogy between the electromagnetic field and the field of the inertial accelerations. In this way it is possible to define for the inertial field a generalized potential analogous to the well known one defined, in the electromagnetic framework, by the previous relation (6). Thanks to EEP, we impose that all inertial potentials can be seen, from the point of view of the accelerated observer, as gravitational potentials. In this way, in our opinion, some effects predicted by the relativistic theory of gravitation, can be understood by students by applying the potentials of classical mechanics without knowing the complex mathematical formalism of general relativity. Obviously, in the context of Lagrangian formalism in classical mechanics, the time is absolute. Therefore the relative velocity in relation (6) is the same for each observer. This is no longer true in relativity. At each point of the accelerated reference frame the time flows differently and the speed of the moving

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point will be different depending on the clock used to calculate it. When we use the potential (6) it is implied that the speed is calculated using the clock carried by the moving body. The paper is organized as follows: in Section 2 we briefly summarize the formal analogy between electromagnetic field and inertial field while in Section 3 we analyze some simple applications. In Section 4 we give the conclusion.

2. Inertial-electromagnetic analogy

Let us recall that, if we consider the electromagnetic field and the following electromagnetic acceleration

$$\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}), \tag{8}$$

then the four Maxwell's equations are satisfied

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases} \tag{9}$$

Furthermore, we have the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{10}$$

the following aforementioned relation

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \tag{11}$$

and

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{12}$$

The term ϕ is the scalar potential while $\mathbf{A}$ is the vector potential and we can write the following electromagnetic potential

$$U(P, \mathbf{v}, t) = \phi(P, t) + \mathbf{A}(P, t) \cdot \mathbf{v}. \tag{13}$$

Now, instead, let us remember that, if we have a free point mass (P, m) in "absolute" motion with respect to an inertial frame of reference $T_O \equiv Oxyz$ and in relative motion with respect to a non-inertial reference frame $T_{O'} \equiv O'x'y'z'$, from Galilean composition of motion we get

$$\mathbf{v}_a = \mathbf{v}' + \mathbf{v}_\tau = \mathbf{v}' + \mathbf{v}_{\mathbf{O}'} + \omega \wedge \mathbf{r}', \tag{14}$$

where $\mathbf{v}_a$, $\mathbf{v}'$ and $\mathbf{v}_\tau = \mathbf{v}_{\mathbf{O}'} + \omega \wedge \mathbf{r}'$ are respectively "absolute", relative and dragging velocity, $\mathbf{v}_{\mathbf{O}'}$ is the velocity of the origin of the non-inertial frame, ω is the angular velocity vector while $\mathbf{r}'$ is the vector $\mathbf{O'P}$. From non-inertial point of view, the point mass is subjected to the apparent forces. It is well known that we have the dragging force [7–9]

$$\mathbf{F}_\tau = m[\mathbf{a}_{O'} + \alpha \wedge \mathbf{r}' + \omega \wedge (\omega \wedge \mathbf{r}')], \tag{15}$$

and the Coriolis one

$$\mathbf{F}_c = 2m\omega \wedge \mathbf{v}', \tag{16}$$

with $\alpha = \frac{d\omega}{dt}$.

In order not to burden the calculations, we consider plane motions and we neglect a possible jerk and that is the rate at which the acceleration could change with respect to time. It is easy to write the following vectors into component form

$$\omega \wedge \mathbf{r}' = (-y'\omega_{z'})\, i' + (x'\omega_{z'})\, j', \tag{17}$$

$$\omega \wedge (\omega \wedge \mathbf{r}') = \left(-x'\omega_{z'}^2\right) i' + \left(-y'\omega_{z'}^2\right) j', \tag{18}$$

$$\alpha \wedge \mathbf{r}' = (-y'\alpha_{z'})\, i' + (x'\alpha_{z'})\, j', \tag{19}$$

$$2\omega \wedge \mathbf{v}' = \left(-2v_y'\omega_{z'}\right) i' + (2v_x'\omega_{z'})\, j', \tag{20}$$

where, obviously, i', j', k' are the unit vectors associated with $T_{O'}$. For this reason, we can write

$$\begin{cases} v_{\tau x} = v_{0'x} - y'\omega_{z'} \\ v_{\tau y} = v_{0'y} + x'\omega_{z'} \end{cases} \tag{21}$$

$$\begin{cases} a_{\tau x} = a_{0'x} - y'\alpha_{z'} - x'\omega_{z'}^2 \\ a_{\tau y} = a_{0'y} + x'\alpha_{z'} - y'\omega_{z'}^2 \end{cases} \tag{22}$$

where $\mathbf{a}_\tau = \frac{\mathbf{F}_\tau}{m}$. For simplicity, now we write ω instead of $\omega_{z'}$ and α in place of $\alpha_{z'}$. It is just as easy to obtain the curl and the divergence of the following vector fields

$$\nabla \wedge \mathbf{v}_\tau = \left(-\frac{\partial v_{\tau y}}{\partial z'}\right) i' + \left(\frac{\partial v_{\tau x}}{\partial z'}\right) j' + \left(\frac{\partial v_{\tau y}}{\partial x'} - \frac{\partial v_{\tau x}}{\partial y'}\right) k' = 2\omega, \tag{23}$$

$$\nabla \cdot 2\omega = 0, \tag{24}$$

$$\nabla \wedge 2\omega = 0, \tag{25}$$

$$\nabla \cdot \mathbf{a}_\tau = -2\omega^2, \tag{26}$$

$$\nabla \wedge \mathbf{a}_\tau = \left(-\frac{\partial a_{\tau y}}{\partial z'}\right) i' + \left(\frac{\partial a_{\tau x}}{\partial z'}\right) j' + \left(\frac{\partial a_{\tau y}}{\partial x'} - \frac{\partial a_{\tau x}}{\partial y'}\right) k' = 2\alpha = \frac{\partial(2\omega)}{\partial t}. \tag{27}$$

Finally we report the following relation

$$\mathbf{a}_\tau = \nabla\left(\frac{1}{2}v_\tau^2\right) + \frac{\partial \mathbf{v}_\tau}{\partial t}. \tag{28}$$

For the proof of (28), the reader can read for example [7]. By establishing the following analogies

$$\begin{cases} \mathbf{E} = -\mathbf{a}_\tau \\ \mathbf{B} = 2\omega \\ \phi = \frac{1}{2}v_\tau^2 \\ \mathbf{A} = \mathbf{v}_\tau \\ \rho = 2\omega^2 \\ \mathbf{J} = \frac{\partial \mathbf{a}_\tau}{\partial t} \end{cases} \tag{29}$$

the relations (8–12) are satisfied for the inertial field. The relation (13), therefore, becomes

$$U(P, \mathbf{v}', t) = \frac{1}{2} v_\tau^2 + \mathbf{v}_\tau \cdot \mathbf{v}'. \quad (30)$$

The most interesting aspect of this analogy, is the fact that it is possible to interpret the term $\omega \wedge \mathbf{r}'$ as the vector potential of Coriolis accelerations and $\mathbf{v}_{0'}$ as the vector potential of $\mathbf{a}_{0'}$. We can split (30) into five parts

$$\begin{cases} U_1 = \frac{v_{0'}^2}{2} \\ U_2 = \frac{1}{2}\left[\omega \wedge \mathbf{r}'\right]^2 \\ U_3 = \mathbf{v}_{0'} \cdot (\omega \wedge \mathbf{r}') \\ U_4 = \mathbf{v}_{0'} \cdot \mathbf{v}' \\ U_5 = (\omega \wedge \mathbf{r}') \cdot \mathbf{v}' \end{cases} \quad (31)$$

noting that U_5 is a generalized potential for the Coriolis field. Indeed

$$U_5 = (\omega \wedge \mathbf{r}') \cdot \mathbf{v}' = -y'\omega v'_{x'} + x'\omega v'_{y'}, \quad (32)$$

and, without loss of generality, we consider a constant angular velocity getting

$$-\frac{\partial U_5}{\partial x'} + \frac{d}{dt}\left(\frac{\partial U_5}{\partial v'_{x'}}\right) = -\omega v'_{y'} - v'_{y'}\omega, \quad (33)$$

$$-\frac{\partial U_5}{\partial y'} + \frac{d}{dt}\left(\frac{\partial U_5}{\partial v'_{y'}}\right) = +\omega v'_{x'} + v'_{x'}\omega. \quad (34)$$

Therefore

$$-\frac{\partial U_5}{\partial r'} + \frac{d}{dt}\left(\frac{\partial U_5}{\partial v'}\right) = \left(-2\omega v'_{y'}\right) i' + \left(2\omega v'_{x'}\right) j' = 2\omega \wedge \mathbf{v}'. \quad (35)$$

Instead

$$U_4 = v_{0x} v'_{x'} + v_{0y} v'_{y'}, \quad (36)$$

and

$$-\frac{\partial U_4}{\partial x'} + \frac{d}{dt}\left(\frac{\partial U_4}{\partial v'_{x'}}\right) = a_{0x}; \; -\frac{\partial U_4}{\partial y'} + \frac{d}{dt}\left(\frac{\partial U_4}{\partial v'_{y'}}\right) = a_{0y}. \quad (37)$$

Finally we have

$$-\frac{\partial U_2}{\partial x'} = -\omega^2 x'; \; -\frac{\partial U_2}{\partial y'} = -\omega^2 y', \quad (38)$$

and

$$-\frac{\partial U_3}{\partial x'} = -v_{0y}\omega; \; -\frac{\partial U_3}{\partial y'} = v_{0x}\omega. \quad (39)$$

It is important to observe that U_1 is independent of r', v' and can be neglected. Moreover U_2 and U_3 are independent of v' but they cannot be regarded as a standard potentials owing to the dependence on time. Finally, the most important thing to note is that it is not possible to distinguish separate contributions to the only Coriolis force and to the only dragging force, respectively. Indeed, U_2, U_3 and U_4 give a partial contribution to the only dragging force while the potential U_5 provides Coriolis acceleration and the residual contribution to the dragging force. In the most general case with $\frac{d\omega}{dt} \neq 0$ and $\mathbf{v}_{0'} \neq 0$, U_5 and U_3 have a term

$$\mathbf{v}_{0'} \wedge \omega, \quad (40)$$

of opposite sign. For more details it is recommended to read section 4 of [8].

3. Gravitational potentials

Many problems can be solved thanks to this inertial-electromagnetic analogy. For example, in [9] is observed that any problem involving closed loops of tubes filled with a fluid, which are moved in a rotating system, can be solved through "inertial" Faraday's law. Another interesting example emphasized by the author of [9] is the precession of an electric charge calculated by analogy with the precession of the Focault pendulum. In this paper, instead, we want to emphasize the possible relativistic applications by using EEP. It is well known that EEP is the starting point that allowed the construction of a metric theory of gravitation [10]. One of the most famous predictions of General Relativity is the gravitational time dilation. The closer the clock is to the source of gravitation, the slower time passes. In other words, time accelerates with increasing gravitational potential. In Einstein's gravitational theory, this phenomenon is related to the metric tensor of spacetime $g_{\mu\nu}$. Furthermore, if there is a weak gravitational field, we have [11]

$$g_{00} = 1 + \frac{2V}{c^2}, \quad (41)$$

obtaining the relation that connects the passage of time between two clocks that are in different points of the gravitational field and that is

$$dt_1 = \sqrt{1 + \frac{2\Delta V}{c^2}} dt_2 \approx \left(1 + \frac{\Delta V}{c^2}\right) dt_2, \quad (42)$$

where ΔV is the gravitational potential difference. The first experimental verification of the gravitational time dilation was made by Pound and Rebka [12]. In this experiment gamma rays were emitted at ground level and measured by a receiver placed on top of a tower, with $h = 22.6$ m above the emitter [13]. The metric of spacetime near Earth is

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2GM}{c^2 r}\right)(dx^2 + dy^2 + dz^2). \quad (43)$$

By remembering the redshift paramenter

$$z = \frac{\lambda_{receiver} - \lambda_{emitted}}{\lambda_{emitted}} = \frac{\nu_{emitted} - \nu_{receiver}}{\nu_{receiver}}, \quad (44)$$

and after some calculations, it is possible to get, as a consequence of gravitational time dilation $t_{receiver} = \left(1+\frac{gh}{c^2}\right) t_{emitted}$, the following shift

$$z \approx \frac{gh}{c^2}. \tag{45}$$

Pound and Rebka measured a redshift in perfect agreement with what spacetime weak field metric predicted. Indeed they found

$$z \approx 2.46 \cdot 10^{-15}. \tag{46}$$

It is easy to verify this by substituting in (40) $g = 9.8$ m/s^2, $h = 22.6$ m and $c = 3 \cdot 10^8$ m/s. Another well-known experiment was that carried out by Kündig, using a rotating Mössbauer absorber, to verify the transverse Doopler effect predicted by special relativity [14–18]. We have a source in the center of a rotating disk and gamma rays start towards the absorber on the rim. Finally there is a stationary detector. Kündig measured a blue-shift in perfect agreement with that predicted by the theory and that is

$$z = -\frac{1}{2}\frac{\omega^2 r^2}{c^2}, \tag{47}$$

where ω is the angular velocity of the rotor, r is the distance between the source and the absorber and c is the speed of light. It is well known that, thanks to the equivalence principle, Einstein understood that it is possible to discover properties of gravitation using transformations between accelerated reference systems. In this way, in fact, he predicted, for example, the deflection of light in a gravitational field [19]. Subsequently, when he understood that gravity is curvature of spacetime, he analyzed the inverse problem, that is, he understood that a reference in free fall is equivalent to an inertial system [20–22]. The equivalence principle is strongly confirmed in every experiment. The scientific debate, about this principle, concerns only the classic problem of a charge that falls in a gravitational field, the famous lift Einstein gedankenexperiment. The debates, however, concern the physical interpretation of the radiation and no one doubts the equivalence between inertia and gravity [23, 24]. Generally, the equivalence principle is used to deduce, in a simple way, the relations (45) and (47) using the physical equivalence between acceleration and gravity. In our opinion it might also be interesting to consider the inertial vector potential (30), defined in analogy with the electromagnetic field, as a gravitational potential. In this way, using the relation (42), we can study all phenomena that occur in non-inertial systems and that is in all fictious gravitational fields. For example, if we have an atom emitting a photon in O' and a detector D separated by a distance h in the frame $O'x'y'z'$ that is accelerating uniformly in the direction of y', with $a = g$, from (42) we can write

$$\begin{aligned} t_D \approx \left(1+\frac{U_4}{c^2}\right) t_{O'} &= t_{O'} + \frac{v_{0'}}{c}\frac{h}{c} = t_{O'} + \frac{g t_{O'}}{c}\frac{h}{c} \\ &= \left(1+\frac{gh}{c^2}\right) t_{O'}, \end{aligned} \tag{48}$$

getting relation (45). Another interesting application is if we consider a platform rotating at angular velocity ω and an observer at rest on the disk at radius r. It is well known that, if the observer sends two light signals around the disk in opposite directions along the circle, the counter rotating ray will arrive earlier than the co-rotating one. This is a simplified version of the well-known Sagnac experiment [25]. The observer measures a difference in the journey times

$$\Delta t \approx \frac{4\pi r^2 \omega}{c^2}. \tag{49}$$

The time delay can be quickly calculated via the Coriolis potential. Let us consider three point masses (A, B, C) at rest with respect to the rotating frame at radius r, and, at the same time, A and C leave in the opposite direction but with the same speed $\mathbf{v}'$ with respect to the platform. A moves in the same direction of rotation, C in the opposite one. From the inertial frame point of view, A must travel a distance greater than the mass travelling in the opposite direction and this is, clearly, the cause of the different duration of the path by applying the relativistic composition of velocities. Instead, from B point of view, the paths have the same length but there are inertial potentials. Thanks to EEP we can apply the relation (42). We have two potentials U_2 and U_5 but, since the three material points always remain at the same r, there is no difference in U_2 between them. In this case, $U_5 = \omega r v'$ because ω and $\mathbf{r}$ are perpendicular and $\omega \wedge \mathbf{r}$ and $\mathbf{v}'$ are parallel or anti parallel. For this reason we have

$$dt_B = \left(1+\frac{\omega r v'}{c^2}\right) dt_A = \left(1+\frac{\omega r v'}{c^2}\right)\frac{r d\theta}{v'}, \tag{50}$$

$$dt_B = \left(1-\frac{\omega r v'}{c^2}\right) dt_C = \left(1+\frac{\omega r v'}{c^2}\right)\frac{r d\theta}{v'}. \tag{51}$$

Therefore, integrating along the two paths and making the difference, we obtain $\left(\frac{2\pi r}{v'}+\frac{2\pi\omega r^2}{c^2}\right) - \left(\frac{2\pi r}{v'}-\frac{2\pi\omega r^2}{c^2}\right)$ and that is the Sagnac delay (49).

4. Conclusion

In this paper we have summarized the formal analogy between electromagnetic forces and fictitious forces. In our opinion, the study of all inertial potentials is very useful from a didactic point of view. Indeed, thanks to EEP, they can be applied to analyze the main "gravitational" effects by simply using (31) and (42).

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