The critical behavior of the BCS order parameter: a straightforward derivation

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Textbooks on Solid State Physics, such as [1–3], include a mandatory chapter on Superconductivity. Usually the basic item to start with is the famous Bardeen-Cooper-Schrieffer model (BCS) [4]. One of the main results concerns the way in which the superconducting order parameter $\Delta(T)$ vanishes at the critical temperature $T_c$, namely as

$$\Delta(T) \sim B(T_c - T)^{1/2}$$  \hspace{1cm} (1)

where the prefactor $B$ is a non-universal coefficient and the exponent has the classical value $\alpha = 1/2$.

Then one may read that this is a standard result for any mean-field theory, although the student may wonder, why it does not follow straightforwardly from the model?

Yet in the literature this outstandingly simple statement is obtained in a rather roundabout manner. Furthermore $\alpha$ and $B$ are computed only in the weak-coupling limit $\hbar \omega_D \gg k_B T_c$, where $\omega_D$ is the Debye frequency. This is certainly an aesthetically not very pleasing situation and I doubt the student really wants to grind through the approximations just to get this simple result.

The following lines show a little trick straightening out this situation. It will hopefully find its way to the textbooks.

In the BCS theory the order-parameter $\Delta(T)$ satisfies the non-linear integral equation [2]

$$1 = g \int_0^{\hbar \omega_D} dx \frac{\tanh \left( \frac{\beta E}{T} \right)}{2E},$$  \hspace{1cm} (2)

with $E = \sqrt{\epsilon^2 + \Delta^2}$, $\beta = 1/k_B T$ and $g$ is some coupling constant.

We extract the critical behavior of the order parameter straightforwardly and without approximations. For this purpose we choose $\Delta$ to be real and parametrize it as

$$\Delta(\beta) = a \left( \frac{\beta - \beta_c}{\beta_c} \right)^{\alpha}; \beta \sim \beta_c.$$  \hspace{1cm} (3)

This yields for the derivative $\partial_\beta \Delta^2 \equiv \frac{\partial \Delta^2}{\partial \beta}$:

$$\lim_{T \to T_c} \partial_\beta \Delta^2 = \begin{cases} 0 & \alpha > 1/2 \\ a^2/\beta_c & \alpha = 1/2 \\ \infty & \alpha < 1/2 \end{cases}$$  \hspace{1cm} (4)

The non-linear integral equation (2) for the order parameter has the solution $\Delta(\beta, \omega_D, g)$, depending on three parameters. Substituting this solution into equation (2) yields an identity. Differentiating this identity with respect to $\beta$ easily yields the following relation

$$\partial_\beta \Delta^2(\beta, \omega_D, g) = \frac{\int_0^{\hbar \omega_D} dx \tanh \left( \frac{2E}{\beta} \right)}{\int_0^{\hbar \omega_D} dx \left( \frac{\beta E}{2} - \frac{\beta E}{2 \cosh^2 \frac{x}{2}} \right)}.$$  \hspace{1cm} (5)

Taking the limit $T \to T_c, \Delta \to 0$, we obtain

$$0 < a^2 = \int_0^{\hbar \omega_D \beta_c} dx \left( \frac{\beta E}{2} - \frac{x}{2 \cosh^2 \frac{x}{2}} \right) < \infty$$  \hspace{1cm} (6)

implying $\alpha = 1/2$. Notice that the above integrand is finite at $x = 0$. As illustration we evaluate the integral for $\hbar \omega_D \beta_c = 10$ to get

$$\Delta(T) = 3.10 \cdot k_B T_c \left( 1 - \frac{T}{T_c} \right)^{1/2}, \quad T \lesssim T_c.$$  \hspace{1cm} (7)

References


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1 Although traditionally the exponent is referred to as $\beta = 1/2$, we use $\alpha$ to avoid confusions with $\beta = 1/k_B T$.

2 See e.g. equation (23.20) or [2] equation (6.28).
