Kirchhoff's voltage law corrected for radiating circuits
(Correção da lei das malhas de Kirchhoff para circuitos radiantes)

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When a circular loop composed by a *RLC* circuit begins to oscillate, the oscillation will eventually vanish as an exponentially decaying current, even considering superconducting wires, due to the emission of electric and magnetic dipole radiations. In this work we propose a modification of the Kirchhoff's voltage law by adding the radiative contributions to the energy loss as effective resistances, whose values are relatively small when compared to typical resistance values, but are fundamental to describe real circuits correctly. We also analyse the change in the pattern of the radiation spectra emitted by the circuit as we vary the effective and electric resistances.

Keywords: oscillating circuits, Kirchhoff's voltage law.

Quando uma espira circular formada por um circuito *RLC* começa a oscilar, a oscilação irá desaparecer, com uma corrente decaindo exponencialmente, mesmo considerando fios supercondutores, devido à emissão de radiações de dipolos eléctricos e magnéticos. Neste trabalho, propomos uma modificação da lei da voltagem de Kirchhoff, adicionando as contribuições radiativas associadas à perda de energia, através de resistências efetivas, cujos valores são relativamente pequenos quando comparados com os valores de resistências típicas, mas que são fundamentais para descrever circuitos reais de forma correta. Nós também analisamos as mudanças no espectro de radiação emitida pelo circuito à medida que se variam as resistências elétricas reais e efetivas.

Palavras-chave: circuitos oscilantes, lei das malhas de Kirchhoff.

1. Introduction

The Kirchhoff law of voltage is a simple yet powerful rule that is presented in virtually all Electromagnetism books. This law is a direct application of the energy conservation, yet there are other assumptions that are not normally discussed. Most of the literature argue that the typical size of circuit must be much smaller than the wavelength of the emitted radiation [1], as a condition for the validity of this law but some other assumptions have to be taken into account.

One of them is based on Faraday's law that states that the electromotive force, defined as the line integral of the electric field vector around the circuit closed loop, is proportional to the time rate of change of the magnetic flux through the loop, and can be written as the following:

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{s}, \qquad (1)$$

and if the current varies in the time the electromotive force is not zero, leading to an additional term in the Kirchhoff law. This can be circumvented by assuming the area of the circuit, as well as the variation of the magnetic field, are small enough or by assuming that the current is slowly varying [1]. The second assumption, not normally stated but related to the first one, assumes that the energy radiated by the circuit is negligible when compared with other energy scales involved in the problem.

In this work we intend to overcome these hypotheses and present a simple model that hopefully clarifies the situation to some extent. We consider a simple *LC* circuit. If we assume ideal capacitor, inductor and wires in the classical description, one obtains a perpetually oscillating current. However, intuitively we known that this result is not very realistic. If we consider a circular (or a square) mesh (see Fig. 1), on the other hand, we could add into the recipe a term related to the magnetic and electric dipole radiations, given by the Larmor formula [2]. In doing so, we obtain an effective resistance, which depends basically on the geometry of the mesh, of its natural frequency and the inductor geometry. The overall electric effective resistance is related to all sort

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of loss of energy, by the mechanical work done over the current carrying electrons and by the energy that flows out in an irreversible way in the form of radiation, as stated by the Poynting's theorem.

We have analysed elsewhere [4] the problem of the two capacitors, the discharge of a capacitor associated with an identical and initially discharged, where the final state of equilibrium as well as the energy dissipated in the process are always the same, regardless the presence of a resistor in the system. We have made clear how this system reaches equilibrium, and we have discussed the dissipation mechanisms and the role played by both the Joule effect and radiation. In this sense the present paper generalize the former one and goes beyond in order to complement the Kirchhoff's circuit law.

What motivate us to write this article is the fact that most of the students do not well understand the fact that when the electric circuit radiates the traditional equation of the RLC circuit is no longer valid and must undergo correction. Here we try to give a direction for how to understand the problem and how to approach it, giving a support for students and teachers in this subject that can be quite tricky [3].
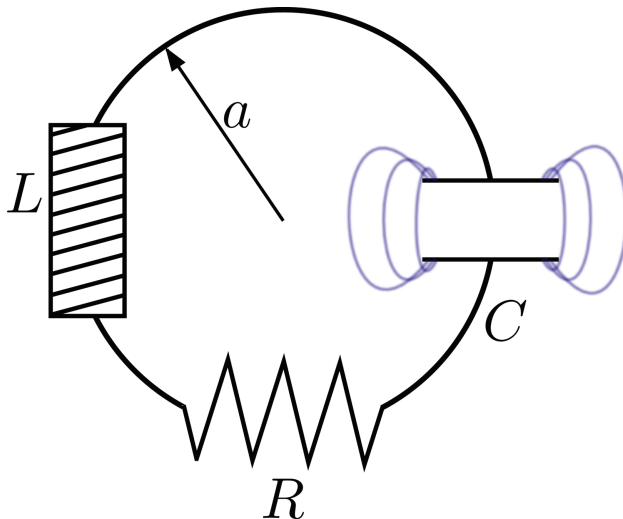

Figura 1 - A circular loop with radius a composed by a RLC circuit showing the capacitor radiating as electric dipole. The loop and the inductor also radiate (not shown) as magnetic dipole.

2. Usual description of the RLC circuit

The Kirchhoff's circuit law states that for a structure such as a mesh or a circuit loop, starting from a point and adding voltages difference across the circuit elements, after describing a complete loop the same potential as that of the starting point is obtained, and its mathematical formulation is written as

$$\sum V_i = 0 \,, \qquad (2)$$

where V_i is the potential difference between the terminals of an element in the circuit.

This result can be derived from the general energy balance. We can think in terms of the Poynting's theorem, related to the conservation of energy in the system formed by the electromagnetic field and matter. In its integral form this theorem can be written as [2]

$$\int_V \mathbf{J} \cdot \mathbf{E} \, dV + \oint_A \mathbf{S} \cdot d\mathbf{A} = -\frac{d}{dt} \int_V \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \, dV \,, \qquad (3)$$

where $\mathbf{J}$ is the current density vector, $\mathbf{E}$ and $\mathbf{B}$ the electric and magnetic fields, $\mathbf{S}$ the Poynting's vector, and $\mathbf{D}$ and $\mathbf{H}$ are, respectively, the electric displacement and auxiliar magnetic field vectors.

This equation states that the time variation of electromagnetic energy in the field, the right hand side, is due to the work carried out on the charges, represented here by the Joule effect and by the work done over the charges by the battery, the first term in the left-hand side of Eq. (3), and the energy that flows to the outside, represented by the surface integral of the Poynting vector, that represents the energy radiated by the circuit. This last term is normally neglected in order to calculate the current flowing in the circuit, as an approximation.

For an RLC circuit forced by a battery, for example, with the assumption that the system do not radiates, the Poynting theorem can be cast in following form

$$-RI^2 - \frac{d}{dt}\left(\frac{LI^2}{2}\right) - \frac{d}{dt}\left(\frac{Q^2}{2C}\right) + \mathcal{E}I = 0 \,, \qquad (4)$$

where $\mathcal{E}I$ is the power supplied by the battery, $LI^2/2$ is the magnetic energy stored in the inductor, $Q^2/2C$ is the electric energy stored in the capacitor and RI^2 is the power dissipated in the resistor. Using the definition of current $I = dQ/dt$ we get the standard circuit law

$$RI + L\frac{dI}{dt} + \frac{Q}{C} = \mathcal{E} \,. \qquad (5)$$

In the derivation of the above equation the hypothesis that the system does not radiate is implicit, and there is only one mechanism of energy dissipation due to the Joule effect. This assumption is not so bad when the electrical resistance is large enough, that is, much greater than the radiation loss from the circuit, but when we are dealing with a circuit without electric resistance such as an LC circuit, we are induced to think that the current never stops, even without a battery source.

In order to clarify this discussion we could remember a curious fact about superconducting (i.e., with no Joule dissipation at all) current carrying wires, which is somewhat related to the issues discussed in this work. A steady electric current flowing in a superconducting circular loop does not radiate, thus maintaining this current constant. One could believe that due to the

presence of accelerated charges this system would radiate, but the fact is that stationary currents do not radiate (for a detailed proof for the case of a superconducting circular loop see, for instance, Ref. [5]).

However, for the case of an LC circuit formed by superconducting wires we find that the current $I(t)$ slowly decays as the time passes, and therefore is not stationary. Even more, we find a slowly decaying oscillating current, in a similar way as one can see in an RLC circuit with small electrical resistance. As we will see later, this current leads to oscillating magnetic and electric dipoles. As a consequence, even an ideal LC circuit radiates, which means that the current eventually vanishes. The main question to be addressed here is what sort of changes we have to make in the Kirchhoff's law in order to include radiation from the circuit, and derive a differential equation that allows us to solve and get the circuit current as a function of time.

3.   Dissipation mechanisms

There are some different mechanisms for dissipation that could be included in an RLC circuit aside the Joule effect, like the magnetic and electric dipole radiations. The electric dipole radiation would be due to the oscillating charge in capacitor, and the magnetic counterpart would be due to the geometry of the mesh and to the inductor, both behaving like an oscillating magnetic dipole. These terms are very similar, as we will see on the next subsections.

3.1.   Joule effect

The first contribution that we could take into account is due to the Joule effect. As stated by the law that bears the same name, when an electric current I passes through a conductor, there is an irreversible energy transfer from the conductor to the medium in which it is embedded, whose power is given by $P_R = RI^2$, where R is the electric resistance of the conductor.

This contribution is well understood and established, and leads, in the case of an RLC circuit, to an electric current that decays exponentially.

3.2.   Electric dipole radiation

Here we go into another well understood dissipation mechanism, although not frequently included in circuit analysis. Considering the accumulated charge on the capacitor plates, we have an electric dipole which is given by

$$\mathbf{p}(t) = q(t)\mathbf{d} , \qquad (6)$$

where d is the distance between the plates of the capacitor (we consider a simple capacitor of parallel plates). As the current varies the electric dipole p is also modified, and we could include a dissipated power due to this electric dipole for the radiation zone ($r \gg a$, where

a is the radius of the RLC mesh), which is given by the Larmor formula. The total radiated power is given by [2]

$$P_E = \frac{\mu_0}{6\pi c}\left[\ddot{\mathbf{p}}(t)\right]^2 . \qquad (7)$$

Since $\mathbf{p}(t)$ varies harmonically, this quantity can be easily calculated.

3.3.   Magnetic dipole radiation

For the magnetic dipole radiation, there are two different sources. First, we can consider that the circular loop constitutes a magnetic dipole $\mathbf{m}_M(t)$, which intensity is given by

$$m_M = I(t)A , \qquad (8)$$

where $A = \pi a^2$ is the area of the circular loop.

But there is also a magnetic dipole due to the inductor, that would be given by

$$m_I = NI(t)A' , \qquad (9)$$

where N is the number of turns of the inductor and A' is the area of each one of the turns. Depending on the ratio between NA' and A, one of the terms could be more important than the other. However, for both of them, we can calculate the emitted radiation power by using Larmor formula, given in this case by [2]

$$P_M = \frac{\mu_0}{6\pi c^3}\left[\ddot{\mathbf{m}}(t)\right]^2 . \qquad (10)$$

With Eqs. (7) and (10) we can extend the Kirchhoff law of voltage, which is derived in the next section.

4.   Generalization of Kirchhoff's Law

As previously discussed, the usual RLC circuit leads to an exponentially decaying current. If, in the other hand, we consider an LC circuit, one obtains a perpetual oscillating current. However, this last result is somewhat artificial. In order to clarify this question, as pointed previously, one could consider a circular loop of radius a (and a corresponding area $A = \pi a^2$). Related to this circuit, the rate of energy dissipation, due to the electric and magnetic dipole radiations, is given by Eqs. (7) and (10) (this latter has two terms, see Eq. (8) and Eq. (9)),

$$P_{rad} = \frac{\mu_0}{6\pi c^3}\left[\ddot{\mathbf{m}}_I(t)\right]^2 + \frac{\mu_0}{6\pi c^3}\left[\ddot{\mathbf{m}}_M(t)\right]^2 + \frac{\mu_0}{6\pi c}\left[\ddot{\mathbf{p}}(t)\right]^2 , \qquad (11)$$

in SI units.

If we require energy conservation again, by adding this time the power formula given by Eq. (11) we obtain

$$RI^2 + \frac{d}{dt}\left(\frac{LI^2}{2}\right) + \frac{d}{dt}\left(\frac{Q^2}{2C}\right) = \mathcal{E}I - P_{rad} . \qquad (12)$$

It is important to remark that we are neglecting retardation effect of the emitted field over the electric current dynamics in the circuit. This is a reasonable

assumption since the circuit is much greater than the emitted wavelength, as stated before.

If we assume that the current in the circuit still oscillate in the form $I(t) = I_0 e^{-\gamma t} \cos \omega t$, where ω is the frequency of the external electromotive force, and γ is the damping term, all terms in Eq. (11) leads to

$$P_{rad} \propto \left[\ddot{I}(t)\right]^2 \propto I^2. \tag{13}$$

This means that both, the electric and magnetic terms are responsible for the circuit's radiation, and are very similar to the typical dissipated power due to an electric resistance. With this in mind, albeit the radiative terms appear to be much more complicated, from the form of the radiative power, we again obtain an exponentially decaying current, and we can put all types of dissipation together in just one term in the differential equation.

Actually, in antenna theory, this is very well known subject, since antennas are designed to radiate [6]. In this context, one needs to consider a radiation resistance R_{rad}, due to the energy loss in Eq. (13). Besides, in order to correctly describe radiating circuits we also need to modify the Kirchhoff's voltage law, which is somehow new for physicists.

In the next section we perform a comparison between the contributions of each of these terms. For now, all we can say is, although we have followed the same steps normally used on the derivation of the Kirchhoff Law of Voltage, if one includes the radiation terms the perpetually oscillating current no longer holds, giving rise to an exponential decaying current.

5. Comparison of dissipative contributions

To perform a comparison between each of radiative terms and the Joule effect contribution itself, first we need to obtain an expression for the effective resistance due to the radiation terms. So, we need to write the dissipated power in terms of $R_{ef}I^2$. The R_{ef} constant would be the effective resistance.

For the Eq. (7), one finds that

$$P_E = \frac{\mu_0}{6\pi c}\left[\ddot{\mathbf{p}}(t)\right]^2 = \frac{\mu_0}{6\pi c}d^2\omega^2 I^2 , \tag{14}$$

since $\mathbf{p}(t) = q(t)\mathbf{d}$ and we have assumed that $I(t) = I_0 e^{-\gamma t} \cos \omega t$. This leads to the following effective electric radiation resistance,

$$R_{ef}^E = \frac{\mu_0}{6\pi c}d^2\omega^2 . \tag{15}$$

Similarly, the Eq. (10) gives

$$R_{ef}^M = \frac{\mu_0\omega^4}{6\pi c^3}\left[(NA')^2 + A^2\right] . \tag{16}$$

As previously stated, the relative relevance of the two terms of the magnetic dipole radiation depends on the ratio between NA' and A. The ratio between R_{ef}^E and R_{ef}^M is given by

$$\frac{R_{ef}^E}{R_{ef}^M} = \left(\frac{dc}{\omega}\right)^2 \frac{1}{(NA')^2 + A^2}. \tag{17}$$

We can perform some estimations for the parameters that appears in Eqs. (15) and (16). Using S.I. units we can also compare their contributions with typical values for the electric resistance R, since all of then would be measured in ohms. Under the following values ($\mu_0 \sim 4\pi \times 10^{-7}$H m^{-1}, $c = 3 \times 10^8$ m/s, $d = 10^{-3}$ m, $C = 9 \times 10^{-13}$ F, $L = 100$ μH, $A' = 0.002$ m^2 and $A = 0.05$ m^2) one finds

$$R_{ef}^E \sim 10^{-9}\Omega \;\; , \;\; R_{ef}^M \sim 10^{-7}\Omega. \tag{18}$$

As expected, both effective radiating resistances are small when compared to typical values of electric resistance. However, appropriate choices for the parameters would make them comparable, enabling one to observe experimentally a decaying current contribution due to radiation in a superconducting circuit, or even in a "real" suitable circuit. This is in fact what happens in an efficient antenna. The radiation resistance are designed to be much greater than the ohmic resistance [6]. An antenna has a more complicated geometry and electric current on it is not uniform. Although, the mechanism is essentially the same.

6. Time scales and spectrum

Given that the system heats and radiates it is interesting to know the spectrum associated with these losses and what the contribution of each event, that is, we would like to calculate the total power dissipated by Joule effect and radiation in a given frequency range.

The power spectrum, or the dissipated power per unit frequency is given by the expression [7]

$$P(\omega) = Re\left[(V(\omega))^* I(\omega)\right] =$$
$$Re\left[(Z(\omega)I(\omega))^* I(\omega)\right] = R_T I(\omega) I(-\omega) , \tag{19}$$

where $R_T = R + R_{rad}$ is the total resistance present in the circuit.

A particular solution of Eq. (5) can be obtained by Fourier transform method. Setting the battery voltage source as $\mathcal{E}(t) = V_0 \cos \omega t$ we get the following power spectrum

$$P(\omega) = \frac{\frac{1}{2}R_T V_0^2}{R_T^2 + (\omega L - 1/\omega C)^2} , \tag{20}$$

where V_0 is the amplitude of battery voltage (see Fig. 2).
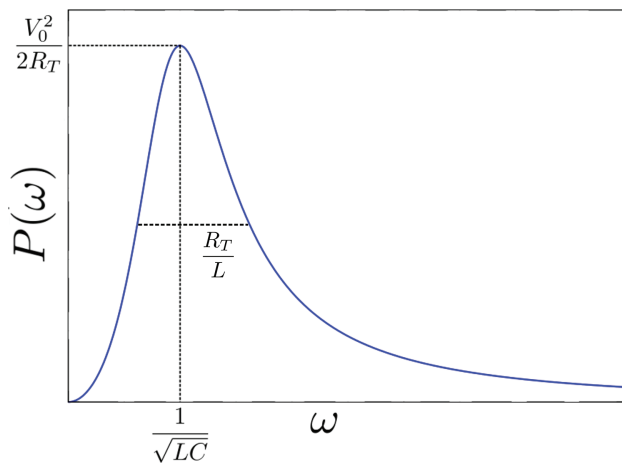
Figura 2 - The power spectrum showing its line width as function of the total resistance that include magnetic and electric radiation.

It is interesting to compare this spectrum for different loss mechanisms, when the Joule effect is more important, and when the radiation is preponderant.

The width of the distribution is determined by the resistance R_T that is present in the circuit, and it become very narrow when the electrical resistance is zero, in which case the width of the spectral line is entirely defined by the radiation resistance, as can be seen in Fig. 2.

7. Conclusions

In this work we have made a modification in the Kirchhoff's law of voltage, by including the Larmor power formula for both electric and magnetic dipole radiations in the energy conservation law, that gives rise to a differential equation for the current, which takes into account both Joule and radiative dissipation effects.

It is interesting to note the similarity in the expressions associated with the Joule dissipation, and the dipole radiation given by the Larmor formula, both proportional to the square of the current. It is precisely this similarity that allows for a simple unified analysis for dissipation.

It is also important to have some awareness of how one can go beyond simple small antennas to compute the radiation resistance for circuits of arbitrary complexity. If we have used another radiation system, other than a dipole that emits just in one frequency, we would have a different equation for the current, and its solution would be not so simple, but in any case the presented change in the Kirchhoff's voltage law would still be valid. This is a challenge that we try to confront in the next future.

Acknowledgments

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