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Faster or slower? Body constrained traveling along vertical curved path

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In this paper, we consider a body motion in the presence of both external gravity force and the holonomic constraint. We find that in the case of vertical curved path in the form of a "slide", the total travel time is always greater than the corresponding travel time along the horizontal section of the track. In the case of "valley", this time can be either more or less than the travel time along the horizontal section. This topic allows the reader to test the free particle model applying simple numerical calculations.

Keywords: Free particle, holonomic constraint, gravity force, numerical analysis.

1. Introduction

In physics, a free particle is the model of a particle (or body) that is not exerted by any external force. In other words, this particle is not in a region, where its potential energy varies. As a consequence, such a particle moves with constant velocity in absolute value and direction. The free particle model is an extreme degree of idealization and has proven to be very useful in explaining the Newton's laws.

The problem of a quantitative and qualitative description of the motion of a particle along a vertical curved path in the presence of gravity has many variations in physics education [1–3]. In this note, we present an intriguing topic allowing the reader to test the free particle model on the example of a body motion in the presence of both external gravity force and the holonomic constraint (that is, in the presence of the position constraint equation).

2. The Problem

We consider the body (bead) of mass m, which slides initially horizontally at constant speed $\vec{v}_0$ along a spoke. Let this spoke be further bent in the vertical plane in the form of a one-dimensional slide (or valley) with the smooth profile defined by function $y(x)$ (Fig. 1).Assuming no friction and the condition that the body is able to overcome the top of the slide, it is necessary to determine the total travel time τ on the slide (valley). A similar problem is considered in Ref. [1].

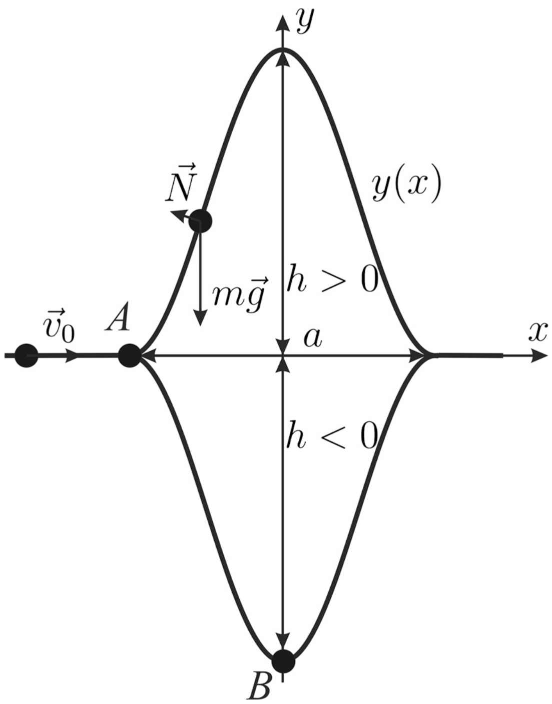

Figure 1: Geometry of the problem.

3. Solution and Numerical Analysis

The forces acting on a body of mass m are the gravity force $m\vec{g}$ and the normal force $\vec{N}$. The normal force holds the body on the given trajectory $y(x)$. In the frictionless case, this force is the only force causing the change in the horizontal component of the body's velocity v_x. Applying the work-energy theorem, we write:

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = -mgy, \quad (1)$$

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where v is the speed at height y. The infinitesimal time interval spent on the curved path is

$$dt = \frac{dx}{v_x} = \frac{dx}{v\cos\alpha}, \tag{2}$$

where α is the current tangential angle of curve $y(x)$ $(0 < \alpha < \pi/2)$. Using relation $\tan\alpha = y'(x)$ (the prime in y denotes the derivative with respect to x), we get:

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+y'^2}}. \tag{3}$$

Taking into account Equations (1)–(3), we derive:

$$\tau = \int dt = \frac{1}{v_0}\int\sqrt{\frac{1+y'^2}{1-\frac{2g}{v_0^2}y}}dx. \tag{4}$$

As a model profile, we can choose function $y(x)$ in the form:

$$y(x) = \frac{h}{2}\left[\cos\left(\frac{2\pi x}{a}\right)+1\right], \tag{5}$$

where h is the height of curve $y(x)$ ($h > 0$ for the slide and $h < 0$ for the valley); a is the curve width ($-a/2 < x < a/2$). The function (5) has all the characteristic features of the curved profiles shown in Fig. 1. It is convenient to proceed to variable $\varphi = 2\pi x/a$ in Equations (4) and (5). In this case $dx = ad\varphi/2\pi$. Using additionally the property that graph $y(x)$ is symmetric about the vertical y-axis, we finally obtain:

$$\tau = \frac{\tau_0}{\pi}\int_0^\pi\sqrt{\frac{1+\delta\sin^2\varphi}{1-\omega(\cos\varphi+1)}}d\varphi, \tag{6}$$

where $\tau_0 = a/v_0$ is the total time within the free particle model; $\delta = \pi^2h^2/a^2$ $(\delta \geq 0)$; $\omega = gh/v_0^2$ $(-\infty < \omega \leq 1/2)$.

The integral in Equation (6) is not expressed in terms of elementary functions, but it can be easily evaluated using various mathematical software [4]. The results of such calculations at different values of δ and ω are presented in Fig. 2. For very small absolute values of ω ($v_0 \to \infty$), the laws of classical mechanics are not applicable, so in this region we show graphs $\tau(\omega)$ with dotted lines.

We see that for the case of a slide ($\omega > 0$) $\tau > \tau_0$ always. In the case of the valley, τ can be either more or less than τ_0. At low values of v_0 ($\omega \ll 0$), the normal force is always directed at the section AB in such a way that it causes an increase in the horizontal component v_x. As v_0 increases, the force $\vec{N}$ can change its direction to the opposite on the convex part of the section AB (in the absence of constraint (the spoke), the body would leave the trajectory $y(x)$ [2]). This circumstance can lead to an increase in the total time τ up to the values greater than τ_0.

At a fixed value of δ, function $\tau(\omega)$ turns to zero and infinity (in this latter case, the body stops at the top of the slide) at the ends of the allowed range of ω values. As δ increases, the values of $\tau(\omega)$ increase too.

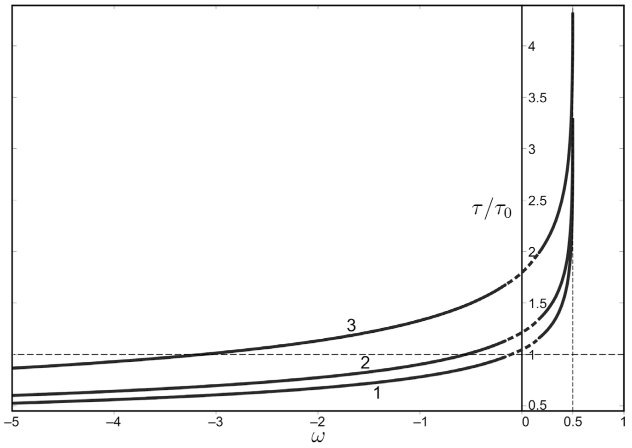

Figure 2: Dependence $\tau(\omega)$ according to Equation (6). (1) $\delta = 0.2$; (2) $\delta = 1$; (3) $\delta = 5$.

The above problem can be used in project-based learning and facilitates the development of students' critical thinking about the idealized models in physics.

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