On the capacitance of a sphere in the presence of another nearby conducting sphere

Vladimir Ivchenko*1

^{1}Kherson State Maritime Academy, Department of Natural Sciences Training, Kherson, Ukraine

We present the numerical analysis of the capacitance dependence of a conducting sphere on the distance to another sphere and its relative size. Among other results, we find that the greatest deviations from the model of a secluded sphere are observed for the case of the presence of infinite conducting plane near it. We provide a simple physical explanation of this fact. For example, our numerical calculations show that the value of relative error in the determination of the sphere capacitance is equal to 5% if the distance from the sphere center to the infinite plate is ten times more than its radius. The consideration of this problem will be useful for advanced undergraduates, who study the methods for solving electrostatic problems.
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1. Introduction

It is well known that capacitance C of a single conductor is the ratio of the change in its electric charge to the corresponding change in its electric potential. This definition should be distinguished from the other two definitions, relating to the system of two conducting objects (that is, for a capacitor) [1]. The capacitance of the isolated (secluded) single conductor depends on its geometric shape, sizes, and the dielectric permittivity ϵ of the surrounding medium. For example, for conducting sphere in SI units: $C = 4\pi\epsilon\epsilon_0 R$, where ϵ_0 is the vacuum permittivity; R is the sphere radius. In reality, the capacitance of a single conductor also depends on the presence of other nearby bodies (mainly conducting bodies). A striking example of the application of this property is the Theremin electronic musical instrument [2].

Different spherical conductors are quite common both in nature and in technology. As an example, we may mention Earth, the hollow metal globe of a van de Graaff generator, and the metal sphere of an electroscope. The electrostatics of two close conducting spheres presents a challenging problem in mathematical physics, which has been considered among others by Poisson, Lord Kelvin, Kirchoff, Maxwell, and Russell [3, 4]. First, this system has a simple geometry. Despite this fact, the derivation of capacitance coefficients is a rather cumbersome procedure [5]. Moreover, the numerical calculation of the values of these coefficients requires significant computer resources, especially, as the two spheres come closer. Finally, this system can demonstrate non-trivial behavior (there are the regions of attraction between like-charged conducting spheres) [6].

In this paper we present the numerical analysis of the capacitance dependence of a conducting sphere on the distance to another sphere and its relative size. The consideration of this problem will be useful for advanced undergraduates, who study the methods for solving electrostatic problems.

2. Theoretical Consideration

Let us consider two distant from other bodies isolated conducting spheres of radii R_1 and R_2 with charges q_1 and q_2. Below we consider two possible cases: the first, when the charge of the second sphere is fixed, and the second, when the potential of the second sphere is fixed. We denote the distance from the center of the first sphere to the nearest point of the surface of the second sphere as d ($d > R_1$; see Figure 1). The charges of these spheres can be represented as follows [7]:

$$q_1 = C_{11}V_1 + C_{12}V_2, \tag{1}$$

$$q_2 = C_{21}V_1 + C_{22}V_2, \tag{2}$$
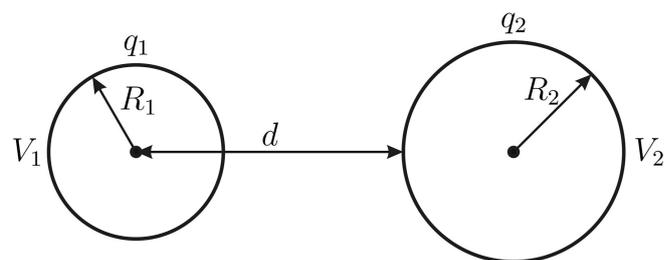
Figure 1: Geometry of the problem.

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* Correspondence email address: reterty@gmail.com

where V_1 and V_2 are the potentials of these spheres; C_{11} and C_{22} are their self capacitances; $C_{12} = C_{21}$ is their mutual capacitance. Using equation (1) and (2), we get:

$$q_1 = \left(C_{11} - \frac{C_{12}^2}{C_{22}} \right) V_1 + \frac{C_{12}}{C_{22}} q_2, \qquad (3)$$

Then, the capacitance of the first sphere for the case of fixed charge q_2 is

$$C_1 = \frac{dq_1}{dV_1} = C_{11} - \frac{C_{12}^2}{C_{22}}. \qquad (4)$$

Using equation (1), for the case of fixed potential V_2 we obtain:

$$C_1 = \frac{dq_1}{dV_1} = C_{11}. \qquad (5)$$

Therefore, the capacitance of a single conductor is equal to its self capacitance only if the potentials of other conductors are kept constant. If the surrounding conductors have fixed charges, then $C_1 < C_{11}$ (see equation (4)).

The calculations of capacitance coefficients, using the method of images (involving an infinite series of the images in this case), yield the following expressions [5, 7]:

$$C_{11} = C_{1\infty} y \sinh \alpha \sum_{n=1}^{\infty} \frac{1}{y \sinh(\alpha n) + \sinh[\alpha(n-1)]}, \qquad (6)$$

$$C_{12} = -C_{1\infty} \frac{y \sinh \alpha}{x+y} \sum_{n=1}^{\infty} \frac{1}{\sinh(\alpha n)}, \qquad (7)$$

$$C_{22} = C_{1\infty} y \sinh \alpha \sum_{n=1}^{\infty} \frac{1}{\sinh(\alpha n) + y \sinh[\alpha(n-1)]}, \qquad (8)$$

where $C_{1\infty} = 4\pi\epsilon_0 R_1$ is the capacitance of the secluded first sphere; $x = d/R_1 > 1$ is the dimensionless distance between the center of the first sphere and the nearest point of the surface of the second sphere; $y = R_2/R_1$ $(0 < y < \infty)$ is the sphere radii ratio. The parameter α is related to x and y by expression

$$\cosh \alpha = \frac{x(x+2y) - 1}{2y}. \qquad (9)$$

The quantity $C_1 \to C_{1\infty}$ as $x \to \infty$ (the secluded conductor model for the first sphere) or $y \to 0$ (the point charge model for the second sphere). For the finite value of y and $x \gg 1$ we can use the known expansions [8] of the capacitance coefficients for the case of two widely separated spherical conductors. Then, considering equations (4) and (5), we derive:

$$C_1 \approx C_{1\infty} \left(1 + \frac{y^3}{x^4} \right) (q_2 = \text{const}), \qquad (10)$$

$$C_1 \approx C_{1\infty} \left(1 + \frac{y}{x^2} \right) (V_2 = \text{const}). \qquad (11)$$

When the spheres are close $(x \to 1)$, one can use the approximate expressions [1] for the capacitance coefficients. Then, for such a near approach, we immediately obtain:

$$C_1 \approx \frac{C_{1\infty} y}{1+y} \frac{\left[\psi\left(\frac{1}{1+y}\right) + \psi\left(\frac{y}{1+y}\right) - 2\psi(1) \right]}{\ln\left(\frac{2}{\alpha}\right) + \psi^2(1) - \psi\left(\frac{1}{1+y}\right) \psi\left(\frac{y}{1+y}\right)}{\psi\left(\frac{1}{1+y}\right) - \ln\left(\frac{2}{\alpha}\right)}$$

$$(12)$$

if $q_2 = \text{const}$,

$$C_1 \approx \frac{C_{1\infty} y}{1+y} \left[\ln\left(\frac{2}{\alpha}\right) - \psi\left(\frac{y}{1+y}\right) \right] \qquad (13)$$

if $V_2 = \text{const}$. Here $\psi(z)$ is the logarithmic derivative of the gamma function [9].

According to equations (12), (13), the capacitance of the first sphere in the case of its contact with the second sphere $(x = 1)$ formally takes a finite value at $q_2 = \text{const}$ and diverges at $V_2 = \text{const}$. The first circumstance follows from the fact that the electrical energy of a limited body with a finite charge should take a finite value. However, our consideration (equations (4)–(8)) is not physically correct as x is very close to 1, since in a dielectric medium the electrical breakdown becomes inevitable. In the vacuum the effects of field electron emission are possible. We also note that according to equations (9) and (10) $C_1 > C_{1\infty}$ for considered above limiting cases. In the following section we give simple physical explanation of this fact for general $(x > 1)$ case.

3. Numerical Results and Discussion

In Figures 2 and 3 we plot relative difference (relative error) $\delta_{C_1} = (C_1 - C_{1\infty})/C_1$ as a function of relative distance x, constructed at different values of y for two cases: $q_2 = \text{const}$ and $V_2 = \text{const}$. Since the series in equations (5)–(7) converge, we can hold a large but finite number of terms for numerical calculations. We increase the number of terms in the series until the change in the value of δ_{C_1} at the last iteration does not exceed 0.1%. This procedure can be realized using some mathematical software (Mathematica, Maple, etc.).

It follows from Figures 2 and 3 that $\delta_{C_1} > 0$ always (that is, $C_1 > C_{1\infty}$). This fact in the case of $q_2 = \text{const}$ is physically explained by the phenomenon of electrostatic induction. Due to the electric field of charge q_1, a redistribution of electric charge in the second sphere takes place. On the nearest side of the second sphere, the induced charges of opposite sign appear, while the far side is charged with the same sign. The resulting electric field of the second sphere causes a decrease in the absolute value of the potential of the first sphere. This ultimately leads to some increase in its capacitance C_1.

As distance x increases, the magnitude of the nearest charge induced on the second sphere and its field decrease. As a result, the value of δ_{C_1} decreases too (Figure 2).
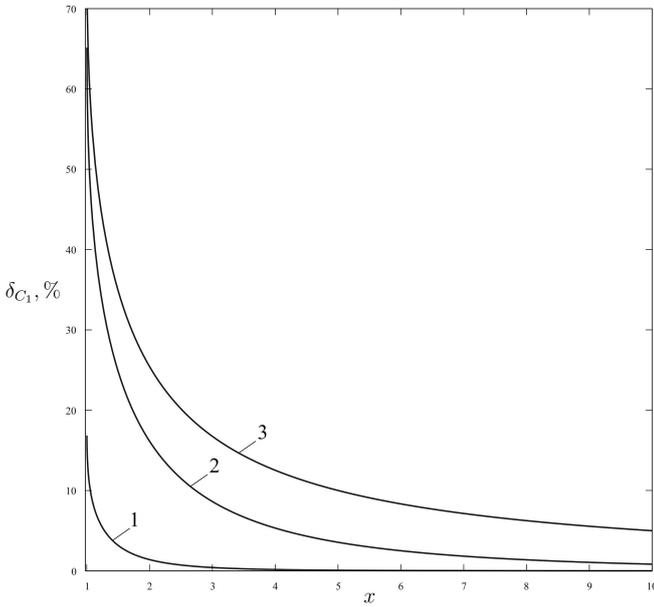
Figure 2: Relative difference δ_{C_1} as a function of x for $q_2 =$ const. (1) $y = 1$, (2) $y = 10$, and (3) $y \to \infty$.
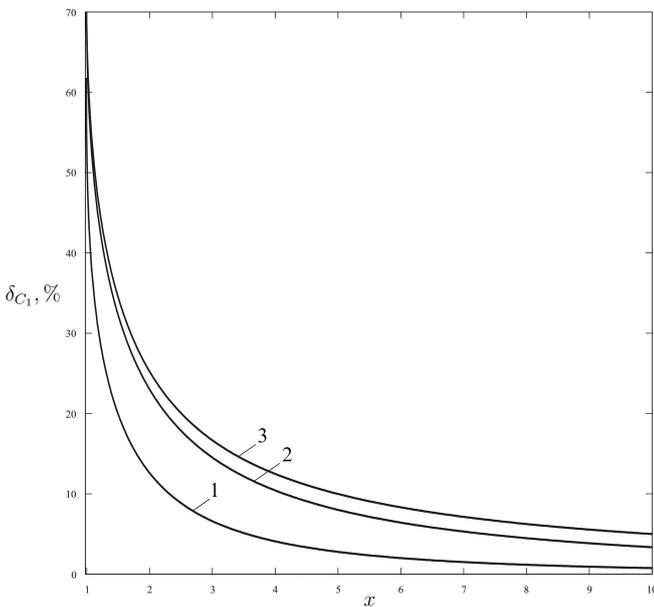
Figure 3: Relative difference δ_{C_1} as a function of x for $V_2 =$ const. (1) $y = 1$, (2) $y = 10$, and (3) $y \to \infty$.

With an increase in y (that is, increase of the second sphere radius) the relative effective distance between its induced charges of opposite sign increases. This causes the increasing of the resulting electric field of the second sphere and the appropriate increasing of the capacitance of the first sphere. Therefore, the greatest deviations from the model of a secluded sphere are observed for the case of the presence of infinite conducting plane ($y \to \infty$; Figure 2) near it. For example, in this case, the value of $\delta_{C_1} = 5\%$ for $x = 10$.

For the case of a grounded sphere ($V_2 =$ const) the electric field of charge q_1 causes the inflow of charge of opposite sign q_2 from ground to the second sphere. This latter will also contribute to reduction of the potential of the first sphere and leads to some increase in capacitance C_1. The process of charging second sphere has somewhat greater influence on the capacitance of the first sphere than its redistribution (see equations (4), (5) and Figures 2, 3).

The solution of the considered in this paper electrostatic problem can be applied in the capacitive displacement sensors (or in the projected capacitive touch technology), when we measure the capacitance of a spherical electrode depending on the distance to another electrode (it is convenient in this case to use the flat surface of the Earth, whose influence on the capacity of the spherical electrode will be the greatest).

4. Conclusion

The capacitance of a conductor is a topic that is possibly not explored deep enough in undergraduate physics degree curricula. In this paper we try to fill this gap. First, we derive the general expression for the capacitance of a sphere in the presence of nearby another conducting sphere and consider it in two limiting cases. Next, we perform the numerical analysis of the capacitance dependence of a conducting sphere on the distance to another sphere and its relative size. Our analysis is accompanied by simple physical explanations employing the phenomenon of electrostatic induction. We hope that our consideration should help readers to probe the limits of applicability of the secluded conductor model and can be used in undergraduate courses or projects.

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