Revista Brasileira de Ensino de Física, vol. 47, e20250204 (2025)
www.scielo.br/rbef
DOI: https://doi.org/10.1590/1806-9126-RBEF-2025-0204

Localizability of charge and current distributions in the regime of electromagnetostatics and the mathematical proofs of Coulomb and Biot-Savart laws

Antony M.M. Polito*[1]

[1]Universidade de Brasília, Instituto de Física, Brasília, DF, Brasil.

Received on May 29, 2025. Revised on September 29, 2025. Accepted on October 3, 2025.

We investigate the problem of those conditions that, once imposed on the spatial localization of electric charge and current distributions, become sufficient for Coulomb and Biot-Savart laws to provide well-defined and unique representations for the solutions of electromagnetostatic field equations. When charge and current distributions are infinitely extended, these laws consist of improper integrals whose convergence is conditioned precisely by the way these distributions are spatially localized. The procedures that currently allow these laws to be proved consist of using the Helmholtz theorem – or, equivalently, the Poisson equation – and require an explicit definition of the scalar and vector potentials in terms of certain improper integrals. For this reason, these procedures appear to require conditions of sufficiency regarding the localization of the distributions that are unnecessarily restrictive. To overcome these limitations, we present alternative deductions of Coulomb and Biot-Savart laws. These new deductions imply a set of necessary conditions that must be satisfied by surface integrals located at infinite, the satisfaction of which depends exclusively on the asymptotic behavior of the fields. It can be shown that this asymptotic behavior coincides with the conditions of regularity that guarantee the uniqueness of the solutions of the field equations.

Keywords: Coulomb Law, Biot-Savart Law, Helmholtz Theorem, Poisson Equation, Spatial Localization, Infinitely Extended Charge and Electric Current Distributions.

1. Introduction

It is well known – but no less curious – that, in the regime of electromagnetostatics, the so-called *Coulomb* and *Biot-Savart laws* constitute useful *methods* to provide, in terms of elementary functions, respectively, the electrostatic and magnetostatic fields corresponding to certain *infinitely extended* distributions of electric charges and currents. For obvious reasons, we will call Coulomb and Biot-Savart laws *volume integral laws*.

It is also well known that these cases of successful applications of the volume integral laws always involve charge or current distributions that display simple *symmetries*. For this reason, instead of directly integrating the Coulomb and Biot-Savart laws, it is usually simpler to obtain the fields using what we can call *boundary integral laws* – effectively, *Gauss* and *Ampère laws*. So, just like the volume integral laws, the boundary integral laws constitute useful *methods* for finding solutions to the electrostatic and magnetostatic fields.

Although they are, admittedly, *methods* for obtaining the fields, nothing, of course, prevents us from interpreting the volume integral laws, on the one hand, or the boundary integral laws, on the other, as alternative sets of *fundamental axioms,* based on which the theory of electromagnetostatics can be constituted.

Axiomatic is, nevertheless, changeable and polymorphic[1]. From another point of view, we can interpret both volume and boundary integral laws not as fundamental, but as results of necessary implications (*theorems*) obtained from another set of axioms constituted by the inhomogeneous and the homogeneous local differential equations for the fields.

As we know, in electromagnetostatics, the inhomogeneous field equations are those that locally define the divergence of the electrostatic field and the curl of the magnetostatic field, respectively, in terms of the charge and current densities, taken as known sources. The homogeneous field equations are those that restrict the admissible functional forms of the fields by setting constraints on the curl of the electrostatic field and on the divergence of the magnetostatic field, regardless of what your sources are.

*Correspondence email address: antony.polito@gmail.com
Editor-in-Chief: Marcello Ferreira

[1] With this statement, we are not saying that an axiomatic structure is vague or devoid of criteria. Incidentally, once a theory is considered to be finally axiomatized, the axioms must satisfy the requirements of being, as a whole, sufficient and necessary for the deduction of all the results of the theory. Once rigorously constituted, any changes made to any of the axioms generally imply the need to reformulate the axiomatization completely.

If we look closely at a comprehensive set of available textbooks in English language [1–24], in which the subject of electromagnetostatics is developed, we will come to the conclusion that the methods based on volume integral laws never seem to establish themselves on the same level of justification as the methods based on boundary integral laws – even though both can be interpreted as *theorems* and both are effective for obtaining the fields as solutions of the fundamental axioms given by the local differential equations.

Indeed, on one hand, when it comes to proving the boundary integral laws, the hypotheses that eventually must be made about the *localization of charge and current distributions* never appear explicitly. These hypotheses are precisely those that involve *how the charge and current distributions are (finitely or infinitely) extended throughout Euclidean space*.

On the other hand, when it comes to proving the volume integral laws, the hypotheses about localization *always* appear explicitly. However, in this case, the problem that arises has a somewhat *opposite* nature. In fact, in the way the mathematical proofs are generally presented, readers are led to believe, unduly, that there are limitations on the scope of validity of these laws that are *stronger* than the actual ones.

Of course, it would be particularly intriguing if the proofs of the boundary integral laws, on the one hand, and the proofs of the volume integral laws, on the other, were carried out under completely different assumptions on the localization of the charges and the currents. It would therefore be essential to investigate the processes of mathematical deduction of these laws in the search for their common foundations. In this paper, we will not dwell deeply on the conditions of validity of the boundary integral laws, but we will point out some problems that deserve further investigation.

In general, the authors of the above-mentioned textbooks [1–24] that deal with electromagnetism – at very different levels of sophistication, indeed – *justify* the validity of Coulomb and Biot-Savart laws in two ways which are, obviously, compatible with each other, but are quite different.

Most of the time, the authors choose, at least initially, not *to deduce* these laws, precisely because they do not take, *prima facie*, the field equations as fundamental axioms. Instead, with one or two minor variations in the procedures, the main rule is to start from the concept of Coulomb force between point charges (or between two infinitesimal elements of charge), or from the concept of Ampère forces between steady linear currents (or infinitesimal elements of current), in order to extract from them the concepts of electrostatic field and magnetostatic field. (A substantial number of authors choose, in fact, to assume Biot-Savart law – but never the Coulomb law – without further justification, beyond its empirical adequacy, even before defining the magnetic force between current circuits).

Ultimately, therefore, the concepts of force between elements of charge and elements of current are considered by these authors to be fundamental. From there, by applying the superposition principle and going to the limit of the continuum, the *improper* integral expressions of the fields associated with continuous distributions of charge and current – in other words, Coulomb and Biot-Savart laws – are obtained. Then, with greater or lesser justification, the truth of the field equations is established by demonstrating that these volume integral laws are *formal* expressions of their solutions. Because they proceed from the particular to the general, these forms of presentation could be called "inductive".

On the other hand, in a minority of cases, the authors actually choose to start from the field equations as fundamental axioms and dedicate themselves to developing *proofs* of Coulomb and Biot-Savart laws. These forms of presentation can therefore be called *deductive*.

There are two standard ways in which these proofs are presented, but both involve defining and constructing the scalar and vector potentials, as well as using the fact that the field equations are invariant by gauge transformations performed on them. For this reason, both ways end up sharing the same assumptions regarding the localization of charge and current distributions. In this sense, they can be considered essentially equivalent.

The first way involves proving that the scalar potential and the components of the vector potential must necessarily satisfy the *Poisson equation* – the solution to which is obtained, using the Green's function method, in terms of certain improper integrals. The second way involves using the *Helmholtz theorem*. In both cases, Coulomb and Biot-Savart laws are obtained, respectively, as the results of calculating the (negative of the) gradient of the scalar potential and the curl of the vector potential.

The Helmholtz theorem requires the imposition of hypotheses on the *asymptotic behavior* of the divergences and curls of the fields. As in the case of the method based on the Poisson equation, these hypotheses are chosen to ensure that the improper integrals that define the potentials are effectively convergent, thus guaranteeing their existence. Ultimately, they are expressed in terms of *conditions of sufficiency on the global localization of charge and current distributions*, which we will call, for reasons that will be explained later, *strong conditions*.

The "weirdness" arises precisely when we see that these strong global localization conditions exclude certain configurations of charge and current which, by means of the Coulomb and Biot-Savart laws, produce *well-defined and unique solutions* of the field equations.

This situation naturally leads to the question of whether it is possible to find *alternative deductions* for Coulomb and Biot-Savart laws that can show that the commitment to strong conditions is not necessary. The hope is that these new deductions will provide a better explanation of why the volume integral laws are in fact successful, especially in situations where the infinitely extended distributions exhibit symmetries.

In this article, we present these alternative deductions. We will see that they explicitly require the satisfaction of a set of *necessary conditions* for the volume integral laws to be valid. These conditions consist of the requirement that the *asymptotic behavior* of certain *closed surface integrals* involving the potentials – and ultimately the fields – be such that they *converge to zero*.

These alternative deductions of the volume integral laws and the establishment of their necessary conditions of validity are, in fact, elementary, requiring no more than the judicious application of vector identities.

In an extensive investigation of English-language textbooks [1–24], we didn't even find any mention of the fact that Coulomb and Biot-Savart laws can be mathematically demonstrated in the way presented here. We also found no mention of the necessary conditions we obtained, the fulfillment of which is a requirement without which the volume integral laws cannot be valid and unique representations of the solutions of the field equations.

Among the articles published in teaching (or similar) journals, also in English, we found only one paper, by J. Heras [25], which is closely related to the present work. In the context of a different problem, Heras developed alternative proofs for the so-called *Jefimenko equations* – also known as *generalized Coulomb and Biot-Savart laws* [26–30]. His purpose was precisely to prove that Jefimenko equations are obtained exclusively by considering Maxwell equations, that is, without resorting to the equations that must be satisfied by the potentials.

Heras' demonstration starts with the identity that expresses the fields in terms of the Green's function associated with the potentials, then uses the field equations and finally arrives at the Jefimenko equations, as long as certain conditions on the fields defined in surfaces located at infinite are fulfilled.

The proofs presented here are, in a sense, carried out in the "opposite direction", since *they start from the field equations*. Then, using Green's function, the Coulomb and Biot-Savart laws are obtained – also as long as certain conditions on the fields defined in surfaces located at infinite are fulfilled. Despite the differences, both proofs become essentially equivalent if we restrict Maxwell equations to the context of static fields.

More importantly, however, is the fact that Heras does not direct his discussion toward addressing the problem of the localization of charge and current distributions, presumably because he does not concern himself with the analysis of the necessary conditions he obtained, nor with the search for conditions of sufficiency for their satisfaction.

As will be shown, in the case of so-called *globally localized distributions*, the conditions of sufficiency that will be obtained are in fact less restrictive than those that support the proofs based on the effective construction of the potentials. As a consequence, we will also obtain more satisfactory justifications for why the Coulomb and Biot-Savart laws constitute valid methods for obtaining well-defined and unique solutions to the field equations, even in the case of infinitely extended distributions.

We will also see that, because the conditions on the closed surface integrals located at infinity are *necessary*, they must be satisfied even in situations where the charge and current distributions are only *partially localized*, if we expect the fields generated by these distributions to be obtained by the Coulomb and Biot-Savart laws. However, in this paper, we will not investigate in depth the problem of partially localized distributions, which should be the subject of a more detailed study for forthcoming publication.

Before we begin, it might be interesting to justify why we are interested in discussing infinitely extended distributions. We maintain that it is not a question of claiming that distributions that are infinitely extended have no "physical" meaning. If, in the context of the theory being considered, physical space is postulated to be infinite and, by hypothesis, any part of the space can be occupied, then any attempt to limit the distributions of matter (or fields) to finite parts of space is nothing more than *ontological prejudice*.

It is even possible to argue that, for the purposes of application or experimentation, infinite distributions can only be the result of idealization processes. This is only because there are practical limitations to their effective realization. However, *theories are not objects that deal with practical possibilities*. The content that can be articulated within them is limited exclusively by virtue of the *logical and mathematical consistency of their own constitution* and, more objectively, by virtue of *the rules that they themselves lay down for the construction of their models*.

In any case, even if the reader doesn't share this view, it shouldn't diminish their interest in an exercise in mathematical physics which, I hope, is interesting in itself, given its role in classifying mathematical models and solidifying the logical consistency of a theory.

This work is divided into eight sections. The first section consists of this introduction. Sections 2 and 3 are dedicated to investigating the problem of the localization of electric charge and current distributions. In particular, we will be interested in obtaining sufficient conditions for Coulomb and Biot-Savart laws to provide *well-defined* solutions to the field equations.

In the fourth section, we will briefly discuss the problem related to the so-called *regularity conditions*, that is, the necessary conditions that must be imposed on the asymptotic behavior of the fields so that the solutions of the field equations are *unique*. Next, we will comment on the problem related to how the conditions of regularity on the fields and the conditions of localization of distributions relate to each other.

In the fifth section, we briefly explain the limitations of the proofs of Coulomb and Biot-Savart laws found in textbooks. We deal more specifically with the Helmholtz theorem and justify the need to obtain alternative proofs to overcome these limitations.

In the sixth and seventh sections, we present the alternative deductions of Coulomb and Biot-Savart laws and obtain the necessary conditions for them to express solutions of the field equations. Next, we will discuss the sufficient conditions to be asymptotically satisfied by the fields for the satisfaction of these necessary conditions. As we will see these conditions coincide with the regularity conditions that must be satisfied for the representations of the solutions to be *unique.* Finally, we will discuss how the results obtained in these last two sections relate to the results obtained in sections 2 and 3, regarding the conditions of localization on the distributions.

In the last section, we make some brief concluding remarks.

2. Field Equations of Electromagnetostatics and Global Localization of Charge and Current Distributions

In this and the next section, we will systematically discuss the problem of localization of *infinitely extended* charge and current distributions, successively taking into account the following aspects: (i) the *good definition* of the *field equations*; (ii) the *formal validity* of the Coulomb and Biot-Savart laws, as *solutions* to the field equations and the (iii) the *good definition* of the *improper integrals* that represent the Coulomb and Biot-Savart laws.

That said, we must also establish that, in the context of this work, we are exclusively interested in discussing electromagnetostatics in *vacuum.* Therefore, the electrically charged matter will be considered non-polarizable, non-magnetizable and non-conductive. To maintain the discussion as simple as possible, we do not consider also charge and current densities defined on surfaces.

Let's assume that the fundamental *axioms* of electromagnetostatics are the so-called *field equations*, that is, the *local laws* that determine the form of the relationships between the electrostatic field, $\boldsymbol{E}(\boldsymbol{r})$, and the magnetostatic field, $\boldsymbol{B}(\boldsymbol{r})$, and their respective *sources*, namely the electric charge density, $\rho(\boldsymbol{r})$, and the electric current density, $\boldsymbol{J}(\boldsymbol{r})$.

The mathematical representation of these local laws is given by the well-known set of differential equations that specify, in the entire Euclidean space, the divergence and the curls of the fields:

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\varepsilon_0}\rho(\boldsymbol{r}) \quad \text{(i)}; \qquad \nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \quad \text{(ii)}; \quad (1)$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \quad \text{(i)}; \qquad \nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}(\boldsymbol{r}) \quad \text{(ii)}; \quad (2)$$

where the constants ε_0 and μ_0 are respectively the electric permittivity and the magnetic permeability of vacuum.

We can't stress enough how these equations should be interpreted. The charge and current densities must be considered as *prescribed functions* or, as we say, according to a more empiricist bias, they are "*sources whose spatial disposition is under the possibility of our control*".

At first glance, eqs. (1) and (2) do not, in fact, seem to place any restrictions on our ability to prescribe how charges and currents are distributed in space. If the distributions are, as we will call them later, *absolutely localized*, in fact, there is little reason to doubt this.

However, this is not the case if we are willing to consider distributions that are *infinitely extended.* In this case, it is worth asking whether there aren't *principles* in the theory that naturally impose restrictions on the ways in which these infinite distributions can be spatially arranged and asking what exactly these restrictions are.

The answer to the first question is simple (and well known): indeed, there is only one *fundamental principle*, namely, that *the forces produced by the fields acting on test charges must be finite and uniquely defined at all points in the Euclidean space.* From this last requirement it follows that *the forces must also be continuously extended throughout Euclidean space.* On the other hand, the answer to the second question is not so straightforward and is basically the content of this article.

Let's start this discussion by establishing the concept of *good definition* of the field equations (1–2). By definition, the field equations will be *well-defined* if *the sources are prescribed in such a way that the generated fields produce forces on test charges that are* (i) *continuously extended throughout the entire Euclidean space*; (ii) *finite at almost all points in space; and* (iii) *smoothly extended at almost all points in space*[2].

By "almost" in the requirement (ii) we mean that, *for now*, we will assume that the only possible exceptions are points belonging to *closed surfaces "located at infinity"*[3].

[2] Therefore, what we call *good definition*, in this context, does not include the requirement of *uniqueness.* The uniqueness conditions depend, as is well known, on setting appropriate *boundary conditions* on the fields, and we will deal with them separately, in section 4.

[3] There is no such a thing like surfaces "located at infinity". Without risk of falling into inconsistency, we will make unrestricted use of this current abuse of language to refer to a *limiting process* according to which closed surfaces S grow unlimitedly, *in all directions of space*, such that the volumes enclosed by them approach the entire Euclidean space. We denote this limiting process by $S \to \infty$. However, it is important to note that the way in which this limiting process should be performed must be specified on a case-by-case basis, since its calculation depends on the geometric shape of surface S and the way in which its infinitesimal surface elements depend on the spatial coordinates that will (eventually) serve as those parameters that must be taken to the infinite limit.

We'll go back to these exceptions in section 4, where the *uniqueness* requirement is considered.

Since the forces are produced by the fields, we conclude that the requirement of good definition of the field equations means that we must demand on the sources that the electrostatic and magnetostatic fields generated by them be (i) *continuous* at all points of Euclidean space; (ii) *smooth* at almost all points in Euclidean space; and (iii) *bounded* (above and below), that is, finite throughout the Euclidean space, with the possible exceptions of points belonging to closed surfaces located at infinity.

Under these requirements on the fields, the field equations demand that *the charge and current distributions* be (i) *bounded* (above and below) *at all points in space – with the possible exception of points located on a closed surface located at infinity* – and be (ii) *at least piecewise continuous throughout the Euclidean space.* Thus, certain *finite discontinuities on the distribution of the sources* (and a finite number of them) *can be tolerated* on surfaces that represent interfaces between regions with different charge or current densities[4].

However, special attention should be paid to Ampère law, eq. (2.ii), as it necessarily implies a constraint on the divergence of the current density:

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = 0. \tag{3}$$

This constraint guarantees – in view of the principle of conservation of electric charge – that the magnetostatic regime is compatible with the electrostatic regime, insofar as the zero divergence of the current densities implies the independence of the charge densities with respect to time. But above all *divergenceless currents must be assumed as a precondition without which Ampère law would simply be inconsistent.*

The requirement that the null divergence constraint (3) be satisfied at all points in space generally implies some conditions on the continuity of the components of the current densities but does not necessarily imply that they must all be continuous. In any case, we must consider – in addition to *boundedness* and *piecewise continuity* – the *null divergence condition* (3) as part of the requirements on charge and current distributions for the good definition of the field equations.

By applying the divergence theorem, an important consequence is implied by the constraint (3). If we require that it be satisfied at all points in space, then *the total flux of currents through any closed surface must be zero.* In particular, this must apply also to a closed surface S located at infinity:

$$\oint_S \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} = 0 \Rightarrow \oint_{S\to\infty} \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} = 0. \tag{4}$$

Eq. (4) should be understood as a *limiting process*, which defines an improper surface integral with a well defined asymptotic behavior. Note, however, that in the magnetostatic regime, the condition that the total flux of currents must be zero must be satisfied *on all closed surfaces*, not just on closed surfaces located at infinity.

Typically, infinitely extended current distributions that satisfy constraint (3) (and (4)) are those that display symmetries that guarantee the conservation of the total amount of charge inside any volume arbitrarily placed in space.

Intuitive examples are the cases of distributions of currents infinitely extended in one direction, such as distributions that exhibit *full cylindrical symmetry* – that is, whose current densities are invariant by translations and rotations with respect to a spatial axis – or the cases that exhibit *azimuthal symmetry* – with current densities invariant only under rotations with respect to a spatial axis. Also intuitive are the cases of current distributions that display *rectangular symmetry.* In all these cases, it is quite clear that the divergence of the current densities is always zero everywhere.

2.1. A classification of the distributions according to localization criteria

Now is a good time to introduce a number of concepts and terms that will be used from now on. There are infinite ways in which charge and current distributions can be arranged in Euclidean space. However, we are mostly interested in situations that exhibit some kind of symmetry.

By definition, we will say that *infinitely extended charge distributions* are *globally localized* if, and only if, the magnitudes of their charge densities *decay to zero*, at the limit of infinity – that is, with $r \to \infty$, $r = |\boldsymbol{r}|$ being the distance from the origin of the coordinate system –, *regardless of direction in the Euclidean space*, but *not* necessarily in the same way.

We mathematically express this idea in the following way. Given the functional form of the charge density in spherical coordinates, $\rho(\boldsymbol{r}) = \rho(r, \theta, \varphi)$, the distribution is globally localized if, and only if, for any fixed direction of space, $\lim_{r\to\infty} |\rho(r, \theta, \varphi)| = 0$. It is *not* necessary, for global localization, that the functions $\rho(r, \theta, \varphi)$, defined for each fixed direction, have *the same asymptotic behavior*, unless it is stated that the distributions are *asymptotically symmetric.*

Likewise, we will say that *infinitely extended current distributions* are *globally localized* if, and only if, the magnitudes of the Cartesian components of their current

[4] This is why some components of the fields may not be smooth at all points in space, since discontinuities of the charge and current densities imply discontinuities in the derivatives of some components of the fields. Discontinuities in charge and current densities on interface surfaces can be tolerated, in principle, because they can be treated as mere mathematical conveniences, designed to avoid a detailed description of the fields in regions where densities vary very fast, on a very short length scale, compared to the typical scales of variation of densities in other regions. Singularities represented by charge and current distributions on surfaces or by "point charges", however, are ruled out.

densities *decay to zero*, at the limit of infinity, regardless of direction in the Euclidean space, but not necessarily in the same way.

Mathematically, we say that, given the functional form of the Cartesian components of the current density in spherical coordinates, $J_\mu(r,\theta,\varphi)$, with $\mu = 1,2,3$, the distribution is globally localized if, and only if, for any fixed direction of space, $\lim_{r\to\infty} |J_\mu(r,\theta,\varphi)| = 0$. Again, it is *not* necessary that the functions $J_\mu(r,\theta,\varphi)$, defined for each fixed direction, have the same asymptotic behavior.

On the other hand, we say that *infinitely extended distributions* are *partially localized* if, and only if, the magnitudes of the charge (or the magnitudes of the Cartesian components of the current) densities *decay to zero*, at the limit of infinity, but not in all directions of space. Although not essential, for the sake of simplicity, we will assume that partially localized distributions are such that they display some kind of *global* symmetry. The main examples we have in mind are as follows.

By definition, *rectangular partially localized* distributions of *charge* decay to zero, at infinity, but *only in a privileged direction defined by a privileged axis*, say, without loss of generality, the z-axis. Moreover, we will require that this kind of distribution has not only an axis, but also a privileged plane, perpendicular to the privileged axis (the x–y plane), taken as a *plane of symmetry by reflection.* Finally, it is supposed that the charge densities remain uniform in any plane perpendicular to the privileged axis (and parallel to the plane of symmetry). We mathematically express these ideas saying that, in Cartesian coordinates, $\rho(\boldsymbol{r}) = \rho(x,y,z) = \rho(|z|)$ and $\lim_{z\to\pm\infty} \rho(z) = 0$.

Rectangular partially localized distributions of *current* are defined in analogous way, but it is supposed that they have only one direction in the space, say, without loss of generality, the x-direction. Thus, they are such that $J_x(\boldsymbol{r}) = J_x(x,y,z) = J_x(|z|)$ and $\lim_{z\to\pm\infty} J_x(z) = 0$.

By the same token, we define *full cylindrical partially localized* distributions of *charge* by demanding that, in cylindric coordinates, $\rho(\boldsymbol{r}) = \rho(s,\varphi,z) = \rho(s)$ and $\lim_{s\to\infty} \rho(s) = 0$. That is, they decay to zero, at infinity, but *only in the directions that are perpendicular to a privileged axis*, say, without loss of generality, the z-axis. The z-axis is, obviously, an *axis of symmetry by rotation* around it and *axis of symmetry by translation* along it. Clearly, full cylindrical partially localized distributions have not only an axis, but also infinite (perpendicular) planes of symmetry by reflection, and it is supposed that the charge densities remain uniform in any plane perpendicular to the privileged axis.

Full cylindrical partially localized distributions of *current* are defined in analogous way, but it is supposed that they have only one non-zero component, in the cylindric basis. These non-zero components can be the z-component (*axial symmetry*) or the φ-component (*azimuthal symmetry*). In each case, we must have $J_{z,\varphi}(\boldsymbol{r}) = J_{z,\varphi}(s,\varphi,z) = J_{z,\varphi}(s)$ and $\lim_{s\to\infty} J_{z,\varphi}(s) = 0$[5].

If infinite charge and current distributions are not globally or partially localized, they are simply said to be *non-localized at all.*

In the present context, we are mainly interested in discussing globally localized distributions. And within this category, we are particularly interested in distributions that exhibit *asymptotic behavior in the form of inverse power laws.* To this end, we should define a convenient set of categories in order to build a classification that is better fitted to handle these types of distributions.

By definition, charge and current distributions are said to be *globally localized, in the semi-strong sense*, if, and only if, their asymptotic behaviors, in the limit of infinity, are such that, *for any fixed direction in the space*, *the magnitudes of the corresponding densities go to zero faster than* $1/r$. Therefore, in addition to $\lim_{r\to\infty} |\rho(r,\theta,\varphi)| = 0$, or $\lim_{r\to\infty} |J_\mu(r,\theta,\varphi)| = 0$, we require from charge or current densities even stronger asymptotic decays, such that they are *asymptotically dominated* by the decay of the inverse power function $1/r$. Therefore, semi-strong global localization requires that $\lim_{r\to\infty} |\rho(r,\theta,\varphi)|r = 0$ or $\lim_{r\to\infty} |J_\mu(r,\theta,\varphi)|r = 0$[6].

By opposition, we define the category of *globally localized distributions in the weak sense.* The charge or current distributions belong to this category if, and only if, their asymptotic behaviors are such that the *magnitudes of the corresponding densities go to zero, for any fixed direction in the space, slower than or as slow as* $1/r$. This is the same as requiring that the charge and current densities *dominate asymptotically* (or, yet, have the *same asymptotic behavior* of) the inverse power function $1/r$. Therefore, weak global localization requires that $\lim_{r\to\infty} |\rho(r,\theta,\varphi)| = 0$, but also that $\lim_{r\to\infty} |\rho(r,\theta,\varphi)|r \to \infty$ or $\lim_{r\to\infty} |\rho(r,\theta,\varphi)|r = cte \neq 0$. The same mathematical definitions extend to the (Cartesian components of the) current densities.

By force of the adopted definitions, weak and semi-strong localization conditions are complementary and mutually exclusive concepts.

Within the category of semi-strong globally localized distributions, we define the category of *globally localized distributions in the strictly semi-strong sense.* Distributions belong to this category if, and only if, their asymptotic behaviors are such that, for any fixed direction in space, *the magnitudes of the corresponding densities go*

[5] *Azimuthal partially localized* distributions are a special case of cylindric symmetry applied to currents distributions, such that, by definition, the only non-zero component is the azimuthal component, such that $J_\varphi(\boldsymbol{r}) = J_\varphi(r,\theta,\varphi) = J_\varphi(r,\theta)$, with $\lim_{r\to\infty} J_\varphi(r,\theta) = 0$.

[6] Cf. Appendix A, to the definitions of asymptotic dominances or asymptotic equivalence.

to zero faster than $1/r^{1+\varepsilon}$, for *some* arbitrary *positive* constant $\varepsilon > 0$, that can be chosen as small as we want.

Clearly, strictly semi-strong localization means that a globally localized distribution is *asymptotically dominated* by an inverse power law that, in turn, approaches indefinitely close, from below, to $1/r$, in the process of taking the positive parameter ε approaching zero indefinitely. Therefore, strictly semi-strong global localization requires that $\lim_{r\to\infty} |\rho(r,\theta,\varphi)|r^{1+\varepsilon} = 0$ or $\lim_{r\to\infty} |J_\mu(r,\theta,\varphi)|r^{1+\varepsilon} = 0$, for *some* arbitrary *positive* constant $\varepsilon > 0$.

The categories listed above have as fundamental criteria for their definitions the asymptotic relationships that the charge and current densities have with the inverse power law $1/r$. We can replicate the same classification structure by assuming, instead of $1/r$, the inverse power laws $1/r^2$ and $1/r^3$. Each of them produces its own set of categories, generically designated by the terms *strong* (associated with $1/r^2$) and *super-strong* (associated with $1/r^3$). Note that these sets of categories are not necessarily mutually exclusive.

Thus, within the category of strictly semi-strong globally localized distributions, we define the categories of *globally localized distributions in the strong sense and in the strictly strong sense.* Distributions belong to these categories if, and only if, their respective asymptotic behaviors are such that, for any fixed direction in space, *the magnitudes of the corresponding densities go to zero faster than* $1/r^{2+\varepsilon}$, for $\varepsilon = 0$ (strong) or for *some* $\varepsilon > 0$ (strictly strong).

By the same token, we define that, within the category of strictly strong globally localized distributions, distributions are *globally localized in the super-strong sense and in the strictly super-strong sense*, if, and only if, their respective asymptotic behaviors, in the limit of infinity, are such that, for any fixed direction in the space, *the magnitudes of the corresponding densities go to zero faster than* $1/r^{3+\varepsilon}$, for $\varepsilon = 0$ (super-strong) or for *some* $\varepsilon > 0$ (strictly super-strong).

To complete the classification, within the category of strictly super-strong globally localized distributions, we define the *absolutely localized distributions.* They correspond to distributions that are *finitely extended*, that is, they are completely contained inside a sphere of sufficient large radius R, such that $\rho(r > R, \theta, \varphi) = 0$ or $J_\mu(r > R, \theta, \varphi) = 0$.

In Table 1, the reader can observe the nested structure of the classification established for globally localized distributions, with their respective definition criteria.

2.2. The total flux of currents of a distribution

Let's return to the limiting process defined by eq. (4), which defines what we can call the *total flux of currents*, particularly associated with an infinitely extended divergenceless current distribution. Equation (4) defines a *necessary condition* on the asymptotic behavior of the current distributions, in the context of electromagnetostatics. Furthermore, given its simplicity, it constitutes a good first example of the use of the classification we built above that will help us to understand the concepts of global localization.

It can be seen that a *sufficient condition* to the satisfaction of the necessary condition (4) is that the current distributions are globally localized, in the *strong sense*, since $\lim_{r\to\infty} |J_\mu(r,\theta,\varphi)|r^2 = 0 \Rightarrow \oint_{S\to\infty} \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} = 0,$ *as*

Table 1: The nested classification of globally localized distributions. The function $f(r,\theta,\varphi)$ is a placeholder representing the charge density or the Cartesian components of the current density.

Category	Criterion
Globally Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert = 0.$
Weakly Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r \to \infty$ or $\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r = cte \neq 0.$
Semi-Strongly Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r = 0.$
Strictly Semi-Strongly Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r^{1+\varepsilon} = 0, \quad \varepsilon > 0.$
Strongly Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r^2 = 0.$
Strictly Strongly Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r^{2+\varepsilon} = 0, \quad \varepsilon > 0.$
Super-Strongly Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r^3 = 0.$
Strictly Super-Strongly Localized	$\lim_{r\to\infty} \lvert f(r,\theta,\varphi)\rvert r^{3+\varepsilon} = 0, \quad \varepsilon > 0.$
Absolutely Localized	$f(r > R,\theta,\varphi) = 0, \quad R > 0.$

long as we take the surface S as a spherical shell centered at the origin and define the limiting process in terms of the unlimited growth of its radius, taken as a parameter. From now on, *all limit processes involving surface integrals located at infinity should be understood exclusively in this sense.*

Indeed, for any fixed direction in space, all the Cartesian components of the vector $d\boldsymbol{a} = r^2 \sin\theta d\theta d\varphi \hat{\boldsymbol{r}}$ have the same asymptotic behavior and diverge as r^2, when $r \to \infty$ – except in the direction of the z-axis, where $d\boldsymbol{a}$ is zero. We denote this asymptotic behavior by $d\boldsymbol{a}(r \to \infty, \theta, \varphi) \sim r^2$.

Thus, for any fixed direction in space, $\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} \sim \sum_\mu J_\mu(r,\theta,\varphi) r^2$, where J_μ denotes the Cartesian components of the current density. Therefore, if the Cartesian components of the current density are localized in the strong sense, that is, if $\lim\limits_{r\to\infty} |J_\mu(r,\theta,\varphi)| r^2 = 0$, then, the surface integral (4) converges to zero. We conclude that *strong global localization is sufficient to satisfy the necessary condition* (4).

We could now ask if global localization (in any sense) would be *necessary* to satisfy eq. (4). We already saw that this is not true, because we explicitly gave examples of *partially* localized distributions that satisfy (4).

The condition (4) is a necessary condition, and it provides a first illustration of what we mean when we refer to *conditions of sufficiency for localization.*

By general definition, *conditions of sufficiency for localization* will be *restrictions on the asymptotic functional forms of charge or current densities* which, *although not strictly necessary*, are enough to (i) guarantee the validity of *necessary conditions* imposed on *surface integrals located at infinity*; or, as we will see, (ii) guarantee *the convergence of improper volume integrals taken on the entire Euclidean space.*

Necessary conditions imposed on surface integrals obviously imply some restriction on the functional forms of the vector fields that are defined on them. The fact is that these restrictions, by themselves, are never strong enough to *uniquely determine* these vector fields – as we had seen, in the case of partially localized distributions.

The best we can do is try to establish conditions as general as possible for the functional form of the fields that are sufficient to satisfy those necessary conditions. Eventually, it is possible that the adoption of additional hypotheses will make it possible to find conditions that are also necessary.

2.3. The total charge of a distribution

To illustrate point (ii) above, let's consider the problem of the *finiteness* of the *total charges* Q of general extended charge distributions:

$$Q = \int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}) d\tau. \tag{5}$$

Such as in the situation involving eq. (4), eq. (5) must be understood as a limiting process, which defines an improper volume integral on the entire Euclidean space. It is interesting to note that strong global localization is still not sufficient to guarantee the finiteness of the total charge Q.

Indeed, we can prove that *a sufficient condition* for finiteness of the total charge is that the charge distributions are *globally localized in a strictly super-strong sense.* A particular case is, of course, the case of *absolutely localized* distributions. In these last cases, the charge densities become zero from a sufficiently large sphere that contains all the distribution, making their total charges necessarily finite.

To prove the result in the general case of *infinitely extended* distributions, we must demonstrate that, if $\lim\limits_{r\to\infty} |\rho(\boldsymbol{r})| r^{3+\varepsilon} = 0$, for any fixed direction in space, then the implication:

$$\lim_{r\to\infty} |\rho(\boldsymbol{r})| r^{3+\varepsilon} = 0 \Rightarrow \int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}) d\tau < \infty; \tag{6}$$

is true, for any positive constant $\varepsilon > 0$, that can be chosen as small as we want.

We can express this implication in the most convenient system of coordinates to deal with globally localized distributions and to define the limiting process, namely, spherical coordinates:

$$\begin{aligned} &\lim_{r\to\infty} |\rho(r,\theta,\varphi)| r^{3+\varepsilon} = 0 \\ &\qquad \Rightarrow \int d\Omega \int_0^{r\to\infty} r^2 dr \rho(r,\theta,\varphi) < \infty; \end{aligned} \tag{7}$$

where $\int d\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi$.

Before continuing, the following point needs to be clarified. To deal with the volume integral (5), we should decide whether we are going to consider the integrations of the angular coordinates before considering the improper integration of the radial coordinate, or whether we are going to do the opposite.

If we perform the integrations of the angular coordinates first, we will obtain the improper radial integral $\int_0^{r\to\infty} h(r) r^2 dr$ of a *new function* $h(r) = \int \rho(r,\theta,\varphi) d\Omega$ – whose meaning is the total electric charge contained in a spherical shell between r and $r + dr$. Thus, we must verify if $h(r)$ and $\rho(r,\theta,\varphi)$ *behaves asymptotically in the same way*, before using this fact in some eventual proof[7].

However, it is unnecessary to follow this route. It is much more convenient to consider the improper integration of the radial coordinate first. In this case, what we have to do is determine the conditions for the existence of the functions $g(\theta,\varphi) = \int_0^{r\to\infty} r^2 \rho(r,\theta,\varphi) dr$,

[7] If we have *infinitely extended* and *globally* localized charge distributions, there is no reason why, *asymptotically*, the total electric charge contained in continuously distributed spherical shells should behave differently from the density itself, as long as we assume that the charge density becomes smoothly *spherically symmetric*, in the asymptotic limit. Assuming that this is the case, it would be safe to consider that $h(r)$ and $\rho(r,\theta,\varphi)$ have the same asymptotic behavior.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2025-0204

that is, determine if, for any fixed direction in space, given by fixed angles (θ, φ), the improper integrals are convergent.

Of course, after that, we should verify if the angular dependence of $g(\theta, \varphi)$ is such as these functions are well behaved in the finite intervals of integration of the angular variables, to obtain finite results. Once the charge density is assumed to be bounded everywhere – and, in particular, in each spherical shell with a radius between r and $r + dr$–, this condition will always be satisfied.

Now, given any bounded function $f(r)$, the most important fact about the improper integral $\int_0^{r\to\infty} f(r)r^2 dr$ is that its convergence is completely determined by the asymptotic behavior of $f(r)$. So, we must investigate the behavior of the integral on the "tail' of the distribution, that is, we must assess the behavior of $\int_a^{r\to\infty} f(r)r^2 dr$, where $a > 0$ is a sufficiently large positive number.

In Appendix A, throughout eqs. (A.5–A.9), we prove that, if $f(r)$ is a bounded function throughout the interval $0 \le r < \infty$, it is asymptotically continuous and positive and decays to zero, at infinite, then, the implication $\lim_{r\to\infty} f(r)r^{3+\varepsilon} = 0 \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < \infty$ is true, for any $\varepsilon > 0$, where $a > 0$ is a sufficiently large positive constant.

Now, if, for any fixed direction in space, we choose $f(r) \sim |\rho(r, \theta, \varphi)|$, then:

$$
\begin{aligned}
&\lim_{r\to\infty} |\rho(r, \theta, \varphi)| r^{3+\varepsilon} = 0 \\
&\quad \Rightarrow \int_a^{r\to\infty} |\rho(r, \theta, \varphi)| r^2 dr < \infty \\
&\quad \Rightarrow \int_a^{r\to\infty} r^2 \rho(r, \theta, \varphi) dr < \infty; \qquad (8)
\end{aligned}
$$

for any $\varepsilon > 0$, where $a > 0$ is a sufficiently large positive constant, and we used the fact that, if a function is *absolutely integrable* on the interval $[a, \infty)$, then it is *integrable* on this interval.

Finally, we can conclude, from eqs. (7) and (8), that the improper integral $\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}) d\tau$ converges, provided that the resulting functions $g(\theta, \varphi; a) = \int_a^{r\to\infty} r^2 \rho(r, \theta, \varphi) dr$ be sufficiently well behaved in the intervals of definition of the angular variables.

This completes the demonstration that strictly super-strong global localization furnishes a sufficient condition to the finiteness of the total charge of a distribution, provided that the charge density is bounded everywhere and is asymptotically continuous.

More precisely, *the strictly super-strong condition tells us that the total charge of any distribution is finite if its radial behavior is asymptotically dominated by an inverse power law that, in turn, approaches infinitely close, from below, to* $1/r^3$ (that defines the super-strong condition) *in the process of taking the positive parameter* ε *approaching zero indefinitely.*

On the other hand, it is *not true* that global localization in the strictly super-strong sense is a *necessary* condition for the finiteness of the total charge of a distribution. In fact, charge distributions that are globally localized, in any sense, or even partially localized, or non-localized at all, can have finite total charges. Configurations with some kind of symmetry involving total or partial cancellation of contributions from the positive and negative parts of the charge distributions can easily be imagined as examples.

If the charge distributions are globally localized, but their densities go to zero *slower* than $1/r^3$, for any fixed direction in the space, then, nothing can be said, in general, about the convergence of the total charge, without assuming additional hypothesis about the asymptotic details of the distribution, in particular, about the arrangements of their positive and negative parts.

Although, in these cases, it is not possible to conclude anything specific regarding the finiteness of the total charge, what can be done is to establish a comprehensive *sufficient condition* for the total charge *to possibly converge or diverge*, but, in the latter case, *slower than* r^2. This result will be useful in section 4.

Indeed, reproducing, in a similar way, the argument we presented above, it is not difficult to see, using the results presented in Appendix A, throughout equations (A.10–A.12), that a sufficient condition for a possible divergence to occur, at most, slower than r^2 is *global localization, in the semi-strong sense.* We express this result as follows:

$$
\lim_{r\to\infty} |\rho(\boldsymbol{r})| r = 0 \Rightarrow \int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}) d\tau < r^2. \qquad (9)
$$

Obviously, according to the results in Appendix A, if the semi-strong global localization is a sufficient condition, then the *strictly semi-strong global localization is also sufficient.*

On the other hand, if we want the total charge to diverge *as slowly as* r^2, then a sufficient condition is *weak global localization*, with the charge density behaving asymptotically like $1/r$, but with the *additional condition* of *asymptotic spherical symmetry:*

$$
|\rho(\boldsymbol{r})| \sim \frac{1}{r} \Rightarrow \int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}) d\tau \sim r^2. \qquad (10)
$$

Asymptotic spherical symmetry ensures that, starting from a spherical shell with a sufficiently large radius, the distribution becomes either strictly positive or strictly negative. Without this additional condition, nothing could be concluded from $|\rho(\boldsymbol{r})| \sim 1/r$, since the total charge could even be finite.

Conditions (4) and (5) are simple and paradigmatic. In fact, most sufficiency conditions for satisfying asymptotic conditions on closed surface integrals or convergence of improper volume integrals fall into one of these two categories.

2.4. The total current of a distribution

To complete the discussion of this section, we will consider the problem of the "total current" $\boldsymbol{J}$ of an extended current distribution, defined by:

$$\boldsymbol{J} \equiv \int_{V\to\mathbf{R}^3} \boldsymbol{J}(\boldsymbol{r})d\tau. \tag{11}$$

The discussion regarding the convergence of the volume integral (11) should be entirely analogous to the discussion of the volume integral (5). Of course, since $\boldsymbol{J}$ is a vector, we must treat each one of their Cartesian components separately. As before, we would conclude that, if a current distribution is globally localized, then, a sufficient condition to the convergence of (11) would be strictly super-strong localization.

However, we cannot forget that $\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = 0$ and this has an important consequence regarding the conditions of convergence of the total current (11). In fact, we can prove that, *if the currents are divergenceless*, then:

$$\boldsymbol{J} \equiv \int_V \boldsymbol{J}(\boldsymbol{r})d\tau = \oint_S \boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a}; \tag{12}$$

where V is a finite volume, with S as its boundary, and $\boldsymbol{r}$ is the non-zero position vector where $\boldsymbol{J}(\boldsymbol{r})$ is evaluated.

We begin the demonstration of (12) with the implication:

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = 0 \Rightarrow \int_V \boldsymbol{r}\nabla \cdot \boldsymbol{J}(\boldsymbol{r})d\tau = 0. \tag{13}$$

The right-hand side of the implication (13) is a vector equation. To prove eq. (12), we can perform, on eq. (13), an integration by parts for each Cartesian component of $\boldsymbol{r}$ to obtain:

$$\begin{aligned}\int_V & x^\mu \nabla \cdot \boldsymbol{J}(\boldsymbol{r})d\tau \\ &= \oint_S x^\mu \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} - \int_V \boldsymbol{J}(\boldsymbol{r}) \cdot \nabla x^i d\tau = 0 \\ &\Rightarrow \oint_S \boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} = \int_V \boldsymbol{J}(\boldsymbol{r}) \cdot \nabla \boldsymbol{r} d\tau. \end{aligned} \tag{14}$$

But we can recognize that:

$$\int_V \boldsymbol{J}(\boldsymbol{r}) \cdot \nabla \boldsymbol{r} d\tau = \int_V (\boldsymbol{J}(\boldsymbol{r}) \cdot \nabla)\boldsymbol{r} d\tau = \int_V \boldsymbol{J}(\boldsymbol{r})d\tau; \tag{15}$$

from which eq. (12) follows.

Eq. (12) has an immediate (and intuitive) consequence. *If the current distributions are absolutely localized, then, their total current are precisely zero*, provided the volume of integration is large enough to contain the entire distribution. Indeed, if the current distribution is absolutely localized, we can always choose a volume V sufficiently large to have $\boldsymbol{J}(\boldsymbol{r}) = 0$ at any surface S that encloses V. Under this condition, we conclude that $\int_V \boldsymbol{J}(\boldsymbol{r})d\tau = 0$.

To assess the total current in the general case of infinitely extended distributions, it is enough to take eq. (12) in the limiting process where V approaches the entire Euclidean space and, therefore, the surface integral is taken located at infinity:

$$\boldsymbol{J} \equiv \int_{V\to\mathbf{R}^3} \boldsymbol{J}(\boldsymbol{r})d\tau = \oint_{S\to\infty} \boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a}; \tag{16}$$

where we take the surface S as a spherical shell centered at the origin and the limiting process defined by its radius growing indefinitely.

To assess the asymptotic behavior of the surface integral in eq. (16), we must remember, from the discussion of the total flux of currents, eq. (4), that, for any fixed direction in space, $\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} \sim \sum_\mu J_\mu(r,\theta,\varphi)r^2$, where J_μ denotes the Cartesian components of the current density. Therefore, the asymptotic behaviors of the Cartesian components of the integrand of the surface integral (16) are given by $x^\mu \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} \sim x^\mu \sum_\alpha J_\alpha(r,\theta,\varphi)r^2$. But, since $x^\mu \sim r$, then, $x^\mu \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} \sim \sum_\alpha J_\alpha(r,\theta,\varphi)r^3$.

Now, we can see that, if *divergenceless* current distributions are *globally localized in the super-strong sense*, such that $\lim_{r\to\infty} |J_\mu(r,\theta,\varphi)|r^3 = 0$, for any fixed direction in the space, then *their total currents converge to zero*. This is a condition of sufficiency.

If divergenceless current distributions are globally localized, but their Cartesian components go to zero like $1/r^3$, for any fixed direction in the space, then, $\boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a}$ goes to a constant, in all directions. Therefore, we can conclude that its closed surface integral, in the limiting process, will be finite, that is, *their total currents converge to a non-zero constant*. This is also a condition of sufficiency.

Nevertheless, if the divergenceless current distributions are globally localized, but their Cartesian components go to zero *slower* than $1/r^3$, for any fixed direction in the space, then, nothing can be said, in general, about the convergence of the total current, without assuming additional hypothesis about the details of the distribution. Indeed, in these cases, we know that the vector integrand $\boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a}$ diverges in all directions, but we cannot conclude from that, in general, that its integral over a closed surface located at infinity will necessarily diverge.

Although we cannot conclude anything when $\boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a}$ diverges, what we can do is to establish that a *sufficient condition* to the total current *to possibly converge or to diverge slower than* r^2 is that divergenceless current distributions are *globally localized in the semi-strong sense*.

Indeed, given the Cartesian components $J_\mu(r,\theta,\varphi)$ of the current density, we have, asymptotically – that is, from $r > a$, where a is a sufficiently large positive constant:

$$\begin{aligned}&\lim_{r\to\infty} |J_\mu(r,\theta,\varphi)|r = 0 \\ &\quad \Rightarrow |J_\mu(r,\theta,\varphi)| < r^{-1} \Rightarrow |J_\mu(r,\theta,\varphi)|r^3 < r^2\end{aligned}$$

$$\Rightarrow \int d\Omega |J_\mu(r,\theta,\varphi)| r^3 < \int d\Omega r^2$$

$$\Rightarrow \oint_{S\to\infty} \boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{a} < 4\pi r^2 |_a^{r\to\infty} \hat{\boldsymbol{r}}. \tag{17}$$

Obviously, according to the results in Appendix A, if the semi-strong global localization is a sufficient condition, then the *strictly semi-strong global localization is also sufficient.* This result will prove useful in section 4.

3. Coulomb and Biot-Savart Improper Integrals as Formal Solutions and the Localization Conditions to Charge and Current Distributions

3.1. Coulomb and Biot-Savart laws as formal solutions

Let's turn to the problem related to the *solutions* of the field equations (1) and (2). In this respect, it can be shown that their *formal solutions* are provided by the so-called *Coulomb and Biot-Savart laws*, respectively, expressed by the following *improper integrals* performed over the entire Euclidean space:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'; \tag{18}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{V\to\mathbf{R}^3} \boldsymbol{J}(\boldsymbol{r}') \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'. \tag{19}$$

We should now emphasize the fact that, mathematically, Coulomb and Biot-Savart laws are, in general, *limiting processes*, because we are assuming the possibility that the charge and current distributions are infinitely extended.

The verification of what we call the *formal validity* of these laws is done by substituting the integral expressions (18) and (19) respectively into the differential equations (1) and (2). As far as Coulomb law is concerned, satisfying formally the local equations (1) does not require any conditions on the localization of the charge distributions.

However, regarding the Biot-Savart law, for the local equations (2) to be formally satisfied – more specifically, Ampère law – a necessary condition is required on the following integral, involving the current densities, on a surface located at infinity [7, p. 233]:

$$\oint_{S\to\infty} \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \boldsymbol{J}(\boldsymbol{r}') \cdot d\boldsymbol{a}' \to 0. \tag{20}$$

Eq. (20) provides the second example of a *necessary condition* expressed by a *closed surface integral located at infinity.* According to the discussion regarding the total flux of currents (4), we already know that eq. (20) cannot impose any specific *necessary* condition on the current densities. On the other hand, if all that was required was its satisfaction, then, *weak global localization* would already be *sufficient.* This result can be demonstrated as follows.

First, note that it is sufficient to satisfy (20) that, for any fixed direction in the space, $\lim_{r'\to\infty} \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \boldsymbol{J}(\boldsymbol{r}') \cdot d\boldsymbol{a}' = 0$. The asymptotic behavior of the geometric factor $\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ is such that, for any fixed direction in the space, $\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(r'\to\infty,\theta',\varphi') \sim \frac{\boldsymbol{r}'}{r'^3} = \frac{1}{r'^2}\hat{\boldsymbol{r}}$. We already know that $\boldsymbol{J}(\boldsymbol{r}') \cdot d\boldsymbol{a}' \sim \sum_\mu J_\mu(r',\theta',\varphi') {r'}^2$, where J_μ denotes the Cartesian components of the current density.

Therefore, the asymptotic behavior of the integrand of the surface integral (20) is given by $\frac{1}{r'^2}\sum_\mu J_\mu(r',\theta',\varphi'){r'}^2$, from which we have $\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(\boldsymbol{r}') \cdot d\boldsymbol{a}' \sim \sum_\mu J_\mu(r',\theta',\varphi')\hat{\boldsymbol{r}}$.

Therefore, it is sufficient to satisfy the condition (20) that, for any fixed direction in space, $\lim_{r'\to\infty} |J_\mu(r',\theta',\varphi')| = 0$, from which we conclude that weak global localization is a sufficient condition.

It may seem particularly strange that a necessary asymptotic condition appears only in the case of formal satisfaction of Ampère law by Biot-Savart law. After all, both Coulomb and Biot-Savart laws exhibit the same behavior with regard to the geometric factor present in their integrands. However, from the point of view of formal satisfaction, the vector content of the integrands matters. The surface term (20) that appears in the specific case of Biot-Savart law is due to the fact that its integrand consists of a vector product.

In any case, as we will see next, the weak global localization of the current densities is still not sufficient to guarantee that the improper integral (19) is convergent – just as the *absence* of any conditions on the localization of the charge distributions does not guarantee that the improper integral (18) is convergent too. This shows that the mere formal satisfaction of the field equations is not a criterion for what we will call the *good definition* of Coulomb and Biot-Savart laws.

3.2. The good definitions of the Coulomb and Biot-Savart laws

Let's move to the problem of the *good definition* of Coulomb and Biot-Savart laws. We said that these laws have merely *formal* validity because the electrostatic and magnetostatic fields are not necessarily mathematically well defined by the expressions (18) and (19) if those improper integrals do not provide finite results, in the limiting process, for all points of space located at a finite distance. We will, therefore, say that *the Coulomb and Biot-Savart laws are well defined precisely if, and only if, the improper integrals that represent them are convergent in the limiting process.*

In addition to the requirement that charge densities be bounded everywhere and (at least piecewise) continuous, an inspection of the form of the improper integral of Coulomb law *suggests* that, in general, *it is a sufficient* *condition for its good definition that the charge densities are globally localized, in the strictly semi-strong sense.*

This result can be demonstrated by following a procedure analogous to the proof that was developed to obtain a sufficient condition for the finiteness of the total charge of a distribution (eq. (5)). To prove it, we should demonstrate that:

$$\lim_{r'\to\infty} |\rho(\boldsymbol{r}')| {r'}^{1+\varepsilon} = 0 \Rightarrow \int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' < \infty; \tag{21}$$

for *any* positive constant $\varepsilon > 0$, that can be chosen as small as we want.

In the Appendix B, we demonstrate, under convenient but mild suppositions, that the convergence of the volume integral of Coulomb law depends on the convergence of the improper integrals $g(\theta', \varphi'; a) = \int_a^{r'\to\infty} \rho(r', \theta', \varphi') dr'$, where $a > 0$ is a sufficiently large positive constant, provided that the resulting functions be sufficiently well behaved in the interval of definition of the angular variables.

In Appendix A, throughout eqs. (A.1–A.4), we demonstrated that, if $f(r')$ is a bounded function throughout the interval $0 \le r' < \infty$, it is asymptotically continuous and positive and decays to zero, at infinite, then, the implication $\lim_{r'\to\infty} f(r'){r'}^{1+\varepsilon} = 0 \Rightarrow \int_a^{r'\to\infty} f(r')dr' < \infty$ is true, for *any* positive constant $\varepsilon > 0$, where $a > 0$ is a sufficiently large positive constant (eq. A.4).

Now, if, for, any fixed direction in space, we choose $f(r') \sim |\rho(r', \theta', \varphi')|$, then:

$$\begin{aligned} &\lim_{r'\to\infty} |\rho(r', \theta', \varphi')| {r'}^{1+\varepsilon} = 0 \\ &\quad \Rightarrow \int_a^{r'\to\infty} |\rho(r', \theta', \varphi')| dr' < \infty \\ &\quad \Rightarrow \int_a^{r'\to\infty} \rho(r', \theta', \varphi') dr' < \infty; \end{aligned} \tag{22}$$

for *any* positive constant $\varepsilon > 0$, where $a > 0$ is a sufficiently large positive constant[8].

Finally, we can conclude, following eq. (B.11) of Appendix B, that the improper integral $\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'$ converges and, therefore, *the Coulomb law is well-defined, provided the charge density is bounded everywhere, is asymptotically continuous, and represents a globally localized distribution, in the strictly semi-strong sense.*

Now, it is not so surprising that global localization, in the strictly semi-strong sense, cannot be a *necessary* condition for the good definition of Coulomb law, because *is not true* that $\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' < \infty \Rightarrow \lim_{r'\to\infty} |\rho(\boldsymbol{r}')| {r'}^{1+\varepsilon} = 0$, in general, even if the distributions are globally localized.

[8] Again, if a function $f(r)$ is absolutely integrable on the interval $[a, \infty)$, then it is integrable on this interval.

An analogous inspection of the Biot-Savart law also shows that, for current distributions given by *general* vector fields $\boldsymbol{J}(\boldsymbol{r}')$ – that is, not necessarily divergenceless –, in addition to satisfying the requirements of boundness and (at least piecewise) continuity, *it is also a sufficient, but not necessary, condition for its good definition that the Cartesian components of the current densities are globally localized in the strictly semi-strong sense.*

By the way, even though current distributions must be divergenceless in magnetostatics, this constraint has nothing to do with the good definition of Biot-Savart law, because it is not "embedded" within the improper integral (19). Of course, the divergenceless of the current densities is implicated by Ampère law, and this requirement must always accompany Biot-Savart law.

As before, the convergence of integral (19) is completely determined by the asymptotic behavior of $|\boldsymbol{J}(\boldsymbol{r}')|$, so we must investigate the behavior of the improper integral in the "tail" of the distribution, that is, investigate $\int d\Omega' \int_a^{r'\to\infty} |\boldsymbol{J}(r', \theta', \varphi')| \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} {r'}^2 dr'$. The reasoning to be developed here is exactly the same as that developed above, which is why it is unnecessary to repeat it.

Note, however, that the presence of the vector product in the integrand would require us to be a little more careful. It is always possible that the Cartesian components of vector products contain differences between two terms that can behave asymptotically in the same way. If this is the case, to deal with these differences it is important to extract common factors that allow the differences to remain finite in the limiting process, allowing asymptotic behavior to be deduced from the behavior of the extracted factors.

Fortunately, there is no reason to be concerned unless the vector products involve purely geometric factors which may be geometrically correlated. In the Biot-Savart law, the arbitrariness of the vector field $\boldsymbol{J}(\boldsymbol{r}')$ excludes, in general, the possibility of correlation and, therefore, excludes the appearance of differences that behave asymptotically in an identical way.

It is important to emphasize once that, *although the strictly semi-strong condition of global localization is sufficient for the good definition of the improper integral expressions* (18) *and* (19), *it is not a necessary condition at all.* The fact is that, in the presence of *symmetries*, it can be proved that some charge or current distributions that are weakly localized, partially localized or even nonlocalized at all can be also sufficient for the integrals of the Coulomb or Biot-Savart laws to converge.

Indeed, in a variety of configurations where symmetries appear, it is always feasible to attain a certain degree of "relaxation" of the strictly semi-strong global localization conditions. However, the nature or the extent of these relaxations are contingent upon the constraints imposed on the asymptotic forms of the fields, as will be briefly illustrated, in the section 4, in the case of spherically symmetric charge distributions.

In certain instances, relaxation can be achieved on the *global* localization requirement itself. Indeed, distributions exhibiting cylindrical or rectangular symmetries can be only partially localized, yet the electrostatic and magnetostatic fields are well defined by their improper integrals.

In other cases, relaxation can be achieved on the *strictly semi-strong* localization requirement. For instance, charge distributions exhibiting *spherical symmetry* can be *weakly* localized, or even *non-localized at all*, yet the electrostatic fields are well defined by Coulomb law.

3.3. The spherically symmetric charge distribution as an example

To illustrate this latter case, consider the following example of a class of spherically symmetric charge densities:

$$\rho(\boldsymbol{r}) = \rho(r) = \begin{cases} \rho_{cte} & (r \leq R); \\ \rho_{cte} f(r) & (r \geq R). \end{cases} \tag{23}$$

In this expression, ρ_{cte} is a constant, which is designed to ensure that the charge density is finite at the origin ($r = 0$). The function $f(r)$ is an arbitrary function, but it is required to be bounded, continuous, and subject to the condition $f(R) = 1$. This last condition is designed to ensure the continuity of the charge density at all points in the Euclidean space[9].

Substituting the charge distribution (23) into Coulomb law (18), followed by integration of the angular variables, yields the following result for $r > R$:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_{cte}}{\varepsilon_0} \frac{1}{r^2} \left\{ \frac{R^3}{3} + g(r;R) + \frac{1}{2} r^2 s(r;\infty) \right\} \hat{\boldsymbol{r}}; \quad r > R; \tag{24}$$

where:

$$\begin{aligned} g(r;R) &= \int_R^r f(r') {r'}^2 dr'; \\ s(r;\infty) &= \int_r^\infty f(r') {r'}^2 k(r,r') dr'; \quad r > R. \end{aligned} \tag{25}$$

In eqs. (24–25), $g(r;R)$ is always a finite function if $r < \infty$, *regardless of the functional form of the function* $f(r)$, since it satisfies the conditions given above. The really noteworthy fact, however, is that *the function* k *is identically equal to zero*: $k(r,r') \equiv 0$, as can be easily verified. Thus, it follows that $s(r;\infty) \equiv 0$.

Therefore, the electrostatic fields corresponding to any spherically symmetric charge distributions given by class (23), are perfectly well defined by:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_{cte}}{\varepsilon_0} \frac{1}{r^2} \left\{ \frac{R^3}{3} + g(r;R) \right\} \hat{\boldsymbol{r}}; \quad r > R. \tag{26}$$

It can be seen that this solution is compatible with the field equations, as well as with the stipulation made about the functional form of the charge density. Therefore, Coulomb law is a well-defined representation of the solution to the field equations, regardless of the specific functional form of the spherically symmetrical charge distribution, even when it is not even localized[10].

This example illustrates the fact that, in general, there are no *necessary conditions* that must be imposed on the localization of the charge distributions for Coulomb law to constitute a well-defined representation of the solutions to the field equations in electrostatics[11]. This conclusion naturally extends to the case of the Biot-Savart law in magnetostatics.

Finally, it is also worth emphasizing that the good definition of the improper integrals that represent Coulomb and Biot-Savart laws does not depend on the fact that these integrals can actually be carried out, in the sense that the fields are represented in terms of *elementary functions*.

On the other hand, if the fields can be obtained, by some other method, in terms of elementary functions, then the Coulomb and Biot-Savart laws must necessarily be able to be effectively integrated.

4. Uniqueness Conditions on the Electrostatic and Magnetostatic Fields

4.1. Regularity conditions

We should now address a final issue: in the absence of additional conditions, *the Coulomb and Biot-Savart laws do not necessarily provide unique solutions to the field equations*. In fact, due to the linearity of the field equations, to the expressions (18) and (19) can be added any arbitrary fields that have zero divergence and zero curl at all points in space.

The uniqueness of the solutions of the field equations is an essential requirement from a physical point of view, since it is the fields that are responsible for the interaction forces between bodies made up of electrically charged matter, in accordance with the fundamental principle established at the beginning of our discussion, in section 2. However, the existence and uniqueness of solutions of partial differential equations depend crucially on the boundary conditions imposed on them.

At this point, let's remember that we are only interested in global solutions of the field equations in the absence of polarizable, magnetizable and, above all,

[9] These requirements are not really necessary, as the charge density could be defined to be piecewise continuous.

[10] The reason for that is a known result from Gauss law: whichever way the electric charge is radially distributed in space, if this distribution is spherically symmetric, the contribution to the field at points $r < R$ from every (possibly infinite) quantity of charge in the spherical shells beyond the radius R of a sphere centered on the origin must be precisely zero.

[11] In addition, obviously, to the necessary condition represented by the convergence of the improper integral itself.

conductive matter – and, therefore, in the absence of surface charge densities formed on equipotential surfaces. For this reason, existence and uniqueness must require boundary conditions with respect to the behavior of the electrostatic and magnetostatic fields on surfaces located at infinity.

Before we proceed any further, it is important to establish the following result. Let $\boldsymbol{F}(\boldsymbol{r})$ be a bounded field, continuous and differentiable everywhere, and that satisfies the following conditions at all points of the Euclidean space: (i) $\nabla \cdot \boldsymbol{F}(\boldsymbol{r}) = 0$; (ii) $\nabla \times \boldsymbol{F}(\boldsymbol{r}) = 0$; (iii) $|\boldsymbol{F}(\boldsymbol{r})| \to 0$, when $r \to \infty$. Then, $\boldsymbol{F}(\boldsymbol{r})$ must be an identically null field.

To reach this conclusion, note that, according to (ii), $\nabla \times (\nabla \times \boldsymbol{F}(\boldsymbol{r})) = 0$ and, using the vector identity that allows to rewrite the double curl, we have $\nabla(\nabla \cdot \boldsymbol{F}(\boldsymbol{r})) - \nabla^2 \boldsymbol{F}(\boldsymbol{r}) = 0$. By condition (i), we conclude that $\nabla^2 \boldsymbol{F}(\boldsymbol{r}) = 0$. Therefore, the Cartesian components of $\boldsymbol{F}(\boldsymbol{r})$ satisfy Laplace's equation at all points in Euclidean space, that is, they are *harmonic functions*. By *Liouville's theorem* [31], harmonic functions that are divergenceless and curless are necessarily constant. So, given the asymptotic condition (iii), $\boldsymbol{F}(\boldsymbol{r})$ must be an identically zero field[12].

In view of this result, it is easy to conclude that sufficient boundary conditions to guarantee the uniqueness of the solutions to the field equations are such that the electrostatic and magnetostatic fields asymptotically vanish, that is, for any fixed direction in the space, $\boldsymbol{E}(r \to \infty, \theta, \varphi) \to 0$ and $\boldsymbol{B}(r \to \infty, \theta, \varphi) \to 0$, *regardless the way they approach zero*. However, we can see that *these conditions are not the most general ones*.

Indeed, suppose the boundary condition is set so that the field takes on a *constant magnitude (zero or non-zero)* at infinity, and a solution to the field equations has been found that satisfies the boundary condition. This solution is unique. In fact, suppose that there is another solution that satisfies the field equations and the fixed (zero or non-zero) constant boundary condition. The difference between the two solutions must be a field which, due to the linearity of the field equations and the boundary condition, has zero divergence and zero curl at all points in space and that also vanish asymptotically. Since only a null field satisfies these conditions, then the second solution is necessarily equal to the first.

Among other examples of configurations whose fields satisfy non-zero asymptotic boundary conditions, we can take the cases for certain *partially localized* charge (or current) distributions that exhibit *rectangular* symmetry. Given a system of Cartesian coordinates, we can consider, as an example, homogeneous and uniform charge (or current) distributions that are infinitely extended in the x and y directions, with precisely the $x - y$ plane as the plane of symmetry, and that are zero when $|z| > h > 0$, where $2h$ is the thickness of the infinite rectangular volume.

[12] See also the proof given by Milton and Schwinger [14, p. 7].

The solutions for this case are well known and can even be easily obtained by integrating Coulomb and Biot-Savart laws. They consist of electrostatic and magnetostatic fields that are uniform, in the empty regions of space, whose asymptotic values at $z \to \pm\infty$ are obviously not zero. These solutions are nevertheless unique, since only a null field could be added to them without violating the field equations and the asymptotic boundary condition.

Weakly localized spherically symmetric distributions of charge can also satisfy constant, non-zero boundary conditions without violating the uniqueness of the solutions, as will be seen below.

From this brief discussion, we conclude that the *necessary* conditions for the *uniqueness* of solutions to field equations consist of imposing boundary conditions such that the fields are *asymptotically finite*, that is, that the fields satisfy the so-called *regularity conditions* [20, p. 168].

In fact, suppose that a boundary condition is set so that the fields diverge asymptotically and solutions have been found that satisfy this condition. These solutions cannot be unique, since any uniform fields that are added to them would provide other solutions that would also satisfy the field equations and the same (infinite) boundary condition.

Having found the asymptotic conditions that must necessarily be satisfied by the fields, we could investigate the kinds of restrictions they possibly impose on the localization of the charge and current distributions.

4.2. Regularity conditions: an illustration with spherically symmetric charge distributions

For example, let's return to the spherically symmetric charge distributions given by eq. (23). It can be seen from a simple inspection of its general solution, given by eq. (26), that for the radial component of the electrostatic field to be asymptotically zero, the asymptotic behavior of the function $g(r; R)$, given in eq. (25), must be such that this function goes to zero at infinity or, at most, diverges slower than r^2.

It is easy to prove that, in this particular case of spherically symmetric distributions, any globally localized distribution in the *semi-strong sense* is already sufficient to satisfy the regularity condition given by $\boldsymbol{E}(r \to \infty) \to 0$. Indeed, according to eq. (A.11), from Appendix A, for $r > 0$ and $\varepsilon = 0$, with $a > 0$ a sufficiently large positive constant, we have:

$$\begin{aligned} &\lim_{r\to\infty} |f(r)| r = 0 \\ &\quad \Rightarrow |f(r)| < r^{-1} \Rightarrow |f(r)| r^2 < r \\ &\quad \Rightarrow \int_a^{r\to\infty} |f(r)| r^2 dr < \int_a^{r\to\infty} r\, dr \sim r^2 \big|_a^{r\to\infty} \\ &\quad \Rightarrow g(r; a) < r^2 \big|_a^{r\to\infty}. \end{aligned} \tag{27}$$

For example, suppose we require $g(r \to \infty; R) \sim r$. To do so, we should impose that $f(r \to \infty) \sim r^{-2}$. The electrostatic field generated by the charge distribution (23), with $f(r) = \frac{R^2}{r^2}$, is given, from eq. (26), by:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_{cte}}{\varepsilon_0}\left\{-\frac{2R^3}{3}\frac{1}{r^2} + R^2\frac{1}{r}\right\}\hat{\boldsymbol{r}}; \quad r > R. \tag{28}$$

If, on the other hand, we impose the regularity condition given by $\boldsymbol{E}(r \to \infty) \to a \neq 0$, it turns out that the *constant* a must be precisely $a = \frac{\rho_{cte}}{\epsilon_0}\frac{R}{2}$ and that $f(r \to \infty) \sim r^{-1}$, which corresponds to the minimum necessary for the distribution to be classified as *weakly localized*. The solution will be given, from eq. (26), by:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_{cte}}{\varepsilon_0}\left\{-\frac{R^3}{6}\frac{1}{r^2} + \frac{R}{2}\right\}\hat{\boldsymbol{r}}; \quad r > R. \tag{29}$$

The weak case with $f(r \to \infty) \sim r^{-1}$ is a threshold one. Any spherically symmetric charge distribution that is weaklier localized will implicate an asymptotically divergent electrostatic field, violating the uniqueness requirement.

4.3. Compatibility between regularity conditions and localization conditions in electrostatics

The various ways in which the conditions of regularity over the fields impose general conditions on the localizability of charge distribution and current distribution are a major issue. To address this problem in a completely general way, it would be necessary to return to field equations (1) and (2) to investigate the asymptotic relationships they establish respectively between charge and current densities and the derivatives of the electrostatic and magnetostatic field components. In principle, once we know how the fields behave asymptotically, we can infer how their derivatives behave and, therefore, what the behaviors of the densities should be.

However, we will not pursue this general goal in this work. That would require its own investigation, that would be too detailed and would divert us to much from our fundamental purpose, namely, to deduce and to obtain comprehensive sufficient conditions for the validity of Coulomb and Biot-Savart laws.

What we can do easily, however, is ask ourselves whether it is possible to find conditions regarding the localization of charge and current distributions that are at least *compatible* with the conditions of regularity over the electrostatic and magnetostatic fields – without, however, establishing, in general, conditions of sufficiency or necessity.

To this end, in the case of electrostatics, instead of trying to analyze the differential equations (1) directly, it is more illuminating to analyze the problem from the perspective of the asymptotic behavior of the boundary integral law that involves the charge distribution, namely, the Gauss law:

$$\oint_{S\to\infty} \boldsymbol{E}(\boldsymbol{r}) \cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0}\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r})d\tau. \tag{30}$$

Naturally, an evaluation of the asymptotic behavior of the closed surface integral in eq. (30) – the total flux of the electrostatic field – involves an evaluation limited to the radial component of the electrostatic field. Thus, we must keep in mind that the total flux depends on the asymptotic behavior of $\boldsymbol{E}(\boldsymbol{r}) \cdot d\boldsymbol{a}$, given by $E_r(r,\theta,\varphi)r^2$, or else $\sum_\mu E_\mu(r,\theta,\varphi)r^2$.

We already know that, according to the analysis of section 2, following eq. (5), if the distributions are globally localized, a sufficient condition to the convergence of the right-hand side of eq. (30) – the total charge – is strictly super-strong global localization, given by eq. (6).

We also know that sufficient conditions for the total charge to be finite or to diverge slower than r^2 is that the distributions are globally localized in the semi-strong or in the strictly semi-strong sense, according to the eq. (9).

Finally, a sufficient condition to the divergence of the total charge as slow as r^2 is that the distributions are globally localized in the weak sense and are asymptotically spherically symmetric, according to the eq. (10).

Based on these results, we can distinguish four interesting cases.

(i) If, for any fixed direction in space, the electrostatic field behaves asymptotically such that the magnitudes of its Cartesian components go to zero faster than $1/r^2$, that is, $\lim_{r\to\infty} |E_\mu(r,\theta,\varphi)|r^2 = 0$, then, the total flux of the electrostatic field goes to zero. To see this, we can resort to the discussion of the total flux of currents, since the behavior of the total flux of a vector field is independent of its nature. From Gauss law, we conclude that the total charge of the distribution must go to zero either. By the way, this is the same as saying that the term of monopole of the entire distribution is zero.

According to the discussion about the finiteness of the total charge, this is (at least) possible if the positive and the negative contributions of *infinitely extended* charge distributions cancel each other out asymptotically faster than $1/r^{3+\varepsilon}$, for some $\varepsilon > 0$, or even, if they cancel each other in *absolutely localized* distributions.

If the distributions are absolutely localized, with zero total charge, it is safe to say that this is sufficient to satisfy the above specified regularity condition on the electrostatic fields.

However, if the distributions are *infinitely extended*, the most that can be said is that *the regularity condition* given by *the electrostatic field going to zero faster than* $1/r^2$ and *the condition of strictly super-strong global localization – whenever their total charges are equal to zero* – are *compatible with each other*.

Therefore, only a compatibility relationship between regularity and localization can be safely established. This is so because Gauss law – without the requirement that the curl of the electrostatic fields must be identically zero and without the imposition of additional conditions on the functional forms of the charge densities – is unable to establish relations of necessity or sufficiency between the asymptotic behaviors of the fields and the charge distributions.

(ii) Similarly, if the electrostatic field behaves asymptotically like $|E_\mu(r,\theta,\varphi)| \sim r^{-2}$ and is asymptotically symmetric, then, the total flux of the field goes to a non-zero constant. So, the total charge of the distribution – that is, the term of monopole of the entire distribution – must go to a non-zero constant too.

Again, if the distributions are absolutely localized, with total charge different from zero, it is safe to say that this is sufficient to satisfy the given regularity condition.

The case of absolute localization can be clearly observed, for example, from the class of spherically symmetric distributions defined by eq. (23). For the electrostatic field, given by eq. (26), to behave asymptotically as r^{-2}, it is sufficient that the integral $g(r;R)$, given in eq. (25), is identically null. However, this is only possible if the function $f(r)$ is zero, which means that the class of distributions (23) would be in fact finitely extended.

On the other hand, if the distributions are infinitely extended, the most we can say from Gauss law alone is that *the condition of regularity with asymptotically symmetric electrostatic fields going to zero like* $1/r^2$ *is compatible with the condition of global localization in a strictly super-strong sense – whenever their total charges are constants different from zero.*

(iii) If, for any fixed direction in space, the electrostatic field behaves asymptotically such that the magnitudes of its Cartesian components go to zero slower than $1/r^2$, that is, $\lim\limits_{r\to\infty}|E_\mu(r,\theta,\varphi)|r^2 \to \infty$, in an asymptotically symmetric manner, then, the total flux of the electrostatic field diverges. Thus, we should conclude that the total charge diverges either.

Nevertheless, although the total flux diverges, it cannot diverge faster than r^2, given the regularity condition $\lim\limits_{r\to\infty}|E_\mu(r,\theta,\varphi)| = 0$ (or a $cte \neq 0$). We should conclude that the total charge diverges but not faster than r^2.

In this case, the most we can say, as before, is that *the condition of regularity with asymptotically symmetric electrostatic fields going to zero slower than* $1/r^2$ *is compatible with the condition of global localization in a semi-strong – or even strictly semi-strong sense.*

(iv) Finally, it remains to be investigated what happens when the electrostatic field goes asymptotically towards a *constant*, in an asymptotically symmetric manner. In this case, the total flux of the field and, consequently, the total charge of the infinitely extended distribution must necessarily diverge quadratically. This is compatible with the asymptotic symmetric global localization in the *weak sense*, with $|\rho(r\to\infty,\theta,\varphi)| \sim r^{-1}$. The globally spherically symmetric example is given by eq. (29).

4.4. Compatibility between regularity conditions and localization conditions in magnetostatics

We can move to the case of magnetostatics. Here, it is also more illuminating to analyze the problem from the perspective of the asymptotic behavior of the volume integral of the Ampère law, establishing a boundary integral law that could be called the "Gauss-Ampère law":

$$\int_{V\to\mathbf{R}^3} \nabla \times \boldsymbol{B} d\tau = \mu_0 \int_{V\to\mathbf{R}^3} \boldsymbol{J}(\boldsymbol{r}) d\tau$$
$$\Rightarrow \oint_{S\to\infty} d\boldsymbol{a} \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \int_{V\to\mathbf{R}^3} \boldsymbol{J}(\boldsymbol{r}) d\tau. \quad (31)$$

To obtain eq. (31), we used the vector identity that allows us to express the volume integral of the curl of a vector field as a closed surface integral of this vector field. Note that, in this way, we have made the total current of the distribution appear on the right-hand side of eq. (31).

As in the previous analysis, only compatibilities between regularity and localization can be safely established. This is so because Gauss-Ampère law – without the requirement that the divergence of the magnetostatic fields must be identically zero and without the imposition of additional conditions on the functional forms of the current densities – is unable to establish relations of necessity or sufficiency between the asymptotic behaviors of the fields and the current distributions.

Naturally, an evaluation of the asymptotic behavior of the closed surface integral in eq. (31) involves an evaluation limited to the transverse components of the magnetostatic field. However, we are interested in the Cartesian components of the surface integral. We know that, while the azimuthal component of the magnetostatic field involves only its x and y Cartesian components, the polar component involves all the Cartesian components of $\boldsymbol{B}(\boldsymbol{r})$. The expansion of the vector product $d\boldsymbol{a} \times \boldsymbol{B}(\boldsymbol{r})$ shows that all the Cartesian components of the integrand of the surface integral behave like$B_\mu(r,\theta,\varphi)r^2$. This means that the analysis of eq. (31) can be carried out analogously to the previous electrostatic case.

Again, due to the presence of the vector product, care should be taken when assessing the asymptotic behavior of the surface integral in eq. (31). But we already know that the arbitrariness of functional form of the vector field $\boldsymbol{B}(\boldsymbol{r})$ excludes, in general, the possibility of correlation with the geometric factor $d\boldsymbol{a}$. Thus, in general, there is no reason to be concerned.

As before, we can distinguish four interesting cases.

(i) If, for any fixed direction in space, the magnetostatic field behaves asymptotically such that the magnitudes of its Cartesian components go to zero faster than $1/r^2$, that is, such that $\lim\limits_{r\to\infty}|B_\mu(r,\theta,\varphi)|r^2 = 0$, then, all the components of $\oint_{S\to\infty} d\boldsymbol{a} \times \boldsymbol{B}(\boldsymbol{r})$ go to zero. Thus, we conclude that all the components of the total current must go to zero either.

Distributions *absolutely localized* have zero total current necessarily. Thus, it is safe to say that absolute localization is sufficient to satisfy the given regularity condition. By the way, we see that this is consistent with the fact that the monopole terms of absolutely localized current distributions are identically zero.

However, if the distributions are *infinitely extended*, the most that can be said is that *the regularity condition* given by *the magnetostatic field going to zero faster than* $1/r^2$ and *the condition of super-strong global localization – sufficient for the total currents to be zero* – are *compatible with each other*, according to the discussion following eq. (16).

(ii) If, for any fixed direction in space, the magnetostatic field behaves asymptotically like $|B_\mu(r,\theta,\varphi)| \sim r^{-2}$, in an asymptotically symmetric manner, then, we see that all the components of $\oint_{S\to\infty} d\boldsymbol{a} \times \boldsymbol{B}(\boldsymbol{r})$ goes to a non-zero constant. So, all the components of the total current of the distribution must go to a non-zero constant too.

No absolutely localized distribution can satisfy this condition. And, if the distributions are infinitely extended, the most that can be said is that *the condition of regularity with the magnetostatic field going to zero like* $1/r^2$, *in an asymptotically symmetric manner, is compatible with the condition of global localization with the Cartesian components of the current density going to zero like* $1/r^3$.

(iii) If, for any fixed direction in space, the magnetostatic field behaves asymptotically such that the magnitudes of its Cartesian components go to zero slower than $1/r^2$, such that $\lim_{r\to\infty} |B_\mu(r,\theta,\varphi)| r^2 \to \infty$, in an asymptotically symmetric manner, then, all the components of $\oint_{S\to\infty} d\boldsymbol{a} \times \boldsymbol{B}(\boldsymbol{r})$ diverge. Thus, we should conclude that all the components of the total current diverge either.

Nevertheless, the components of $\oint_{S\to\infty} d\boldsymbol{a} \times \boldsymbol{B}(\boldsymbol{r})$ cannot diverge faster than r^2, given the regularity condition $\lim_{r\to\infty} |B_\mu(r,\theta,\varphi)| = 0$ (or a $cte \neq 0$). We should conclude that the total current diverges but not faster than r^2.

In this case, the most we can say is that *the condition of regularity with the magnetostatic field going to zero slower than* $1/r^2$, *in an asymptotically symmetric manner, is compatible with the condition of global localization in a semi-strong sense – or even strictly semi-strong sense* –, according to the discussion involving eq. (17).

(iv) Finally, it remains to be investigated what happens when the magnetostatic field goes asymptotically towards a constant, in an asymptotically symmetric manner. In this case, all the components of $\oint_{S\to\infty} d\boldsymbol{a} \times \boldsymbol{B}(r)$ and, consequently, the components of the total current of the infinitely extended distribution, must necessarily diverge quadratically. This is compatible with asymptotic symmetric global localization in the *weak sense*, with $|J_\mu(r \to \infty, \theta, \varphi)| \sim r^{-1}$.

5. Helmholtz Theorem: Sufficient Conditions for Coulomb and Biot-Savart Laws

We have already seen that global localization of charge and current distributions, in the *strictly semi-strong* sense, furnish sufficient conditions for the good definition of Coulomb and Biot-Savart laws. This conclusion was established without considering how these laws can be deduced from the field equations.

The fact is, however, that the standard deduction procedure, based on *Helmholtz theorem*, seems to suggest that strictly semi-strong localizations *are not* sufficient. On the contrary, it leads one to believe that the sufficient conditions must be stricter, namely global localizations in the *strictly strong sense*. Let's see why this is the case.

In simplified terms, Helmholtz theorem states that every vector field $\boldsymbol{F}(\boldsymbol{r})$ that is asymptotically zero can be uniquely determined by its divergence and its curl, as follows:

$$\boldsymbol{F}(\boldsymbol{r}) = -\nabla U(\boldsymbol{r}) + \nabla \times \boldsymbol{W}(\boldsymbol{r}); \tag{32}$$

provided that the scalar function $U(\boldsymbol{r})$ and the vector function $\boldsymbol{W}(\boldsymbol{r})$ are well defined by the following improper integrals:

$$\begin{aligned} U(\boldsymbol{r}) &= \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{\nabla \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d\tau'; \\ \boldsymbol{W}(\boldsymbol{r}) &= \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{\nabla \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d\tau'. \end{aligned} \tag{33}$$

Applied in the context of electromagnetostatics, Helmholtz theorem guarantees that the electrostatic and magnetostatic fields can be obtained, respectively, as derivatives of the scalar potential, $V(\boldsymbol{r})$, and of the vector potential, $\boldsymbol{A}(\boldsymbol{r})$, as follows[13]:

$$\begin{aligned} &\boldsymbol{E}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r}) \quad \text{(i)}; \\ &V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathbf{R}^3} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d\tau' \quad \text{(ii)}; \end{aligned} \tag{34}$$

$$\begin{aligned} &\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) \quad \text{(i)}; \\ &\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\mathbf{R}^3} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d\tau' \quad \text{(ii)}. \end{aligned} \tag{35}$$

By *formally* calculating the derivatives of the potentials given by their respective improper integrals, we obtain, as a corollary, that the (unique) solutions to the field equations are given precisely by Coulomb and Biot-Savart laws.

[13] The fact that Euclidean space is simply connected is sufficient to ensure that the electrostatic and the magnetostatic field can be respectively represented by global and single-valued scalar and vector potentials at all points in space, although to be simply connected is not a strictly necessary condition to that.

The problem now becomes evident, because the electrostatic and magnetostatic fields are only defined if the scalar and vector potentials are *well-defined*, that is, if the improper integrals that represent them converge. But for this to happen, *strictly semi-strong global localizations are not sufficient.*

Indeed, if we repeat, for the improper integrals that define the potentials, the same type of reasoning that allowed us to determine sufficient conditions for the convergence of the previous volume integrals – namely, eqs. (5), (18) and (19) – the only thing we are allowed to conclude, in terms of conditions of sufficiency, is that, if the distributions are globally localized, then, they must be so in a strictly *strong* sense.

By way of illustration, let's return once again to the example of spherically symmetric charge distributions, which we studied in the previous sections. Given the electrostatic potential (eq. 34.ii) and the charge distribution (eq. 23), the integration of the angular variables gives, for $r > R$:

$$V(r) = \frac{\rho_{cte}}{\varepsilon_0}\frac{1}{r}\left\{\frac{R^3}{3} + g(r;R) + rh(r;\infty)\right\}; \quad r > R; \tag{36}$$

where:

$$\begin{aligned} g(r;R) &= \int_R^r f(r')r'^2 dr'; \\ h(r;\infty) &= \int_r^\infty f(r')r'dr'; \quad r > R. \end{aligned} \tag{37}$$

Note that, from the point of view of the localization conditions, the situation involving the good definition of the electrostatic potential (eqs. 36–37) is different from the situation involving the good definition of Coulomb law (eqs. 24–25).

In fact, if, for simplicity, we restrict ourselves to inverse power laws, the convergence of the last integration, given by $h(r;\infty)$, is not guaranteed if we only require that $f(r')$ decays faster than $r^{-1+\varepsilon}$, with $\varepsilon > 0$ being an arbitrary positive constant. On the other hand, convergence is guaranteed if we require that it decays faster than $r^{-2+\varepsilon}$, with $\varepsilon > 0$ being an arbitrary positive constant – that can be as small as we want. That is, convergence is guaranteed if we require strictly *strong* global localization.

Nevertheless, we know that, if the electrostatic field exists, then a global and single-valued potential also exists and is given uniquely – apart from a constant fixed by the value of the potential at an arbitrary reference point O – by:

$$V(\boldsymbol{r}) = -\int_O^{\boldsymbol{r}} \boldsymbol{E}(\boldsymbol{r}') \cdot d\boldsymbol{l}' + V(O). \tag{38}$$

That way, from eq. (38), by using eqs. (25–26), we will have, for $r > R$:

$$V(r) = -\int_{r(O)}^{r} \frac{\rho_{cte}}{\varepsilon_0}\frac{1}{r'^2}\left\{\frac{R^3}{3} + \int_R^{r'} f(r'')r''^2 dr''\right\} \cdot dr' + V(O) \quad r > r' > R. \tag{39}$$

This expression is perfectly well defined, since the results of radial coordinate integrations are always finite, if $r, r' < \infty$, and if the reference point $r(O)$ is finite. And *this happens regardless of the functional form of the function* $f(r)$ – assuming that it is a bounded and continuous (or piecewise continuous) function everywhere. We can see that the potential given by eq. (39) correctly gives back the corresponding electrostatic field, given by eq. (26).

Let's say, from the spherically symmetric example given by the charge distribution (23), that $f(r) = \frac{R^2}{r^2}$. The electrostatic potential generated in this case – which is globally localized, but in a strictly semi-strong sense – will be given, except for a constant, from eq. (39), by:

$$V(r) = -\frac{\rho_{cte}}{\varepsilon_0}\left\{\frac{2R^3}{3}\frac{1}{r} + R^2\ln(r)\right\}; \quad r > R. \tag{40}$$

If we take the gradient of this expression, we can recover the expression given in eq. (28) for the electrostatic field associated with this distribution.

Let's say, again as an example, that $f(r) = \frac{R}{r}$. The electrostatic potential generated in this case – which is globally localized, but in the weak sense – will be given, except for a constant, from eq. (39), by:

$$V(r) = -\frac{\rho_{cte}}{\varepsilon_0}\left\{\frac{R^3}{6}\frac{1}{r} + \frac{R}{2}r\right\}; \quad r > R. \tag{41}$$

If we take the gradient of this expression, we recover the expression given in eq. (29) for the electrostatic field associated with this distribution.

This brief discussion is sufficient to establish a general fact: to deduce Coulomb and Biot-Savart laws as valid representations to the electrostatic and magnetostatic fields, Helmholtz theorem requires that the potentials be well defined by improper integrals. However, this does not allow us to obtain the strictly semi-strong localization conditions that we know to be sufficient for the good definition of the laws. On the contrary, Helmholtz theorem suggests that the sufficient conditions must be more restrictive.

The fact that a procedure for deducing Coulomb and Biot-Savart laws as well-defined and unique solutions to the field equations seems to require a restriction on the localization of the charge and current distributions that neither the field equations nor the solutions themselves require is not without some discomfort.

In view of the discussion we've had, the first suspicion as to why this is the case is the need to define the potentials in terms of the improper integrals. For this reason, we will investigate whether alternative deduction procedures that dispense with explicit integral representations of the potentials are able to remove this apparent difficulty.

6. Deduction of the Coulomb Law and Global Localization Conditions for Charge Distributions

6.1. Deduction of the Coulomb law from the electrostatic field equations

Let's start by deducing Coulomb law. Take the field equation for the divergence of the electrostatic field (eq. (1.i)), multiply both sides by the geometric factor $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ and integrate over a finite volume V:

$$\begin{aligned}\nabla' \cdot \boldsymbol{E}(\boldsymbol{r}') &= \frac{1}{\varepsilon_0}\rho(\boldsymbol{r}') \\ &\Rightarrow \int_V \nabla' \cdot \boldsymbol{E}(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' \\ &= \frac{1}{\varepsilon_0}\int_V \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'. \end{aligned} \tag{42}$$

The idea behind this deduction is to find necessary conditions for the left-hand side of eq. (42) to be equal to $4\pi\boldsymbol{E}(\boldsymbol{r})$, thus obtaining the mathematical demonstration of the Coulomb law.

Before continuing, it is important to clarify that, *under integration*, there is no reason to be concerned about the fact that the geometric factor becomes indeterminate whenever $\boldsymbol{r} = \boldsymbol{r}'$. This simple, but important result is proved, for example, by Stratton [20, p. 170–171].

The field equation eq. (1.ii) for the curl of the electrostatic field necessarily implies the existence of a scalar potential $V(\boldsymbol{r}')$, such that $\boldsymbol{E}(\boldsymbol{r}') = -\nabla' V(\boldsymbol{r}')$[14]. We can therefore write the left-hand side of eq. (42) as follows:

$$\begin{aligned}\int_V \nabla' \cdot \boldsymbol{E}(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' &= -\int_V \nabla^{2'} V(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' \\ &= -\int_V \nabla^{2'} V(\boldsymbol{r}') \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] d\tau'; \end{aligned} \tag{43}$$

where $\nabla^{2'}$ denotes the Laplacian operator with respect to $\boldsymbol{r}'$ and:

$$\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]. \tag{44}$$

[14] As noted in footnote 13, the fact that Euclidean space is simply connected is sufficient to ensure that the electrostatic field can be represented by a global and single-valued scalar potential at all points in space – although, due to gauge symmetry, the potentials cannot be unique.

From this point on, it becomes more convenient to use the covariant notation for Cartesian tensors, so that one can write:

$$\nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] \equiv \nabla' f \equiv (\partial'^{\nu} f)\hat{\boldsymbol{e}}_\nu; \tag{45}$$

$$\nabla^{2'} V(\boldsymbol{r}') \equiv \nabla' \cdot (\nabla' V(\boldsymbol{r}')) \equiv \partial'^{\mu}\partial'_{\mu} V; \tag{46}$$

where $\hat{\boldsymbol{e}}_\nu = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\} = \{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ is the basis of Cartesian unit vectors (constants), which allows an arbitrary vector $\boldsymbol{D}$ to be represented by the linear combination $D^\nu \hat{\boldsymbol{e}}_\nu$ of the elements of the basis. Note that we now assume Einstein's summation convention for Cartesian indices $\mu, \nu = 1, 2, 3$ which are repeated[15].

Using eqs. (44–46), we can write the first term of eq. (43) as:

$$\begin{aligned}\int_V \nabla' \cdot \boldsymbol{E}(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' \\ = -\hat{\boldsymbol{e}}_\nu \int_V (\partial'^{\mu}\partial'_{\mu} V)(\partial'^{\nu} f) d\tau'. \end{aligned} \tag{47}$$

Let's now consider the Green's first identity (in covariant form), valid for arbitrary scalar functions $\varphi(\boldsymbol{r}')$ and $\psi(\boldsymbol{r}')$. Green's first identity makes it possible to relate an integration over a volume V to an integration over the closed surface S that delimits it, as follows:

$$\int_V [\varphi(\partial'^{\mu}\partial'_{\mu}\psi) + (\partial'^{\mu}\psi)(\partial'_{\mu}\varphi)] d\tau' = \oint_S \varphi(\partial'^{\mu}\psi) da'_{\mu}. \tag{48}$$

If we identify $\varphi = \partial'^{\nu} f$ and $\psi = V$, we can use the identity (48) to manipulate separately each component ν of the vector that constitutes the right-hand side of eq. (47), in order to perform a first integration by parts and obtain a first boundary term:

$$\begin{aligned} -\int_V (\partial'^{\nu} f)(\partial'^{\mu}\partial'_{\mu} V) d\tau' \\ = -\oint_S (\partial'^{\nu} f)(\partial'^{\mu} V) da'_{\mu} + \int_V [(\partial'^{\mu} V)(\partial'_{\mu}\partial'^{\nu} f)] d\tau'. \end{aligned} \tag{49}$$

Now, if we identify $\varphi = V$ and $\psi = \partial'^{\nu} f$, we can again use the identity (48) in the second term on the right-hand side of eq. (49), in order to perform a second integration by parts and obtain a second boundary term.

[15] As these are tensors in Euclidean space, we can raise and lower indices without worrying about changes of sign between covariant and contravariant components. The only precautions that must be taken are to always keep in alternating positions (contravariant and covariant) each of the indices that are repeated (contracted in a sum) in the same side of an equation and to preserve the covariance of the equations, so that free indices always appear on their right and left sides in the same positions (contravariant or covariant).

Thus, eq. (49) is written as:

$$
\begin{aligned}
&-\int_V (\partial'^{\nu} f)(\partial'^{\mu}\partial'_{\mu} V)d\tau' \\
&\quad = -\oint_S (\partial'^{\nu} f)(\partial'^{\mu} V)da'_{\mu} \\
&\qquad + \oint_S V(\partial'^{\mu}\partial'^{\nu} f)da'_{\mu} - \int_V V\partial'^{\nu}(\partial'^{\mu}\partial'_{\mu} f)d\tau'.
\end{aligned} \tag{50}
$$

A manipulation performed on the last term of eq. (50) will allow us to realize a third integration by parts, so that we obtain:

$$
\begin{aligned}
&-\int_V (\partial'^{\nu} f)(\partial'^{\mu}\partial'_{\mu} V)d\tau' \\
&\quad = -\oint_S (\partial'^{\nu} f)(\partial'^{\mu} V)da'_{\mu} + \oint_S V\partial'^{\mu}(\partial'^{\nu} f)da'_{\mu} \\
&\qquad - \int_V \partial'^{\nu}(V\partial'^{\mu}\partial'_{\mu} f)d\tau' + \int_V (\partial'^{\nu} V)(\partial'^{\mu}\partial'_{\mu} f)d\tau'.
\end{aligned} \tag{51}
$$

We can now recompose the expression of the vector on the right-hand side of eq. (47) by contracting both sides of eq. (51) with $\hat{\boldsymbol{e}}_{\nu}$. In this way, it will be possible to recognize, from the third term on the right-hand side of eq. (51), the volume integral of the gradient of $V\partial'^{\mu}\partial'_{\mu} f$. This volume integral can be converted into a surface integral, so that a third boundary term is obtained. Thus, the right-hand side of eq. (47) is written, from eq. (51), as follows:

$$
\begin{aligned}
&-\hat{\boldsymbol{e}}_{\nu}\int_V (\partial'^{\nu} f)(\partial'^{\mu}\partial'_{\mu} V)d\tau' \\
&\quad = -\oint_S (\partial'^{\nu} f\hat{\boldsymbol{e}}_{\nu})(\partial'^{\mu} V)da'_{\mu} + \oint_S V da'_{\mu}\partial'^{\mu}(\partial'^{\nu} f\hat{\boldsymbol{e}}_{\nu}) \\
&\qquad - \oint_S V(\partial'^{\mu}\partial'_{\mu} f)\hat{\boldsymbol{e}}_{\nu} da'^{\nu} + \int_V (\hat{\boldsymbol{e}}_{\nu}\partial'^{\nu} V)(\partial'^{\mu}\partial'_{\mu} f)d\tau'.
\end{aligned} \tag{52}
$$

We can now reverse eq. (52) to vector notation, using eqs. (45) and (46), so that the left-hand side of eq. (42) is written as:

$$
\begin{aligned}
&\int_V \nabla'\cdot\boldsymbol{E}(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau' \\
&\quad = -\int_V \nabla^{2'} V(\boldsymbol{r}')\nabla'\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]d\tau' \\
&\quad = -\oint_S \nabla'\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right](\nabla' V(\boldsymbol{r}')\cdot d\boldsymbol{a}') \\
&\qquad + \oint_S V(\boldsymbol{r}')[d\boldsymbol{a}'\cdot\nabla']\left\{\nabla'\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\right\} \\
&\qquad - \oint_S V(\boldsymbol{r}')\nabla^{2'}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]d\boldsymbol{a}' \\
&\qquad + \int_V (\nabla' V(\boldsymbol{r}'))\nabla^{2'}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]d\tau'.
\end{aligned} \tag{53}
$$

Now, let's recompose the right-hand side of expression (53) in terms of the electrostatic field, $\boldsymbol{E}(\boldsymbol{r}') = -\nabla' V(\boldsymbol{r}')$, and in terms of the geometric factor $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ (with eq. (44)). We will also use the fact that the geometric factor $\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$ satisfies the equation for the Green's function:

$$
\nabla^{2'}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}'); \tag{54}
$$

where $\delta(\boldsymbol{r}-\boldsymbol{r}')$ is the Dirac Delta distribution. Thus, eq. (53) is written as:

$$
\begin{aligned}
&\int_V \nabla'\cdot\boldsymbol{E}(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau' \\
&\quad = 4\pi\int_V \boldsymbol{E}(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')d\tau' + \oint_S \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}[\boldsymbol{E}(\boldsymbol{r}')\cdot d\boldsymbol{a}'] \\
&\qquad + \oint_S V(\boldsymbol{r}')[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \\
&\qquad + 4\pi\oint_S V(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')d\boldsymbol{a}'.
\end{aligned} \tag{55}
$$

If we extend the integration volume so that $V\to \mathrm{R}^3$ and the surface $S\to\infty$, we can perform the integrations of the Delta distributions and recognize that the last term of eq. (55) must be zero, since $\boldsymbol{r}$ is an arbitrary point, but always located at a finite distance. In this way, we obtain:

$$
\begin{aligned}
&\int_{\mathrm{R}^3} \nabla'\cdot\boldsymbol{E}(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau' \\
&\quad = 4\pi\boldsymbol{E}(\boldsymbol{r}) + \oint_{S\to\infty} \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(\boldsymbol{E}(\boldsymbol{r}')\cdot d\boldsymbol{a}') \\
&\qquad + \oint_{S\to\infty} V(\boldsymbol{r}')[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.
\end{aligned} \tag{56}
$$

Taking the limit, $V\to\mathrm{R}^3$, on eq. (42), and using eq. (56) allow us to express the electrostatic field as follows:

$$
\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0}\int_{\mathrm{R}^3} \rho(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau' \\
&\quad - \frac{1}{4\pi}\oint_{S\to\infty} \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(\boldsymbol{E}(\boldsymbol{r}')\cdot d\boldsymbol{a}') \\
&\quad - \frac{1}{4\pi}\oint_{S\to\infty} V(\boldsymbol{r}')[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.
\end{aligned} \tag{57}
$$

Finally, eq. (57) makes it clear what are the necessary conditions for the solutions of the electrostatic equations to be given by Coulomb law formula. *These necessary conditions consist of the asymptotic behavior of the surface integrals being such that they converge to zero in the limiting process*:

$$
\oint_{S\to\infty} V(\boldsymbol{r}')[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\to 0; \tag{58}
$$

$$
\oint_{S\to\infty} \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(\boldsymbol{E}(\boldsymbol{r}')\cdot d\boldsymbol{a}')\to 0. \tag{59}
$$

6.2. Regularity and localization: satisfaction of the asymptotic necessary conditions

The satisfaction of the necessary conditions (58) and (59) depends, in turn, exclusively on the asymptotic behavior of the scalar potential and, ultimately, on the asymptotic behavior of the electrostatic field. Precisely because these conditions are imposed on surface integrals "located at infinity", there is an infinite number of functional forms for the fields and potentials that are sufficient to satisfy them. The best we can do is look for sufficient conditions that are as general as possible.

Let's take under analysis condition (58) first. It can be verified that (each Cartesian component of) the geometric factor $[d\boldsymbol{a}' \cdot \nabla'] \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ behaves asymptotically as $1/r'$, when $r' \to \infty$. Thus, a sufficient condition for the satisfaction of condition (58) is that, for any fixed direction in space, the potential decay or diverge slower than r' (sublinearly), that is, $\lim_{r' \to \infty} \frac{V(r',\theta',\varphi')}{r'} = 0$.

This includes, of course, potentials whose asymptotic values are constants – which can be set to zero, given the fact that the potential is defined at less than a constant. It also includes potentials whose asymptotic behaviors are given by logarithmic functions, such as the potential given by eq. (40).

For reasons that will become clear just below, let us restrict the behavior of the potential to functions that *diverge sublinearly.*

Conditions on the potentials imply conditions on the electrostatic fields. Since the components of the electrostatic field are first derivatives of the potential, any potential that diverges asymptotically in a sublinear way implies an electrostatic field whose components $E_\mu(r',\theta',\varphi')$ decrease to zero at infinity, for any fixed direction in space. Indeed, in spherical coordinates:

$$\lim_{r' \to \infty} \frac{V(r',\theta',\varphi')}{r'} = 0 \Rightarrow \lim_{r' \to \infty} \frac{\partial}{\partial r'} V(r',\theta',\varphi') = 0$$
$$\Rightarrow \lim_{r' \to \infty} E_r(r',\theta',\varphi') = 0; \quad (60)$$

$$\lim_{r' \to \infty} \frac{V(r',\theta',\varphi')}{r'} = 0 \Rightarrow \lim_{r' \to \infty} \frac{\frac{\partial}{\partial \theta'} V(r',\theta',\varphi')}{r'} = 0$$
$$\Rightarrow \lim_{r' \to \infty} E_\theta(r',\theta',\varphi') = 0; \quad (61)$$

$$\lim_{r' \to \infty} \frac{V(r',\theta',\varphi')}{r'} = 0 \Rightarrow \lim_{r' \to \infty} \frac{\frac{\partial}{\partial \varphi'} V(r',\theta',\varphi')}{r' \sin\theta'} = 0$$
$$\Rightarrow \lim_{r' \to \infty} E_\varphi(r',\theta',\varphi') = 0; \quad (62)$$

where $\sin\theta' \neq 0$.

In eq. (60), we did employ the L'Hôpital rule, since we have an indeterminate form, since the potential diverges.

In eqs. (61) and (62), we applied the differential operators $\frac{\partial}{\partial \theta'}$ and $\frac{1}{\sin\theta'}\frac{\partial}{\partial \varphi'}$ and commute them with the limits in r' to identify the polar and azimuthal components of the electrostatic field.

Therefore, we conclude that $\lim_{r' \to \infty} \boldsymbol{E}(r',\theta',\varphi') = 0$ is a sufficient condition to satisfy the condition (58). This condition is also exactly sufficient for the satisfaction of the necessary condition given by eq. (59), given the asymptotic behaviors of $d\boldsymbol{a}'(r' \to \infty, \theta', \varphi') \sim {r'}^2$ and of the geometric factor $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(r' \to \infty, \theta', \varphi') \sim {r'}^{-2}$, for any fixed direction in space.

What's more, we know that $\lim_{r' \to \infty} \boldsymbol{E}(r',\theta',\varphi') = 0$ is one of the *regularity conditions* that guarantees the *uniqueness* of the solutions of the field equations.

At this point, by way of recapitulation, we can state three facts.

The first fact is that *the above regularity condition on the electrostatic fields is sufficient to establish the formula of the Coulomb law* (eq. 18) *as the formal improper integral representation of the fields.* For consistency, this implies that *Coulomb law formula must be well-defined and must provide a field that satisfies the condition of regularity.*

The second fact is that, according to the results of the section 3, *strictly semi-strong global localization* of the charge distributions is *a sufficient condition for the improper integral representing Coulomb law to be well-defined.*

The third fact is that, according to the discussion we had in the last part of section 4, *strictly semi-strong global localization is compatible with the regularity condition* $\lim_{r' \to \infty} \boldsymbol{E}(r',\theta',\varphi') = 0$.

By way of conclusion, we can summarize these three facts in a single statement. *The strictly semi-strong global localization of charge distributions and the regularity condition* $\lim_{r' \to \infty} \boldsymbol{E}(r',\theta',\varphi') = 0$ *on the electrostatic fields are mutually compatible conditions that, jointly, are sufficient to ensure that the Coulomb law formula*:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathrm{R}^3} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'; \quad (63)$$

provides well-defined and unique solutions to electrostatic field equations.

At first glance, it would seem that the potential given by eq. (41) and the corresponding asymptotically constant field, given by eq. (29), would be excluded. What happens, however, is that, in this case, *because of the spherical symmetry*, potentials and fields which, respectively, behave asymptotically as $V(r' \to \infty) \sim r'$ and $\boldsymbol{E}(r' \to \infty) \sim cte$, *also satisfy the necessary conditions* (58) and (59), since, in these cases, the integrations of the angular variables cancel out identically.

Nevertheless, we also know that $\lim_{r' \to \infty} \boldsymbol{E}(r',\theta',\varphi') = cte \neq 0$ is also a *regularity condition* that guarantees the *uniqueness* of the solutions of the field equations.

This leads us to conclude that, although a threshold, *the weak global localization condition on spherically symmetric charge distributions, with* $|\rho(r')| \sim 1/r'$, *and the regularity condition* $\lim_{r' \to \infty} \boldsymbol{E}(r',\theta',\varphi') = cte \neq 0$ *on* *the electrostatic fields are mutually compatible conditions that, jointly, are sufficient to ensure that the Coulomb law formula provides well-defined and unique solutions to electrostatic field equations.*

By the way, part of this last conclusion can be easily extended, because regardless of the asymptotic forms of the potentials and the corresponding fields, if they are spherically symmetric, the integrations of the angular variables will be zero and will imply the automatic satisfaction of the asymptotic conditions (58) and (59), so that Coulomb law will provide, as we have already seen, well-defined solutions. However, in these cases, the electrostatic field will be non-regular and, thus, the uniqueness condition will be violated.

The above situation, which involves the weak localization condition, clearly shows again that, *although the strictly semi-strong global localization condition is a sufficient condition for the good definition of Coulomb law, the fact is that it cannot be a necessary condition, in general.*

In fact, apart from the threshold case of weak global localization, we have already mentioned some examples which make it quite clear that Coulomb law is capable of providing well-defined and unique solutions even from partially localized charge distributions.

As in the case of weakly localized distributions that exhibit spherical symmetry, *in all other partially localized cases, the global symmetries of the distributions must imply symmetries for the electrostatic field that, in one way or another, will be sufficient to satisfy the necessary conditions* (58) and (59), *in a manner compatible with the conditions of regularity.* Of course, in order to demonstrate that these are in fact the cases, one must proceed on a case-by-case basis.

By way of conclusion, in this section, we used the results obtained in sections 2 to 4 to show how Coulomb law can be deduced without explicitly defining the potentials. As has become clear, the alternative deduction allows us to include cases of global localization that the deduction based on Helmholtz theorem is unable to include.

7. Deduction of the Biot-Savart Law and Global Localization Conditions for Current Distributions

7.1. Deduction of the Biot-Savart law from the magnetostatic field equations

Let's now investigate the deduction of the Biot-Savart law. Take the equation eq. (2.ii) for the curl of the magnetostatic field, perform a vector multiplication in both sides by the geometric factor $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ and integrate over a finite volume V:

$$\nabla' \times \boldsymbol{B}(\boldsymbol{r}') = \mu_0 \boldsymbol{J}(\boldsymbol{r}')$$

$$\Rightarrow \int_V (\nabla' \times \boldsymbol{B}(\boldsymbol{r}')) \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' = \mu_0 \int_V \boldsymbol{J}(\boldsymbol{r}') \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'. \quad (64)$$

The idea behind the deduction is once again to find necessary conditions for the left-hand side of eq. (64) to be equal to $4\pi\boldsymbol{B}(\boldsymbol{r})$, thus obtaining the demonstration and conditions of validity of the Biot-Savart law.

We should note, however, that in the transition from the left to the right side of implication (64), the information about the zero divergence of current density ends up being concealed. Thus, we must impose, at the end of the proof, that current densities must be necessarily divergenceless.

The field equation eq. (2.i) for the divergence of the magnetostatic field necessarily implies the existence of a vector potential $\boldsymbol{A}(\boldsymbol{r}')$, such that $\boldsymbol{B}(\boldsymbol{r}') = \nabla' \times \boldsymbol{A}(\boldsymbol{r}')$[16]. We can therefore write the left-hand side of eq. (64) as follows:

$$\begin{aligned}
&\int_V (\nabla' \times \boldsymbol{B}(\boldsymbol{r}')) \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' \\
&= \int_V [\nabla' \times (\nabla' \times \boldsymbol{A}(\boldsymbol{r}'))] \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' \\
&= \int_V [\nabla'(\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}')) - \nabla^{2'} \boldsymbol{A}(\boldsymbol{r}')] \times \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] d\tau' \\
&= \int_V \nabla'(\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}')) \times \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] d\tau' \\
&\quad - \int_V \nabla^{2'} \boldsymbol{A}(\boldsymbol{r}') \times \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] d\tau'; \quad (65)
\end{aligned}$$

where we use eq. (44) and the vector identity $\nabla' \times (\nabla' \times \boldsymbol{A}(\boldsymbol{r}')) \equiv \nabla'(\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}')) - \nabla^{2'} \boldsymbol{A}(\boldsymbol{r}')$ – where $\nabla^{2'}$ denotes the Laplacian operator acting on the Cartesian components of $\boldsymbol{A}(\boldsymbol{r}')$.

Each Cartesian component of the last volume integral in eq. (65) is a volume integral of the vector product of two gradients, which can be converted into a volume integral of a curl by means of the identity $\nabla' g \times \nabla' f \equiv \nabla' \times (g \nabla' f)$. We can then use the vector identity $\int_V \nabla \times \boldsymbol{W} d\tau = \oint_S d\boldsymbol{a} \times \boldsymbol{W}$ (see eq. (31)).

These two last steps allow us to write the last volume integral in eq. (65) as a surface integral:

$$\int_V \nabla'(\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}')) \times \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] d\tau' = \int_V \nabla' \times \left\{(\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}')) \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\right\} d\tau'$$

[16] As noted in footnote 13, the fact that Euclidean space is simply connected is sufficient to ensure that the magnetostatic field can be represented by a global and single-valued vector potential at all points in space – although, due to gauge symmetry, the potentials cannot be unique.

$$= -\oint_S (\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}'))\nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] \times d\boldsymbol{a}'$$
$$= -\oint_S (\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}'))\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \times d\boldsymbol{a}'; \qquad (66)$$

where we use eq. (44) to recover the geometric factor in the last equality.

With the result obtained in eq. (66), eq. (65) can be written as follows:

$$\int_V (\nabla' \times \boldsymbol{B}(\boldsymbol{r}')) \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'$$
$$= -\oint_S (\nabla' \cdot \boldsymbol{A}(\boldsymbol{r}'))\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \times d\boldsymbol{a}'$$
$$- \int_V \nabla^{2'} \boldsymbol{A}(\boldsymbol{r}') \times \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] d\tau'. \qquad (67)$$

As we did before in the previous section, to treat the remaining volume integral on the right-hand side of eq. (67), it is more convenient to use the covariant notation for Cartesian tensors.

In particular, to represent vector products and curls, it is extremely convenient to use the Levi-Civita tensor in three dimensions, $\epsilon^{\mu\nu\eta}$, which is the invariant tensor totally antisymmetric by permutations of pairs of indices. The Levi-Civita tensor is conventionally defined as having component $\epsilon^{123} = +1$. The other non-zero components are obtained as permutations of the indices of the ϵ^{123} component. For even permutations, the components are equal to $+1$ and, for odd permutations, they are equal to -1.

Using covariant notation and the Levi-Civita tensor, we can write the following useful definitions:

$$\nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] \equiv \nabla' f \equiv \partial'^{\nu} f \hat{\boldsymbol{e}}_\nu; \qquad (68)$$

$$\nabla^{2'} \boldsymbol{A}(\boldsymbol{r}') \equiv \nabla' \cdot (\nabla' \boldsymbol{A}(\boldsymbol{r}')) \equiv \partial'_\mu \partial'^\mu A^\nu \hat{\boldsymbol{e}}_\nu; \qquad (69)$$

$$\boldsymbol{C} \times \boldsymbol{D} \equiv \epsilon^{\mu\nu\eta} C_\nu D_\eta \hat{\boldsymbol{e}}_\mu; \qquad (70)$$

$$\nabla' \times \boldsymbol{D} \equiv \epsilon^{\mu\nu\eta} \partial'_\nu D_\eta \hat{\boldsymbol{e}}_\mu. \qquad (71)$$

By using these results, we can write the volume integral on the right-hand side of eq. (67) as follows:

$$- \int_V \nabla^{2'} \boldsymbol{A}(\boldsymbol{r}') \times \nabla' \left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right] d\tau'$$
$$= -\int_V \epsilon^{\eta\lambda\nu}(\partial'^\mu \partial'_\mu A_\lambda)(\partial'_\nu f)\hat{\boldsymbol{e}}_\eta d\tau'$$
$$= -\epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \int_V (\partial'^\mu \partial'_\mu A_\lambda)(\partial'_\nu f) d\tau'. \qquad (72)$$

In eq. (72), we recognize the same volume integral which has already appeared in eq. (47), with the component A_λ of the vector potential in place of the scalar potential V. This volume integral can be manipulated in exactly the same way as we did before, so the steps corresponding to eqs. (49) to (51) can be omitted. As a result, we can write the right-hand side of eq. (72) using the result (51), so that:

$$- \epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \int_V (\partial'^\mu \partial'_\mu A_\lambda)(\partial'_\nu f) d\tau'$$
$$= -\epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \oint_S (\partial'_\nu f)(\partial'^\mu A_\lambda) da'_\mu$$
$$+ \epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \oint_S A_\lambda \partial'_\nu (\partial'^\mu f) da'_\mu$$
$$- \epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \int_V \partial'_\nu (A_\lambda \partial'^\mu \partial'_\mu f) d\tau'$$
$$+ \epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \int_V (\partial'_\nu A_\lambda)(\partial'^\mu \partial'_\mu f) d\tau'. \qquad (73)$$

The first term on the right-hand side of eq. (73) can be manipulated, using the antisymmetry of the Levi-Civita tensor, and reverted to vector notation – using eqs. (45), (44) and (70) – as follows:

$$- \epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \oint_S (\partial'_\nu f)(\partial'^\mu A_\lambda) da'_\mu$$
$$= \oint_S \epsilon^{\eta\nu\lambda}(\partial'_\nu f)(\partial'^\mu A_\lambda)\hat{\boldsymbol{e}}_\eta da'_\mu$$
$$= \oint_S \epsilon^{\eta\nu\lambda}(\partial'_\nu f)(da'_\mu \partial'^\mu A_\lambda)\hat{\boldsymbol{e}}_\eta$$
$$= \oint_S \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \times \{[d\boldsymbol{a}' \cdot \nabla']\boldsymbol{A}(\boldsymbol{r}')\}. \qquad (74)$$

The second term on the right-hand side of eq. (73) can be manipulated and reverted to vector notation – using eqs. (45), (44) and (70) – as follows:

$$\epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \oint_S A_\lambda \partial'_\nu (\partial'^\mu f) da'_\mu$$
$$= \oint_S \epsilon^{\eta\lambda\nu} A_\lambda \partial'_\nu (\partial'^\mu f)\hat{\boldsymbol{e}}_\eta da'_\mu$$
$$= \oint_S \epsilon^{\eta\lambda\nu} A_\lambda da'_\mu \partial'^\mu (\partial'_\nu f)\hat{\boldsymbol{e}}_\eta$$
$$= \oint_S \boldsymbol{A}(\boldsymbol{r}') \times \left\{[d\boldsymbol{a}' \cdot \nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right\}. \qquad (75)$$

The third term on the right-hand side of eq. (73) can be manipulated, using the antisymmetry of the Levi-Civita tensor, and reverted to vector notation – using eqs. (45), (44) and (71) – as follows:

$$- \epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta \int_V \partial'_\nu (A_\lambda \partial'^\mu \partial'_\mu f) d\tau'$$
$$= \int_V \epsilon^{\eta\nu\lambda} \partial'_\nu (A_\lambda \partial'^\mu \partial'_\mu f)\hat{\boldsymbol{e}}_\eta d\tau'$$

$$= \int_V \nabla' \times \left[\nabla^{2'}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\boldsymbol{A}(\boldsymbol{r}')\right]d\tau'$$
$$= -\oint_S \nabla^{2'}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\boldsymbol{A}(\boldsymbol{r}')\times d\boldsymbol{a}'$$
$$= 4\pi\oint_S \delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{A}(\boldsymbol{r}')\times d\boldsymbol{a}'; \tag{76}$$

in which we use again, in the fourth equality, the vector identity $\int_V \nabla\times\boldsymbol{W}d\tau = \oint_S d\boldsymbol{a}\times\boldsymbol{W}$, and, in the last equality, the definition (54) of the Dirac Delta distribution.

Finally, the last term on the right-hand side of eq. (73) can be manipulated, using the antisymmetry of the Levi-Civita tensor, and reverted to vector notation – using eqs. (45), (44), (71) and (54) – as follows:

$$\epsilon^{\eta\lambda\nu}\hat{\boldsymbol{e}}_\eta\int_V(\partial'_\nu A_\lambda)(\partial'^\mu\partial'_\mu f)d\tau'$$
$$= -\int_V(\epsilon^{\eta\nu\lambda}\partial'_\nu A_\lambda\hat{\boldsymbol{e}}_\eta)(\partial'^\mu\partial'_\mu f)d\tau'$$
$$= -\int_V(\nabla'\times\boldsymbol{A}(\boldsymbol{r}'))\nabla^{2'}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]d\tau'$$
$$= 4\pi\int_V\boldsymbol{B}(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')d\tau'; \tag{77}$$

where we recognize, in the last equality, the magnetostatic field as the curl of vector potential.

With the results of the expressions (73–77), we can rewrite eq. (72) as follows:

$$-\int_V\nabla^{2'}\boldsymbol{A}(\boldsymbol{r}')\times\nabla'\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]d\tau'$$
$$= \oint_S\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times\{[d\boldsymbol{a}'\cdot\nabla']\boldsymbol{A}(\boldsymbol{r}')\}$$
$$+\oint_S\boldsymbol{A}(\boldsymbol{r}')\times\left\{[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right\}$$
$$+4\pi\oint_S\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{A}(\boldsymbol{r}')\times d\boldsymbol{a}'$$
$$+4\pi\int_V\boldsymbol{B}(\boldsymbol{r}')d(\boldsymbol{r}-\boldsymbol{r}')d\tau'. \tag{78}$$

With this result, eq. (67) can be rewritten as follows:

$$\int_V(\nabla'\times\boldsymbol{B}(\boldsymbol{r}'))\times\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau'$$
$$= -\oint_S(\nabla'\cdot\boldsymbol{A}(\boldsymbol{r}'))\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times d\boldsymbol{a}'$$
$$+\oint_S\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times\{[d\boldsymbol{a}'\cdot\nabla']\boldsymbol{A}(\boldsymbol{r}')\}$$
$$+\oint_S\boldsymbol{A}(\boldsymbol{r}')\times\left\{[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right\}$$
$$+4\pi\oint_S\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{A}(\boldsymbol{r}')\times d\boldsymbol{a}'$$
$$+4\pi\int_V\boldsymbol{B}(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')d\tau'. \tag{79}$$

If we extend the integration volume so that the surface $S\to\infty$, we can perform the integrations of the Delta distributions in eq. (79), recognizing that its penultimate term is zero, since $\boldsymbol{r}$ is an arbitrary point, but always located at a finite distance. Thus:

$$\int_{\mathrm{R}^3}(\nabla'\times\boldsymbol{B}(\boldsymbol{r}'))\times\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau'$$
$$= 4\pi\boldsymbol{B}(\boldsymbol{r})-\oint_{S\to\infty}(\nabla'\cdot\boldsymbol{A}(\boldsymbol{r}'))\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times d\boldsymbol{a}'$$
$$+\oint_{S\to\infty}\boldsymbol{A}(\boldsymbol{r}')\times\left\{[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right\}$$
$$+\oint_{S\to\infty}\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times\{[d\boldsymbol{a}'\cdot\nabla']\boldsymbol{A}(\boldsymbol{r}')\}. \tag{80}$$

From eq. (64), we have, therefore:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{R}^3}\boldsymbol{J}(\boldsymbol{r}')\times\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau'$$
$$+\frac{1}{4\pi}\oint_{S\to\infty}(\nabla'\cdot\boldsymbol{A}(\boldsymbol{r}'))\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times d\boldsymbol{a}'$$
$$-\frac{1}{4\pi}\oint_{S\to\infty}\boldsymbol{A}(\boldsymbol{r}')\times\left\{[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right\}$$
$$-\frac{1}{4\pi}\oint_{S\to\infty}\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times\{d\boldsymbol{a}'\cdot\nabla'\boldsymbol{A}(\boldsymbol{r}')\}; \tag{81}$$

where we recognize that $[d\boldsymbol{a}'\cdot\nabla']\boldsymbol{A}(\boldsymbol{r}') = \{d\boldsymbol{a}'\cdot\nabla' A^\mu(\boldsymbol{r}')\}\hat{\boldsymbol{e}}_\mu \equiv d\boldsymbol{a}'\cdot\nabla'\boldsymbol{A}(\boldsymbol{r}')$, with $A^\mu(\boldsymbol{r}')$ the Cartesian components of the vector potential.

Finally, eq. (81) makes it clear what are the necessary conditions for the solutions of the magnetostatic equations to be given by Biot-Savart law formula. *These necessary conditions consist of the asymptotic behavior of the surface integrals being such that they converge to zero in the limiting process*:

$$\oint_{S\to\infty}(\nabla'\cdot\boldsymbol{A}(\boldsymbol{r}'))\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times d\boldsymbol{a}'\to 0; \tag{82}$$
$$\oint_{S\to\infty}\boldsymbol{A}(\boldsymbol{r}')\times\left\{[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right\}\to 0; \tag{83}$$
$$\oint_{S\to\infty}\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times\{d\boldsymbol{a}'\cdot\nabla'\boldsymbol{A}(\boldsymbol{r}')\}\to 0; \tag{84}$$

7.2. Regularity and localization: satisfaction of the asymptotic necessary conditions

The satisfaction of the necessary conditions (82–84) depends, in turn, on the asymptotic behavior of the vector potentials and, ultimately, on the asymptotic behavior of the magnetostatic fields. Precisely because these conditions are imposed on surface integrals located at infinity, there is virtually an infinite number of functional forms for the potentials (and fields) that are sufficient to satisfy them. The best we can do is look for sufficient conditions that are as general as possible.

Let's take condition (82) first. The asymptotic behavior of the geometric term $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times d\boldsymbol{a}'$ must be considered carefully, because their factors are geometrically correlated. A general inspection shows that, due to the vector product, this term must approach zero indefinitely, because the vectors $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ and $d\boldsymbol{a}'$ become asymptotically parallel. To find out how $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times d\boldsymbol{a}'$ approaches zero, it is sufficient to explicit the Cartesian components of the vector product, from which we can conclude that $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times d\boldsymbol{a}'(r'\to\infty,\theta',\varphi')\sim {r'}^{-1}$, for any fixed direction in space.

Thus, a sufficient condition for the satisfaction of eq. (82) is that, for any fixed direction is space, the divergence of the vector potential decay or diverge slower than r' (sublinearly), that is, $\lim\limits_{r'\to\infty}\frac{\nabla'\cdot\boldsymbol{A}(\boldsymbol{r}')}{r'}=0$. To satisfy this condition, it is sufficient that it is valid for each of the terms of the sum that composes $\nabla'\cdot\boldsymbol{A}(\boldsymbol{r}')$. In spherical coordinates:

$$\lim_{r'\to\infty}\frac{1}{r'}\left\{\frac{1}{{r'}^2}\frac{\partial}{\partial r'}({r'}^2A_r(r',\theta',\varphi'))\right\}=0$$
$$\Rightarrow\lim_{r'\to\infty}\frac{A_r(r',\theta',\varphi')}{{r'}^2}=0; \quad (85)$$
$$\lim_{r'\to\infty}\frac{1}{r'}\left\{\frac{1}{r'}\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}(\sin\theta' A_\theta(r',\theta',\varphi'))\right\}=0$$
$$\Rightarrow\lim_{r'\to\infty}\frac{A_\theta(r',\theta',\varphi')}{{r'}^2}=0; \quad (86)$$
$$\lim_{r'\to\infty}\frac{1}{r'}\left\{\frac{1}{r'}\frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}(A_\varphi(r',\theta',\varphi'))\right\}=0$$
$$\Rightarrow\lim_{r'\to\infty}\frac{A_\varphi(r',\theta',\varphi')}{{r'}^2}=0; \quad (87)$$

where $\sin\theta'\neq 0$.

In eq. (85), we employed the L'Hôpital rule. But this is only possible if we assume that ${r'}^2A_r(r',\theta',\varphi')$ diverge, to constitute an indeterminate form. This implies that $A_r(r',\theta',\varphi')$ can diverge or decay, but slower than $1/{r'}^2$.

In eqs. (86) and (87), we commuted the differential operators $\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}$ and $\frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}\sin\theta'$ with the limits in r'.

From eqs. (85–87), we conclude that, to satisfy the condition (82), it is sufficient that the vector potential decay – slower than $1/{r'}^2$ – or, at most, diverge subquadratically, such that $\lim\limits_{r'\to\infty}\frac{A^\mu(r',\theta',\varphi')}{{r'}^2}=0$. But this implies that it is also sufficient that the vector potential *diverge sublinearly*, that is, $\lim\limits_{r'\to\infty}\frac{A^\mu(r',\theta',\varphi')}{r'}=0$.

Let's take condition (83). Note that the vectors $\boldsymbol{A}(\boldsymbol{r}')$ and $[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ are not correlated geometrically, given the fact that the potential vector is arbitrary. We have already mentioned that the Cartesian components of the geometric factor $[d\boldsymbol{a}'\cdot\nabla']\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ behave asymptotically as $1/r'$, with $\boldsymbol{r}'\to\infty$. This leads to the conclusion that each component of the potential $\boldsymbol{A}(\boldsymbol{r}')$ must either decay to zero, or at most *diverge sublinearly*, that is, $\lim\limits_{r'\to\infty}\frac{A^\mu(r',\theta',\varphi')}{r'}=0$.

Finally, let's take condition (84). Note that the vectors $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ and $d\boldsymbol{a}'\cdot\nabla'\boldsymbol{A}(\boldsymbol{r}')\equiv\{d\boldsymbol{a}'\cdot\nabla'A^\mu(\boldsymbol{r}')\}\hat{\boldsymbol{e}}_\mu$ are not geometrically correlated. Condition (84) imposes that, for any fixed direction in space, $\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\times\{d\boldsymbol{a}'\cdot\nabla'\boldsymbol{A}(\boldsymbol{r}')\}\to 0$. Given the asymptotic behaviors of the geometric factors, this implies that, for any fixed direction is space, the gradients of the Cartesian components of the vector potential must decay at infinity, that is, $\lim\limits_{r'\to\infty}\nabla'A^\mu(r',\theta',\varphi')=0$.

To satisfy this condition, it is sufficient that it is valid for each one of the components of the gradient. In spherical coordinates:

$$\lim_{r'\to\infty}\frac{\partial}{\partial r'}A^\mu(r',\theta',\varphi')=0$$
$$\Rightarrow\lim_{r'\to\infty}\frac{A^\mu(r',\theta',\varphi')}{r'}=0; \quad (88)$$
$$\lim_{r'\to\infty}\frac{1}{r'}\frac{\partial}{\partial\theta'}A^\mu(r',\theta',\varphi')=0$$
$$\Rightarrow\lim_{r'\to\infty}\frac{A^\mu(r',\theta',\varphi')}{r'}=0; \quad (89)$$
$$\lim_{r'\to\infty}\frac{1}{r'}\frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}A^\mu(r',\theta',\varphi')=0$$
$$\Rightarrow\lim_{r'\to\infty}\frac{A^\mu(r',\theta',\varphi')}{r'}=0; \quad (90)$$

where $\sin\theta'\neq 0$.

In eq. (88), we employed the L'Hôpital rule, assuming that $A^\mu(r',\theta',\varphi')$ diverges, to constitute an indeterminate form. In eqs. (89) and (90), we commuted the differential operators $\frac{\partial}{\partial\theta'}$ and $\frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}$ with the limits in r'.

We conclude that, to satisfy the condition (84), it is sufficient that the vector potential diverge sublinearly, such that, $\lim\limits_{r'\to\infty}\frac{A^\mu(r',\theta',\varphi')}{r'}=0$.

The analysis of the necessary conditions (82–84) therefore leads us to conclude that *a sufficient condition for all of them to be satisfied conjointly is that the components of the vector potential diverge sublinearly.*

Conditions on the vector potentials imply conditions on the magnetostatic fields. Since the components of the magnetostatic field are first derivatives of the vector potential, any potential that diverges asymptotically in a sublinear way implies a magnetostatic field whose spherical components $B_\mu(r',\theta',\varphi')$, with $\mu=\{r,\theta,\varphi\}$, decrease to zero at infinity, for any fixed direction in space. Indeed:

$$\begin{aligned}
&\lim_{r'\to\infty} B_r(r',\theta',\varphi') \\
&\quad = \lim_{r'\to\infty} \frac{1}{r'\sin\theta'}\left[\frac{\partial}{\partial\theta'}(\sin\theta' A_\varphi) - \frac{\partial}{\partial\varphi'}A_\theta\right] \\
&\quad = \frac{1}{\sin\theta'}\left[\frac{\partial}{\partial\theta'}\left(\sin\theta' \lim_{r'\to\infty}\frac{A_\varphi}{r'}\right) - \frac{\partial}{\partial\varphi'}\lim_{r'\to\infty}\frac{A_\theta}{r'}\right] = 0;
\end{aligned} \tag{91}$$

$$\begin{aligned}
&\lim_{r'\to\infty} B_\theta(r',\theta',\varphi') \\
&\quad = \lim_{r'\to\infty} \frac{1}{r'}\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}A_r - \frac{\partial}{\partial r'}(r'A_\varphi)\right] \\
&\quad = \frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}\lim_{r'\to\infty}\frac{A_r}{r'} - \lim_{r'\to\infty}\frac{\frac{\partial}{\partial r'}(r'A_\varphi)}{r'} \\
&\quad = \frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}\lim_{r'\to\infty}\frac{A_r}{r'} - \lim_{r'\to\infty}\frac{r'A_\varphi}{r'^2} \\
&\quad = \frac{1}{\sin\theta'}\frac{\partial}{\partial\varphi'}\lim_{r'\to\infty}\frac{A_r}{r'} - \lim_{r'\to\infty}\frac{A_\varphi}{r'} = 0;
\end{aligned} \tag{92}$$

$$\begin{aligned}
&\lim_{r'\to\infty} B_\varphi(r',\theta',\varphi') \\
&\quad = \lim_{r'\to\infty} \frac{1}{r'}\left[\frac{\partial}{\partial r'}(r'A_\theta) - \frac{\partial}{\partial\theta'}A_r\right] \\
&\quad = \lim_{r'\to\infty}\frac{\frac{\partial}{\partial r'}(r'A_\theta)}{r'} - \frac{\partial}{\partial\theta'}\lim_{r'\to\infty}\frac{A_r}{r'} \\
&\quad = \lim_{r'\to\infty}\frac{r'A_\theta}{r'^2} - \frac{\partial}{\partial\theta'}\lim_{r'\to\infty}\frac{A_r}{r'} = 0;
\end{aligned} \tag{93}$$

where $\sin\theta' \neq 0$.

Therefore, we conclude that $\lim_{r'\to\infty} \boldsymbol{B}(r',\theta',\varphi') = 0$ is a sufficient condition to satisfy the necessary conditions (82–84). We know that this is also one of the *regularity conditions* that guarantees the *uniqueness* of the solutions of the field equations.

At this point, by way of recapitulation, we can state four facts.

The first fact is that *the above regularity condition on the magnetostatic fields is sufficient to establish the formula of the Biot-Savart law* (eq. 19) *as the formal improper integral representation of the fields.* For consistency, this implies that *Biot-Savart law formula must be well-defined and must provide a field that satisfies the condition of regularity.*

The second fact is that, according to the results of the section 3, *strictly semi-strong global localization* of the current distributions is *a sufficient condition for the improper integral representing Biot-Savart law to be well-defined.*

The third fact is that, according to the discussion we had in the last part of section 4, *strictly semi-strong global localization is compatible with the regularity condition* $\lim_{r'\to\infty} \boldsymbol{B}(r',\theta',\varphi') = 0$.

The fourth fact is that *current distributions must be divergenceless*, and this is not embedded in the Biot-Savart law formula.

By way of conclusion, we can summarize these four facts in a single statement. *The strictly semi-strong global localization of divergenceless current distributions and the regularity condition* $\lim_{r'\to\infty} \boldsymbol{B}(r',\theta',\varphi') = 0$ *on the magnetostatic fields are mutually compatible conditions that, jointly, are sufficient to ensure that the Biot-Savart law formula*:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{R}^3} \boldsymbol{J}(\boldsymbol{r}') \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau'; \quad \nabla\cdot\boldsymbol{J}(\boldsymbol{r}) = 0; \tag{94}$$

provides well-defined and unique solutions to magnetostatic field equations.

Just as in the case of Coulomb law, although the strictly semi-strong condition of global localization of current distributions is a sufficient condition for Biot-Savart law to be well-defined, the fact is that it cannot be a necessary condition.

In fact, we have already mentioned some examples which make it quite clear that the Biot-Savart law provides well-defined and unique solutions even from *partially* localized charge distributions. In all these cases, *the global symmetries of the current distributions must imply symmetries for the magnetostatic field that, in one way or another, will be sufficient to satisfy the necessary conditions* (82–84), *in a manner compatible with the conditions of regularity.* Of course, in order to demonstrate that this is in fact the case, we have to proceed on a case-by-case basis.

By way of conclusion, in this section, we used the results obtained in sections 2 to 4 to show how Biot-Savart law can be deduced without explicitly defining the potentials. As in the electrostatic case, the alternative deduction allows us to include cases of global localization that the deduction based on Helmholtz theorem is unable to include.

8. Concluding Remarks

In this work, we conduct a systematic investigation about the relationship between the problem of localization of infinitely extended charge and current distributions and the problem of the validity of Coulomb and Biot-Savart laws, insofar as they must provide unique and well-defined solutions to the field equations of electromagnetostatics.

To this end, we developed the study by articulating three main axes: (i) the search for localization conditions sufficiently comprehensive to allow the improper integrals of Coulomb and Biot-Savart laws to be well-defined; (ii) the search for compatibility relationships between the regularity conditions (on the fields) and the localization conditions (on the distributions); and (iii) obtaining mathematical proofs for Coulomb and Biot-Savart laws that did not resort to the improper integral expressions for the potentials.

These deduction procedures led to a set of necessary conditions involving integrals of potentials and fields defined over closed surfaces located at infinity. Sufficient conditions over the fields for the satisfaction of these necessary conditions were then obtained, and it was found that they coincide with the regularity conditions on the fields. In this way, the formulas for Coulomb and Biot-Savart laws could be obtained that provide unique solutions, although the good definition of the improper integrals still depends on conditions to be imposed on the localization of charge and current distributions.

Finally, it was shown that a general class of sufficient localization conditions correspond to what we define as global localization in a strictly semi-strong sense. We have consistently demonstrated that this class is compatible with the fulfillment of regularity conditions.

However, this study is far from complete. Several issues have yet to be properly addressed. The two we consider most important are as follows.

The first question relates to a systematic study of the ways in which regularity conditions impose restrictions on the localization of infinitely extended distributions. We were able to find compatibilities, but a more complete study should be able to obtain more comprehensive sufficient conditions and, if possible, necessary conditions too.

Despite these limitations, this study was able to highlight an important result: the fundamental principle – which converts mathematical axioms into the axioms of a physical theory – implies restrictions on how infinite distributions can be arranged in space.

This may be intriguing at first. After all, in the case of absolutely localized distributions, the fundamental principle does not imply any restrictions on their spatial arrangement. However, in the case of infinitely extended distributions, if they are not localized in at least some sense (whether global or partial), there is a risk of violating the fundamental principle. This occurs for the simple reason that the fields – and consequently the forces – can cease to be uniquely determined and become asymptotically infinite. In physics, we understand that this cannot be tolerated in any way.

In summary, *infinite distribution models can be tolerated, but only to the extent that they are sufficiently localized so as not to violate the fundamental principle.*

The second question concerns the systematic study of distributions that are partially localized. In all the cases of partial localization we have defined in this work, we know that Coulomb and Biot-Savart laws can be effectively integrated and provide solutions for fields that satisfy the regularity conditions. And we also know that this is due to the global symmetries they exhibit. Therefore, it remains to be demonstrated that the necessary conditions given in this work are actually satisfied by symmetric partially localized distributions and to determine the role that symmetries play in these demonstrations. This study is in progress and will be the subject of a future publication.

Acknowledges

The author acknowledges the important contribution that the objections raised and comments made by the reviewers brought to the revision of the original manuscript. They led to modifications that allowed for a substantial improvement of the final work.

Appendix A

Before addressing the main points, it is convenient to establish three important results whose proofs, based on the definition of the improper limit of a function, are immediate. Suppose that $f(r)$ and $g(r)$ are both asymptotically continuous and positive functions. In these conditions:

1. If $\lim_{r\to\infty} \frac{f(r)}{g(r)} = 0$, then there is sufficiently large positive constant $a > 0$ from which $f(r) < g(r)$. In this case, we say that $f(r)$ is asymptotically dominated by $g(r)$;
2. If $\lim_{r\to\infty} \frac{f(r)}{g(r)} \to \infty$, then there is sufficiently large positive constant $a > 0$ from which $f(r) > g(r)$. In this case, we say that $f(r)$ dominates $g(r)$ asymptotically;
3. If $\lim_{r\to\infty} \frac{f(r)}{g(r)} = b$, where b is a constant, then $f(r)$ and $g(r)$ have the same asymptotic behavior, denoted by $f(r) \sim g(r)$.

The first result is obtained by recognizing that $\lim_{r\to\infty} \frac{f(r)}{g(r)} = 0$ is equivalent to the following statement: $\forall \varepsilon > 0 \exists a > 0$ such as $r > a \Rightarrow \left|\frac{f(r)}{g(r)}\right| < \varepsilon$. So, choosing $\varepsilon = 1$, with $f(r)$ and $g(r)$ asymptotically positive, we have: $\exists a > 0$ such as $r > a \Rightarrow \frac{f(r)}{g(r)} < 1$, from which we conclude that $f(r) < g(r)$, from a sufficiently large positive constant $a > 0$.

The second result is obtained by recognizing that $\lim_{r\to\infty} \frac{f(r)}{g(r)} \to \infty$ is equivalent to the following statement: $\forall M > 0 \exists a > 0$ such as $r > a \Rightarrow \left|\frac{f(r)}{g(r)}\right| > M$. So, choosing $M = 1$, with $f(r)$ and $g(r)$ asymptotically positive, we have: $\exists a > 0$ such as $r > a \Rightarrow \frac{f(r)}{g(r)} > 1$, from which we conclude that $f(r) > g(r)$, from a sufficiently large positive constant $a > 0$.

The Improper Integral $\int_a^{r\to\infty} f(r)dr$

That said, suppose that $f(r)$ is a bounded function throughout the interval $0 \leq r < \infty$, and that its asymptotic behavior is such that it decays to zero at infinity, that is, $\lim_{r\to\infty} f(r) = 0$. Since we are interested exclusively in the asymptotic behavior of $f(r)$, let us assume that, from a sufficiently large positive constant $a > 0$, $f(r)$ is a continuous function. Let us assume also that $f(r)$ is asymptotically positive[17].

[17] The assumption that the function $f(r)$ is positive is important because if a function is absolutely integrable on the interval $[a, \infty)$,

Given these conditions, let's investigate briefly the problem involving necessary and sufficient conditions for the convergence of the following improper integral:

$$\lim_{r\to\infty}\int_a^r f(r)dr \equiv \int_a^{r\to\infty} f(r)dr. \qquad (A.1)$$

First, we will demonstrate that one (among infinite) *necessary condition* to the convergence of (A.1) is that the function $f(r)$ is asymptotically dominated by $1/r$, that is, that the function is submitted to the so-called *semi-strong condition* $\lim_{r\to\infty} f(r)r = 0$:

$$\int_a^{r\to\infty} f(r)dr < \infty \Rightarrow \lim_{r\to\infty} f(r)r = 0. \qquad (A.2)$$

To do this, it is enough to prove the contrapositive of this implication, namely that if function $f(r)$ is *not* asymptotically dominated by $1/r$, then $\int_a^{r\to\infty} f(r)dr$ diverges. This can be demonstrated as follows.

Suppose first that $\lim_{r\to\infty} f(r)r \to \infty$. This is the same as saying that $1/r$ is asymptotically dominated by $f(r)$, or that, at the limit $r\to\infty$, $\int_a^{r\to\infty} f(r)dr > \int_a^{r\to\infty}\frac{1}{r}dr$. The last integral diverges logarithmically, so does the first. This completes the first part of the demonstration.

Suppose now that $\lim_{r\to\infty} f(r)r \to b$, where b is a non-zero constant. Then, $f(r) \sim 1/r$, that is, $f(r)$ and $1/r$ have the same asymptotic behavior. Then, $\int_a^{r\to\infty} f(r)dr \sim \int_a^{r\to\infty}\frac{1}{r}dr \sim \ln(r)|_a^{r\to\infty} \to \infty$. This concludes the proof that the semi-strong condition on $f(r)$ is a necessary one to the convergence of the improper integral given by (A.1).

Let's consider now the problem involving *sufficiency*. The first thing we must recognize is that *it is not true that* $\lim_{r\to\infty} f(r)r = 0 \Rightarrow \int_a^{r\to\infty} f(r)dr < \infty$. To see this, it suffices to present a counterexample, namely the function $f(r) = \frac{1}{r\ln r}$, that satisfies the semi-strong condition, but whose improper integral (A.1) obviously does not converge. Therefore, the semi-strong condition is necessary, but it is not sufficient to convergence of the improper integral (A.1).

This tells us that the semi-strong condition is an upper bound for all possible sufficient conditions, but it is not a minimal one. In fact, repeating the same type of reasoning that was developed above, we can see that $\int_a^{r\to\infty} f(r)dr < \infty \Rightarrow \lim_{r\to\infty} f(r)r\ln(r) = 0$. So, to be dominated by the function $\frac{1}{r\ln r}$ provides another necessary (but not sufficient) condition for (A.1) – and one that yields an upper bound asymptotically smaller than that provided by the semi-strong condition, since $\frac{1}{r\ln r}$ is dominated by $\frac{1}{r}$.

Although it is not possible to establish a necessary condition that is minimal in the sense indicated, it is possible to find several classes of sufficient conditions, parameterized by a positive parameter ε, which come unlimitedly close to a given necessary condition when $\varepsilon\to 0$.

In fact, let's suppose the existence of a class of functions $g(r;\varepsilon)$ that are asymptotically continuous, positive, bounded and that decrease monotonically to zero, at infinity, such that, $\lim_{r\to\infty} g(r;\varepsilon) = 0$, for any $\varepsilon > 0$. Suppose too that $f(r)$ is asymptotically dominated by $g(r;\varepsilon)$ and, therefore, for any sufficiently large positive constants $c > a > 0$, we have $\int_a^c f(r)dr < \int_a^c g(r;\varepsilon)dr$. If we take $c\to\infty$, we have $\int_a^{r\to\infty} f(r)dr < \int_a^{r\to\infty} g(r;\varepsilon)dr$. In these conditions, to prove the convergence of (A.1) it is enough to prove the convergence of $\int_a^{r\to\infty} g(r;\varepsilon)dr$.

To this end, it is sufficient that the functions $g(r;\varepsilon)$ have bounded, continuous and smooth first-antiderivatives $G(r;\varepsilon)$, such that $\lim_{r\to\infty} G(r;\varepsilon) = b(\varepsilon)$, where $b(\varepsilon)$ are finite constants. Thus, we know that $\int_a^{r\to\infty} g(r;\varepsilon)dr = G(r;\varepsilon)|_a^{r\to\infty} < \infty$, from which we conclude that $\int_a^{r\to\infty} f(r)dr < \infty$.

Therefore, $g(r;\varepsilon)$ is a class of functions that provides sufficient conditions to the convergence of the improper integral (A.1):

$$\lim_{r\to\infty}\frac{f(r)}{g(r;\varepsilon)} = 0 \Rightarrow \int_a^{r\to\infty} f(r)dr < \infty. \qquad (A.3)$$

Since we are particularly interested in the semi-strong condition, a class of functions that provides sufficient conditions and approaches $1/r$ indefinitely from below is given by the inverse power functions $g(r;\varepsilon) = r^{-(1+\varepsilon)}$, with $\varepsilon > 0$ being an arbitrary positive constant that can be as small as we want:

$$\lim_{r\to\infty} f(r)r^{1+\varepsilon} = 0 \Rightarrow \int_a^{r\to\infty} f(r)dr < \infty; \quad \forall\varepsilon > 0. \qquad (A.4)$$

Each choice of the parameter ε provides a sufficient (but not necessary) condition for the convergence of (A.1). For the sake of clarity and accuracy, we will refer to the condition $\lim_{r\to\infty} f(r)r^{1+\varepsilon} = 0$, for any $\varepsilon > 0$, as the *strictly semi-strong condition.*

Thus, by way of conclusion, we can establish that *the strictly semi-strong condition* (A.4) *is a sufficient (but not necessary) condition for the convergence of* (A.1), *which approaches the semi-strong condition indefinitely, from below, which in turn is necessary (but not sufficient) for the convergence of* (A.1).

The Improper Integral $\int_a^{r\to\infty} f(r)r^2dr$

The same line of reasoning can be used to address the necessary and sufficient conditions for the convergence of the improper integral:

$$\lim_{r\to\infty}\int_a^r f(r)r^2dr \equiv \int_a^{r\to\infty} f(r)r^2dr. \qquad (A.5)$$

with $a > 0$ a sufficiently large positive constant. As before, $f(r)$ will be assumed as a bounded function

then it is integrable on this interval. Note that the converse is not necessarily true.

throughout the interval $0 \leq r < \infty$. We assume either that $f(r)$ is asymptotically continuous, positive and that it decays to zero at infinity.

As before, we will demonstrate that one (among infinite) *necessary condition* to the convergence of (A.5) is that the function $f(r)$ is asymptotically dominated by $1/r^3$, behavior that we will call *super-strong condition* $\lim\limits_{r\to\infty} f(r)r^3 = 0$:

$$\int_a^{r\to\infty} f(r)r^2 dr < \infty \Rightarrow \lim_{r\to\infty} f(r)r^3 = 0. \qquad \text{(A.6)}$$

To do this, it is enough to prove the contrapositive of this implication, namely that if function $f(r)$ is *not* asymptotically dominated by $1/r^3$, then $\int_a^{r\to\infty} f(r)r^2 dr$ diverges.

Suppose first that $\lim\limits_{r\to\infty} f(r)r^3 \to \infty$. This is the same as saying that, at the limit $r \to \infty$, $f(r) > 1/r^3 \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr > \int_a^{r\to\infty} \frac{1}{r} dr$. The last integral diverges logarithmically, so does the first.

Suppose now that $\lim\limits_{r\to\infty} f(r)r^3 \to b$, where b is a non-zero constant, such that, $f(r) \sim 1/r^3$. Then, $\int_a^{r\to\infty} f(r)r^2 dr \sim \int_a^{r\to\infty} \frac{1}{r} dr \sim \ln(r)|^{r\to\infty} \to \infty$. This concludes the proof that the super-strong condition on $f(r)$ is a necessary one to the convergence of the improper integral given by (A.5).

Let's consider now the problem involving *sufficiency*. Again, *it is not true that* $\lim\limits_{r\to\infty} f(r)r^3 = 0 \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < \infty$. To see this, it suffices to take the function $f(r) = \frac{1}{r^3 \ln r}$, that satisfies the super-strong condition, but whose improper integral (A.5) does not converge. So, the super-strong condition is necessary, but it is not sufficient to convergence of (A.5).

The super-strong condition is an upper bound for all possible sufficient conditions, but, as before, it is not a minimal one. Repeating the same type of reasoning, we can see that $\int_a^{r\to\infty} f(r)r^2 dr < \infty \Rightarrow \lim\limits_{r\to\infty} f(r)r^3 \ln(r) = 0$. So, to be dominated by the function $\frac{1}{r^3 \ln r}$ provides a stricter necessary (but still not sufficient) condition for (A.5).

Although it is not possible to establish a necessary condition that is minimal, it is possible to find several classes of sufficient conditions, parameterized by a positive parameter ε, which come unlimitedly close to a given necessary condition when $\varepsilon \to 0$.

Let's suppose the existence of a class of functions $g(r;\varepsilon)$ that are asymptotically continuous, positive, bounded and that decrease monotonically to zero, at infinity, such that, $\lim\limits_{r\to\infty} g(r;\varepsilon) = 0$, for any $\varepsilon > 0$. Suppose too that $f(r)$ is asymptotically dominated by $g(r;\varepsilon)$ and, therefore, for any sufficiently large positive constants $c > a > 0$, we have $\int_a^c f(r)r^2 dr < \int_a^c g(r;\varepsilon)r^2 dr$. If we take $c \to \infty$, we have $\int_a^{r\to\infty} f(r)r^2 dr < \int_a^{r\to\infty} g(r;\varepsilon)r^2 dr$. In these conditions, to prove the convergence of (A.5) it is enough to prove the convergence of $\int_a^{r\to\infty} g(r;\varepsilon)r^2 dr$.

To this end, it is sufficient that the functions $g(r;\varepsilon)$ have bounded, continuous and smooth third-antiderivatives $G(r;\varepsilon)$, such that $\lim\limits_{r\to\infty} G(r;\varepsilon) = b(\varepsilon)$, $\lim\limits_{r\to\infty} G(r;\varepsilon)r = c(\varepsilon)$ and $\lim\limits_{r\to\infty} G(r;\varepsilon)r^2 = d(\varepsilon)$, where $b(\varepsilon)$, $c(\varepsilon)$ and $d(\varepsilon)$ are finite constants.

Thus, we know that:

$$\begin{aligned}
&\int_a^{r\to\infty} g(r;\varepsilon)r^2 dr \\
&\quad = \int_a^{r\to\infty} G'''(r;\varepsilon)r^2 dr \\
&\quad = G''(r;\varepsilon)r^2|_a^{r\to\infty} - \int_a^{r\to\infty} G''(r;\varepsilon)(2r)dr \\
&\quad = G''(r;\varepsilon)r^2|_a^{r\to\infty} \\
&\qquad - \left\{2G'(r;\varepsilon)r|_a^{r\to\infty} - \int_a^{r\to\infty} 2G'(r;\varepsilon)dr\right\} \\
&\quad = G''(r;\varepsilon)r^2|_a^{r\to\infty} \\
&\qquad - 2G'(r;\varepsilon)r|_a^{r\to\infty} + 2G(r;\varepsilon)|_a^{r\to\infty} < \infty. \qquad \text{(A.7)}
\end{aligned}$$

from which we conclude that $\int_a^{r\to\infty} f(r)r^2 dr < \infty$.

Therefore, $g(r;\varepsilon)$ is a class of functions that provides sufficient conditions to the convergence of the improper integral (A.5):

$$\lim_{r\to\infty} \frac{f(r)}{g(r;\varepsilon)} = 0 \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < \infty. \qquad \text{(A.8)}$$

Since we are particularly interested in the super-strong condition, a class of functions that provides sufficient conditions and approaches $1/r^3$ without limit is given by the inverse power functions $g(r;\varepsilon) = r^{-(3+\varepsilon)}$, with $\varepsilon > 0$ being an arbitrary positive constant that can be as small as we want:

$$\lim_{r\to\infty} f(r)r^{3+\varepsilon} = 0 \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < \infty; \quad \forall\varepsilon > 0. \qquad \text{(A.9)}$$

Each choice of the parameter ε provides a sufficient (but not necessary) condition for the convergence of (A.5). For the sake of clarity and accuracy, we will refer to the condition $\lim\limits_{r\to\infty} f(r)r^{3+\varepsilon} = 0$, for any $\varepsilon > 0$, as the *strictly super-strong condition*.

Thus, by way of conclusion, we can establish that *the strictly super-strong condition* (A.9) *is a sufficient (but not necessary) condition for the convergence of* (A.5), *which approaches the super-strong condition indefinitely, which in turn is necessary (but not sufficient) for the convergence of* (A.5).

To finish, we will demonstrate two last results that will be very useful. Instead of requiring the convergence of the integral (A.5), let us look for a sufficient condition for it to *possibly converge* or *to diverge*, but *slower than* r^2. We can easily prove that sufficient conditions for that are that $f(r)$ satisfies the *semi-strong* or the *strictly semi-strong* conditions, but now, with $0 < \varepsilon \leq 2$, that is, for both conditions:

$$\lim_{r\to\infty} f(r)r^{1+\varepsilon} = 0 \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < r^2|_a^{r\to\infty}; \quad 0 \le \varepsilon \le 2. \tag{A.10}$$

Indeed, from a sufficiently large positive constant $a > 0$, we have, for $0 \le \varepsilon < 2$:

$$\begin{aligned}
&\lim_{r\to\infty} f(r)r^{1+\varepsilon} = 0 \\
&\quad \Rightarrow f(r) < r^{-(1+\varepsilon)} \Rightarrow f(r)r^2 < r^{1-\varepsilon} \\
&\quad \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < \int_a^{r\to\infty} r^{1-\varepsilon} dr \\
&\quad \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < r^{2-\varepsilon}|_a^{r\to\infty} \\
&\quad \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < r^2|_a^{r\to\infty}.
\end{aligned} \tag{A.11}$$

For $\varepsilon = 2$, we have:

$$\begin{aligned}
&\lim_{r\to\infty} f(r)r^3 = 0 \\
&\quad \Rightarrow f(r) < r^{-3} \Rightarrow f(r)r^2 < r^{-1} \\
&\quad \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < \int_a^{r\to\infty} r^{-1} dr \\
&\quad \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < \ln r|_a^{r\to\infty} \\
&\quad \Rightarrow \int_a^{r\to\infty} f(r)r^2 dr < r^2|_a^{r\to\infty}.
\end{aligned} \tag{A.12}$$

By the way, note that, if $\varepsilon > 2$, (A.10) is also satisfied, but the integral will converge *necessarily*, because we reach the *strictly super-strong* condition.

Finally, if we look for a sufficient condition for the integral *to diverge*, but *as slowly as* r^2, it is also easy to verify that $f(r)$ should behave asymptotically like $1/r$, that is $f(r) \sim 1/r$. When a function decays more slowly or as slowly as $1/r$, we say that it satisfies the *weak condition.*

Appendix B

To discuss the good definition of Coulomb law, we can express the volume integral in spherical coordinates, taking care to establish an order between the integration of angular variables and the integration of the radial variable.

Let us first consider the order in which we take the integrals in the angular variables and only then consider the improper integration in the radial variable:

$$\begin{aligned}
&\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' \\
&= \int_0^{r'\to\infty} dr' \int d\Omega' \rho(r',\theta',\varphi') \frac{{r'}^2}{[{r'}^2 + A(\theta',\varphi')r' + r^2]^{\frac{3}{2}}} \\
&\quad \times \{[r'\sin\theta'\cos\varphi' - x]\hat{\boldsymbol{\imath}} \\
&\quad + [r'\sin\theta'\sin\varphi' - y]\hat{\boldsymbol{\jmath}} + [r'\cos\theta' - z]\hat{\boldsymbol{k}}\};
\end{aligned} \tag{B.1}$$

where $\boldsymbol{r} = (x, y, z)$ is a constant vector position with magnitude given by $r^2 = x^2 + y^2 + z^2$; $\int d\Omega' = \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi'$ represents the integration on a spherical shell of radius between r and $r + dr$; and $A(\theta',\varphi') = -2\{\sin\theta'[x\cos\varphi' + y\sin\varphi'] + z\cos\theta'\}$.

Now, we can write:

$$\frac{{r'}^2}{[{r'}^2 + A(\theta',\varphi')r' + r^2]^{\frac{3}{2}}} = \frac{1}{r'\left[1 + \frac{1}{r'}A(\theta',\varphi') + \frac{r^2}{{r'}^2}\right]^{\frac{3}{2}}}. \tag{B.2}$$

With (B.2), we can express (B.1) as:

$$\begin{aligned}
&\int_0^{r'\to\infty} dr' \int d\Omega' \rho(r',\theta',\varphi') \frac{1}{r'\left[1 + \frac{1}{r'}A(\theta',\varphi') + \frac{r^2}{{r'}^2}\right]^{\frac{3}{2}}} \\
&\quad \times \{[r'\sin\theta'\cos\varphi' - x]\hat{\boldsymbol{\imath}} \\
&\quad + [r'\sin\theta'\sin\varphi' - y]\hat{\boldsymbol{\jmath}} + [r'\cos\theta' - z]\hat{\boldsymbol{k}}\} \\
&\Rightarrow \int_0^{r'\to\infty} dr' \int d\Omega' \rho(r',\theta',\varphi') \frac{1}{\left[1 + \frac{1}{r'}A(\theta',\varphi') + \frac{r^2}{{r'}^2}\right]^{\frac{3}{2}}} \\
&\quad \times \left\{\left[\sin\theta'\cos\varphi' - \frac{x}{r'}\right]\hat{\boldsymbol{\imath}}\right. \\
&\quad \left. + \left[\sin\theta'\sin\varphi' - \frac{y}{r'}\right]\hat{\boldsymbol{\jmath}} + \left[\cos\theta' - \frac{z}{r'}\right]\hat{\boldsymbol{k}}\right\}.
\end{aligned} \tag{B.3}$$

Therefore, the improper integral of the Coulomb law can be expressed by:

$$\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' = \int_0^{r'\to\infty} \boldsymbol{f}(r';\boldsymbol{r}) dr'; \tag{B.4}$$

where the vector function $\boldsymbol{f}(r';\boldsymbol{r})$, given by:

$$\begin{aligned}
\boldsymbol{f}(r';\boldsymbol{r}) &= \int d\Omega' \rho(r',\theta',\varphi') \frac{1}{\left[1 + \frac{1}{r'}A(\theta',\varphi') + \frac{r^2}{{r'}^2}\right]^{\frac{3}{2}}} \\
&\quad \times \left\{\left[\sin\theta'\cos\varphi' - \frac{x}{r'}\right]\hat{\boldsymbol{\imath}}\right. \\
&\quad \left. + \left[\sin\theta'\sin\varphi' - \frac{y}{r'}\right]\hat{\boldsymbol{\jmath}} + \left[\cos\theta' - \frac{z}{r'}\right]\hat{\boldsymbol{k}}\right\};
\end{aligned} \tag{B.5}$$

represent the integral of the charge density performed on a spherical shell of radius between r' and $r' + dr'$.

To go any further, we should be careful. In the first place, we should verify if the Cartesian components of $\boldsymbol{f}(r';\boldsymbol{r})$ are bounded everywhere as a general first condition to the convergence of (B.4).

Next, we must verify if their asymptotic behaviors meet the necessary conditions for convergence, according to the discussion in the Appendix A. That is, it is necessary to verify if $\boldsymbol{f}(r';\boldsymbol{r})$ is sufficiently well defined and well behaved whenever $r' > a$, for any large constant $a > 0$. This means that we demand that the spherical shells of continuously increasing radius r' indeed define a function $\boldsymbol{f}(r';\boldsymbol{r})$ that is bounded everywhere, asymptotically continuous and globally localized, at least, in the semi-strong sense.

All of this depends exclusively on the angular dependence of the charge density, but we will not pursue any further in this direction, because that will not be necessary, as we will see soon.

Even so, there is another problem which, although it does not affect the convergence of the Coulomb law, is important as regards the question of the sufficient conditions for this. We are referring to the fact that the function $\boldsymbol{f}(r';\boldsymbol{r})$ can be *identically null*, beyond a certain radius r'.

This is exactly what *happens*, for example, if the infinitely extended charge density is *spherically symmetric*. In this case, the charge densities do not depend on the angular variables at all. Thus, we will have the result $\int_0^{\pi}\sin\theta' d\theta' \int_0^{2\pi} d\varphi' \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \equiv 0$, if we perform the integration at any spherical shell whose radius r' is bigger than r. Of course, this doesn't change the fact that the integration of the radial variable will be convergent. It just tells us that the argument to be developed below no longer applies – not least because the convergence of the Coulomb law is already guaranteed, in this case.

To proceed any further and avoid this last kind of situation, it is sufficient to demand from the charge density that the "loss" of its angular dependence occurs smoothly, in the limit $r' \to \infty$.

With these caveats in mind, let us take now the opposite approach to addressing the issue of convergence of (B.1), simply by taking the integrals over the angular variables and the improper integral over the radial variable in the commuted order of what was previously considered. So, from eq. (B.3):

$$\int d\Omega' \int_0^{r'\to\infty} dr' \rho(r',\theta',\varphi') \frac{1}{\left[1+\frac{1}{r'}A(\theta',\varphi')+\frac{r^2}{r'^2}\right]^{\frac{3}{2}}} \times \left\{ \left[\sin\theta'\cos\varphi' - \frac{x}{r'}\right]\hat{\boldsymbol{\imath}} + \left[\sin\theta'\sin\varphi' - \frac{y}{r'}\right]\hat{\boldsymbol{\jmath}} + \left[\cos\theta' - \frac{z}{r'}\right]\hat{\boldsymbol{k}} \right\}. \quad \text{(B.6)}$$

Therefore, the improper integral of the Coulomb law can be expressed by:

$$\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau' = \int \boldsymbol{g}(\theta',\varphi';\boldsymbol{r}) d\Omega'; \quad \text{(B.7)}$$

where the vector function $\boldsymbol{g}(\theta',\varphi';\boldsymbol{r})$, given by:

$$\boldsymbol{g}(\theta',\varphi';\boldsymbol{r}) = \int_0^{r'\to\infty} dr' \rho(r',\theta',\varphi') \frac{1}{\left[1+\frac{1}{r'}A(\theta',\varphi')+\frac{r^2}{r'^2}\right]^{\frac{3}{2}}} \times \left\{ \left[\sin\theta'\cos\varphi' - \frac{x}{r'}\right]\hat{\boldsymbol{\imath}} + \left[\sin\theta'\sin\varphi' - \frac{y}{r'}\right]\hat{\boldsymbol{\jmath}} + \left[\cos\theta' - \frac{z}{r'}\right]\hat{\boldsymbol{k}} \right\}; \quad \text{(B.8)}$$

exists if the improper integral of the charge density performed along any fixed direction of space given by (θ',φ') is convergent.

Now, remember that, for a charge density bounded everywhere, the convergence of (B.8) depends entirely on the behavior of the radial integral evaluated at his "tail"[18]:

$$\int_a^{r'\to\infty} dr' \rho(r',\theta',\varphi') \frac{1}{\left[1+\frac{1}{r'}A(\theta',\varphi')+\frac{r^2}{r'^2}\right]^{\frac{3}{2}}} \times \left\{ \left[\sin\theta'\cos\varphi' - \frac{x}{r'}\right]\hat{\boldsymbol{\imath}} + \left[\sin\theta'\sin\varphi' - \frac{y}{r'}\right]\hat{\boldsymbol{\jmath}} + \left[\cos\theta' - \frac{z}{r'}\right]\hat{\boldsymbol{k}} \right\}; \quad \text{(B.9)}$$

where $a > 0$ is a sufficiently large positive constant. An inspection of the integrand of eq. (B.9) shows that we can take a large enough so that it makes sense to take the limit $r' \to \infty$ to approximate the integral (B.9) as closely as we want to:

$$\{\sin\theta'\cos\varphi'\hat{\boldsymbol{\imath}} + \sin\theta'\sin\varphi'\hat{\boldsymbol{\jmath}} + \cos\theta'\hat{\boldsymbol{k}}\} \times \int_a^{r'\to\infty} \rho(r',\theta',\varphi') dr'. \quad \text{(B.10)}$$

From (B.10), we can see that the convergence of $\int_{V\to\mathbf{R}^3} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau'$ depends on the convergence of the following volume integral performed on spherical shells from a sufficiently large radius $a > 0$:

$$\int d\Omega' \left\{\sin\theta'\cos\varphi'\hat{\boldsymbol{\imath}} + \sin\theta'\sin\varphi'\hat{\boldsymbol{\jmath}} + \cos\theta'\hat{\boldsymbol{k}}\right\} \times \int_a^{r'\to\infty} \rho(r',\theta',\varphi') dr'. \quad \text{(B.11)}$$

This demonstrates, therefore, that the *good definition* of Coulomb law depends on the convergence of the improper integrals $\int_a^{r'\to\infty} \rho(r',\theta',\varphi') dr'$, evaluated for all fixed directions in space, where $a > 0$ is a sufficiently large positive constant, provided that the resulting functions:

$$g(\theta',\varphi';a) = \int_a^{r'\to\infty} \rho(r',\theta',\varphi') dr'; \quad \text{(B.12)}$$

be sufficiently well behaved in the intervals of definition of the angular variables.

Data Availability

The entire set of data supporting the results of this study was published in the article itself.

References

[1] A.C. Eringen and G.A. Maugin, *Electrodynamics of Continua I: Foundations and Solid Media* (Springer-Verlag, New York, 1990).

[18] Note, returning to (B.1), that there is no real problem at $r' = 0$, if we assume that the charge density is limited everywhere.

[2] W. Hauser, *Introduction to the Principles of Electromagnetism* (Addison-Wesley, Massachusetts, 1971).
[3] A. Zangwill, *Modern Electrodynamics* (Cambridge University Press, New York, 2012).
[4] A. Sommerfeld, *Electrodynamics: Lectures on Theoretical Physics* (Academic Press, New York, 1952), v. 3.
[5] C.A. Brau, *Modern Problems in Classical Electrodynamics* (Oxford University Press, Oxford, 2004).
[6] D. Radu and I. Merches, *Solved Problems in Classical Electrodynamics and Theory of Relativity* (CRC Press, Taylor and Francis Group, Boca Raton, 2023).
[7] D. Griffths, *Introduction to Electrodynamics* (Pearson Education, Upper Saddle River, 2013), 4 ed.
[8] E.M. Purcell and D.J. Morin, *Electricity and Magnetism* (Cambridge University Press, Cambridge, 2013).
[9] J.R. Reitz, F.J. Milford and R.W. Christy, *Foundations of Electromagnetic Theory* (Addison Wesley, Massachusetts, 1992).
[10] J.D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, Hoboken, 1999), 3 ed.
[11] M. Maggiore, *A Modern Introduction to Classical Electrodynamics* (Oxford University Press, Oxford, 2023).
[12] I.S. Grant and W.R. Phillips, *Electromagnetism* (John Wiley & Sons, Hoboken, 1975).
[13] H.C. Ohanian, *Classical Electrodynamics* (Allyn and Bacon, Boston, 1988).
[14] K. Milton and J. Schwinger, *Classical Electrodynamics* (CRC Press, Taylor and Francis Group, Boca Raton, 2024).
[15] K. Lechner, *Classical Electrodynamics: A Modern Perspective* (Springer International Publishing, Cham, 2018).
[16] M.A. Heald and J.B. Marion, *Classical Electromagnetic Radiation* (Saunders College Publishing, Philadelphia, 1995).
[17] M. Chaichian, I. Merches, D. Radu and A. Tureanu, *Electrodynamics: An Intensive Course* (Springer-Verlag, Berlim, 2016).
[18] O.D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966).
[19] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, 1962).
[20] J.A. Stratton, *Electromagnetic Theory* (MacGraw-Hill Book Company, New York, 1941).
[21] P. Lorrain and D.R. Corson, *Electromagnetism: Principles and Applications* (W.H. Freeman and Company, San Francisco, 1979).
[22] P. Lorrain and D.R. Corson, *Electromagnetic Fields and Waves* (W.H. Freeman and Company, New York, 1988).
[23] W.V. Rosser, *Interpretation of Classical Electromagnetism* (Kluver Academic Publishers, Dordrecht, 1997).
[24] W. Greiner, *Classical Electrodynamics* (Springer-Verlag, New York, 1998).
[25] J.A. Heras, Am. J. Phys. **63**, 10 (1995).
[26] O.D. Jefimenko, Eur. J. Phys. **25**, 287 (2004).
[27] O.D. Jefimenko, Am. J. Phys. **60**, 890 (1992).
[28] D.J. Griffiths and M.A. Heald, Am. J. Phys. **59**, 2 (1991).
[29] T. Ton, Am. J. Phys. **59**, 6 (1991).
[30] K.T. McDonald, Am. J. Phys. **65**, 11 (1997).
[31] E. Nelson, Proc. Amer. Math. Soc. **12**, 995 (1961).

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2025-0204