Positive Polynomials on Closed Boxes

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ABSTRACT. We present two different proofs that positive polynomials on closed boxes of $\mathbb{R}^2$ can be written as bivariate Bernstein polynomials with strictly positive coefficients. Both strategies can be extended to prove the analogous result for polynomials that are positive on closed boxes of $\mathbb{R}^n$, $n > 2$.

Keywords: positive polynomials, unit box, Bernstein polynomials.

1 INTRODUCTION

The goal of this paper is to show that real polynomials that are strictly positive on closed boxes have a representation with positive coefficients when written using Bernstein's polynomial basis. More specifically, we will prove the result for the unit box $I = [0,1] \times [0,1]$, i. e. we present new proofs for the following theorem:

Theorem 1.1. *If $p : \mathbb{R}^2 \to \mathbb{R}$ is such that*

$$p(x_1,x_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j}\, x_1^i x_2^j \tag{1.1}$$

and, for every $(x_1,x_2) \in I$, $p(x_1,x_2) > 0$, then there exist $q_1 \geq n_1, q_2 \geq n_2$ and $C_{i,j} > 0$, $(i,j) \in Q_1 \times Q_2$, such that

$$p(x_1,x_2) = \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} C_{i,j}\, x_1^i (1-x_1)^{q_1-i} x_2^j (1-x_2)^{q_2-j},$$

where $Q_1 = \{0,1,\ldots,q_1\}$ and $Q_2 = \{0,1,\ldots,q_2\}$. Furthermore, we constructively derive the values of q_1 and q_2.

Theorem 1.1 is an extension of similar results obtained for positive polynomials on compact intervals and multidimensional simplexes by, respectively, Bernstein [1], Hausdorff [4] and Pólya

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[6]. We are aware that, using a different proof strategy, Cassier [2] has proven a general result from which a similar version of Theorem 1.1 follows. We discuss this more extensively at the final section.

We provide two proofs of Theorem 1.1. The first one is supported by results for the univariate version of Theorem 1.1, proved by Powers and Reznick [7]. The second proof extends the approach in Garloff [3] and Rivlin [8].

The paper is organized as follows. Section 2 establishes notation and brings the relevant definitions used in the paper. In Section 3 we present the auxiliary results. These results are used in one of the proofs of Theorem 1.1, given in Section 4. Section 5 brings an alternative proof, based on [3] and [8].

2   DEFINITIONS AND NOTATION

Definition 1. *Let $\mathscr{P}_n$ be the linear space of polynomials of degree n, i.e.*

$$\mathscr{P}_n = \{p : \mathbb{R} \to \mathbb{R}, \ where \ \exists a_i \in \mathbb{R}, 0 \leq i \leq n : p(x) = \textstyle\sum_{i=0}^{n} a_i x^i\}.$$

Definition 2. *For any $p \in \mathscr{P}_n$ we define its Goursat transform $\tilde{p}$ by*

$$\tilde{p}(x) = (2x)^n p\left(\frac{1-x}{x}\right).$$

Definition 3. *Let $\mathscr{B}_n^+$ be the set of polynomials of degree n that can be written with non-negative coordinates in the Bernstein basis,*

$$\mathscr{B}_n^+ = \{p \in \mathscr{P}_n, \ where \ \exists A_i \geq 0 : p(x) = \textstyle\sum_{i=0}^{n} A_i \, x^i (1-x)^{n-i}\}.$$

Similarly, let $\mathscr{B}_n^{+,*}$ be the set of polynomials of degree n that can be written with positive coordinates in the Bernstein basis,

$$\mathscr{B}_n^{+,*} = \{p \in \mathscr{P}_n, \ where \ \exists A_i > 0 : p(x) = \textstyle\sum_{i=0}^{n} A_i \, x^i (1-x)^{n-i}\}.$$

Definition 4. *For every $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $m \geq n$ and $0 \leq i \leq m$, let*

$$A_{i,m}(a) = \sum_{j=0}^{\min(n,i)} \binom{m-j}{m-i} a_j.$$

Definition 5. *For every $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let*

$$B_k(a) = \sum_{(i,j)\in\mathbb{N}^2 : i-j=n-k} \left(2^n (-1)^j \binom{i}{j} a_i\right).$$

Notice that $B_k(a)$ is a linear combination of a.

Definition 6. *For each $0 \leq i \leq n_1$ and $0 \leq j \leq n_2$, let $a_{i,j} \in \mathbb{R}$. For each $0 \leq i \leq n_1$, define $a_i : \mathbb{R} \to \mathbb{R}$ as*

$$a_i(x_2) = \sum_{j=0}^{n_2} a_{i,j} x_2^j.$$

Also define $a(x_2) = (a_0(x_2), \ldots, a_{n_1}(x_2))$.

Definition 7. *For each $0 \leq i \leq n_1$ and $0 \leq j \leq n_2$, let $a_{i,j} \in \mathbb{R}$. For each $m \geq n_1$ and $0 \leq k \leq n_2$, define*

$$b_{k,i,m}(a) = \sum_{j=0}^{\min(n_1, i)} \binom{m-j}{m-i} a_{j,k}.$$

Also define $b_{i,m}(a) = (b_{0,i,m}(a), \ldots, b_{n_2,i,m}(a))$.

3   AUXILIARY RESULTS

Lemma 3.1. *If $p \in \mathscr{P}_n$, $p(x) = \sum_{i=0}^{n} a_i x^i$, then, for every $m \geq n$,*

$$p(x) = \sum_{i=0}^{m} A_i \, x^i (1-x)^{m-i}$$

if and only if

$$A_i = A_{i,m}(a), \quad a = (a_1, \ldots, a_n). \tag{3.1}$$

 Proof. Applying the Binomial Theorem to the identity $x^i = x^i(1-x+x)^{m-i}$, it follows that

$$x^i = \sum_{j=i}^{m} \binom{m-i}{j-i} x^j (1-x)^{m-j}.$$

From this expression, we obtain that

$$p(x) = \sum_{i=0}^{m} A_{i,m}(a) \, x^i (1-x)^{m-i}.$$

The proof that the A_i's are unique follows from observing that, $\{x^i(1-x)^{m-i} : 0 \leq i \leq m\}$ is a basis for $\mathscr{P}_m$. $\qquad\qquad\square$

The following theorem is a consequence of Theorem 6 in [7].

Theorem 3.2. *Let $p \in \mathscr{P}_n$ be such that $p(x) > 0$ for all $x \in [0,1]$. Let $\lambda = \min_{x \in [0,1]} p(x)$ and e_j be such that $\tilde{p}(x) = \sum_{i=0}^{n} e_j x^j$. If $q \geq 3n + \frac{2n^2 \max_j |e_j|}{\lambda} + 1$, then $p \in \mathscr{B}_q^+$.*

Lemma 3.3. *Let $p \in \mathscr{P}_n$ be such that $p(x) > 0$ for all $x \in [0,1]$. Let $\lambda = \min_{x \in [0,1]} p(x)$ and e_j be such that $\tilde{p}(x) = \sum_{i=0}^{n} e_j x^j$. Let $q = 3n + \lceil \frac{2n^2 \max_j |e_j|}{\lambda} \rceil + 1$, where $\lceil y \rceil = \min\{n \in \mathbb{N} : n \geq y\}$. Then, for every $q^* \geq 2q$, $p \in \mathscr{B}_{q^*}^{+,*}$.*

Proof. It follows from Theorem 3.2 that there exist $A_i \geq 0$ such that

$$p(x) = \sum_{i=0}^{q} A_i x^i (1-x)^{q-i}.$$

Note that

$$
\begin{aligned}
p(x) &= \sum_{i=0}^{q} A_i x^i (1-x)^{q-i} \\
&= \sum_{i=0}^{q} A_i x^i (1-x)^{q-i} (x+1-x)^{q^*-q} \\
&= \sum_{i=0}^{q} A_i x^i (1-x)^{q-i} \sum_{j=0}^{q^*-q} \binom{q^*-q}{j} x^j (1-x)^{q^*-q-j} \\
&= \sum_{k=0}^{q^*} \left(\sum_{l=\max(0,k+q-q^*)}^{\min(q,k)} \binom{q^*-q}{k-l} A_l \right) x^k (1-x)^{q^*-k}.
\end{aligned}
$$

Observe that, for every k, $\sum_{l=\max(0,k+q-q^*)}^{\min(q,k)} \binom{q^*-q}{k-l} A_l \geq \min(A_0, A_q) > 0$, since $A_0 = p(0) > 0$ and $A_q = p(1) > 0$. Therefore, $p \in \mathscr{B}_{q^*}^{+,*}$. $\square$

Lemma 3.4. *If $p(x) = \sum_{i=0}^{n} a_i x^i$ and $a = (a_1, \ldots, a_n)$, then*

$$\tilde{p}(x) = \sum_{k=0}^{n} B_k(a) x^k.$$

Proof.

$$
\begin{aligned}
\tilde{p}(x) &= (2x)^n p\left(\frac{1-x}{x} \right) \\
&= (2x)^n \sum_{i=0}^{n} a_i \left(\frac{1-x}{x} \right)^i \\
&= \sum_{i=0}^{n} 2^n a_i (1-x)^i x^{n-i} \\
&= \sum_{i=0}^{n} 2^n a_i \sum_{j=0}^{i} \binom{i}{j} (-1)^j x^{n-i+j} \\
&= \sum_{k=0}^{n} \sum_{(i,j):i-j=n-k} \left(2^n (-1)^j \binom{i}{j} a_i \right) x^k = \sum_{k=0}^{n} B_k(a) x^k.
\end{aligned}
$$

$\square$

4   PROOF OF THEOREM 1.1

The main idea behind this proof is to use twice the positive representation result for univariate polynomials (Lemma 3.3). For every fixed value in one of the coordinates of a bivariate polynomial, the function of the free coordinate is a univariate polynomial. This polynomial admits a positive Bernstein representation. Furthermore, the coefficients of this representation are univariate polynomials on the coordinate that was fixed, allowing another application of the positive Bernstein representation theorem for univariate polynomials. As a result of both applications, a positive Bernstein representation for the bivariate polynomial is obtained. This strategy can be extended by induction to arbitrary n-variate polynomials.

Proof. For a given $x_2 \in [0,1]$, obtain from definition 6 that

$$p_{x_2}(x_1) = p(x_1, x_2) = \sum_{i=0}^{n_1} a_i(x_2) x_1^i,$$

Thus, $p_{x_2} \in \mathscr{P}_{n_1}$ and $p_{x_2}(x_1) > 0$ for all $x_1 \in [0,1]$. From this observation, one can obtain two facts. First, since I is compact, then $\lambda = \inf_{(x_1,x_2)\in I^2} p(x_1, x_2) > 0$ and

$$\lambda_{x_2} = \inf_{x_1 \in [0,1]} p_{x_2}(x_1) \geq \lambda > 0. \tag{4.1}$$

Second, it follows from Lemma 3.4 that

$$\tilde{p}_{x_2}(x_1) := \sum_{i=0}^{n_1} B_i(a(x_2)) x_1^i.$$

Since each B_i is a linear combination of the elements of $a(x_2)$ and each element of $a(x_2)$ is a polynomial on x_2, $B_i(a(x_2))$ is a polynomial on x_2. Since $[0,1]$ is compact, there exists $L < \infty$ such that

$$\sup_{x_2 \in [0,1]} \max_i |B_i(a(x_2))| = L. \tag{4.2}$$

Therefore, it follows from Lemma 3.3 and Equations (4.1) and (4.2) that, taking $q_1 = 2\left(3n_1 + \left\lceil \frac{2n_1^2 \sup_{x_2 \in [0,1]} \max_i |B_i(a(x_2))|}{\inf_{(x_1,x_2)\in I^2} p(x_1,x_2)} \right\rceil + 1 \right)$, one obtains that, for all $x_2 \in [0,1]$, $p_{x_2} \in \mathscr{B}_{q_1}^{+,*}$. Therefore, it follows from Lemma 3.1 that, for all $x_2 \in [0,1]$,

$$p(x_1, x_2) = p_{x_2}(x_1) = \sum_{i=0}^{q_1} A_{i,q_1}(a(x_2)) x_1^i (1-x_1)^{q_1-i} \tag{4.3}$$

where $A_{i,q_1}(a(x_2)) > 0$. Notice that

$$A_{i,q_1}(a(x_2)) = \sum_{j=0}^{\min(n_1,i)} \binom{q_1-j}{q_1-i} a_j(x_2)$$

$$= \sum_{j=0}^{\min(n_1,i)} \binom{q_1-j}{q_1-i} \sum_{k=0}^{n_2} a_{j,k} x_2^k$$

$$= \sum_{k=0}^{n_2} \left(\sum_{j=0}^{\min(n_1,i)} \binom{q_1-j}{q_1-i} a_{j,k} \right) x_2^k = \sum_{k=0}^{n_2} b_{k,i,q_1}(a) x_2^k \in \mathscr{P}_{n_2}$$

It follows from Lemma 3.3 that, taking $q_2 = 2\left(3n_2 + \max_i \left\lceil \frac{2n_2^2 \max_j |B_j(b_{i,q_1}(a))|}{\inf_{x_2 \in I} A_{i,q_1}(a(x_2))} \right\rceil + 1\right)$, one obtains that

$$A_{i,q_1}(a(x_2)) = \sum_{j=0}^{q_2} C_{i,j}\, x_2^j (1-x_2)^{q_2-j}, 0 \le i \le q_1 \tag{4.4}$$

where $C_{i,j} > 0$. By applying Equation (4.4) to Equation (4.3), one obtains

$$p(x_1,x_2) = \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} C_{i,j}\, x_1^i (1-x_1)^{q_1-i} x_2^j (1-x_2)^{q_2-j}.$$

$\square$

5   ALTERNATIVE PROOF

We consider, as before, the bivariate polynomial p given in (1.1) and $\lambda = \inf_{(x_1,x_2) \in I} p(x_1,x_2)$. For $q_1, q_2 \ge 1$, let us define the bivariate polynomial

$$b_{k,l}^{(q_1,q_2)}(x_1,x_2) = \binom{q_1}{k} x_1^k (1-x_1)^{q_1-k} \binom{q_2}{l} x_2^l (1-x_2)^{q_2-l}, \tag{5.1}$$

where $k \in Q_1$ and $l \in Q_2$. The set of polynomials $\{b_{k,l}^{(q_1,q_2)}(x_1,x_2), k \in Q_1, l \in Q_2\}$ are the Bernstein polynomials of degree q_1 and q_2 and form a basis for the linear space of all bivariate polynomials of the form (1.1) with $n_1 = q_1$ and $n_2 = q_2$.

Lemma 5.1. *If $i \in Q_1$ and $j \in Q_2$, then*

$$x_1^i x_2^j = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} \frac{\binom{k}{i}\binom{l}{j}}{\binom{q_1}{i}\binom{q_2}{j}} b_{k,l}^{(q_1,q_2)}(x_1,x_2), \tag{5.2}$$

where it is assumed that $\binom{m}{v} = 0$ for integers m and v such that $m < v$.

Proof. The result follows by applying the Binomial Theorem to the identity $x_1^i x_2^j = x_1^i (1-x_1 + x_1)^{q_1-i} x_2^j (1-x_2+x_2)^{q_2-j}$. $\square$

Henceforth, we shall consider $q_1 \ge n_1, q_2 \ge n_2$. Then, it follows from Lemma 5.1 that $p(x_1,x_2)$ given in (1.1) can be rewritten as

$$p(x_1,x_2) = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} c_{k,l}^{q_1,q_2} b_{k,l}^{(q_1,q_2)}(x_1,x_2), \tag{5.3}$$

where

$$c_{k,l}^{q_1,q_2} = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j} \frac{\binom{k}{i}\binom{l}{j}}{\binom{q_1}{i}\binom{q_2}{j}}. \tag{5.4}$$

The $c_{k,l}^{(q_1,q_2)}$ are the Bernstein coefficients and (5.3) is the Bernstein form of $p(x_1,x_2)$. In the sequel, we denote by

$$c^{(q_1,q_2)} = \min_{(k,l)\in Q_1\times Q_2} c_{k,l}^{(q_1,q_2)}$$

the smallest Bernstein coefficient of $p(x_1,x_2)$.

Theorem 5.2. *If p is given by* (1.1)*, then*

$$\lambda - c^{(q_1,q_2)} \geq 0. \tag{5.5}$$

Proof. Since $b_{k,l}^{(q_1,q_2)}(x_1,x_2) \geq 0$ for all $(x_1,x_2) \in I$, then

$$\begin{aligned}
c^{(q_1,q_2)} &= \sum_{k=0}^{q_1}\sum_{l=0}^{q_2} c^{(q_1,q_2)} b_{k,l}^{(q_1,q_2)}(x_1,x_2) \\
&\leq \sum_{k=0}^{q_1}\sum_{l=0}^{q_2} c_{k,l}^{(q_1,q_2)} b_{k,l}^{(q_1,q_2)}(x_1,x_2) \\
&= p(x_1,x_2),
\end{aligned}$$

for all $(x_1,x_2) \in I$, which implies the assertion. $\square$

Theorem 5.3. *If p is given by* (1.1)*, $q_1 \geq n_1$ and $q_2 \geq n_2$, then*

$$\lambda - c^{(q_1,q_2)} \leq \gamma_1 \frac{(q_1-1)}{q_1^2} + \gamma_2 \frac{(q_2-1)}{q_2^2},$$

where

$$\gamma_1 = \frac{1}{2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} |a_{i,j}| i(i-1), \qquad \gamma_2 = \frac{1}{2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} |a_{i,j}| j(j-1).$$

Proof. For any real function $f(x_1,x_2)$, define its Bernstein approximation on I by

$$B_{q_1,q_2}(f;x_1,x_2) = \sum_{k=0}^{q_1}\sum_{l=0}^{q_2} f\left(\frac{k}{q_1},\frac{l}{q_2}\right) b_{k,l}^{(q_1,q_2)}(x_1,x_2). \tag{5.6}$$

For $0 \leq i \leq n_1$ and $0 \leq j \leq n_2$, let $\delta_{k,l}^{q_1,q_2}(i,j)$, $(k,l) \in Q_1 \times Q_2$, be the Bernstein coefficients of the polynomial $B_{q_1,q_2}(x_1^i x_2^j;x_1,x_2) - x_1^i x_2^j$, i.e.,

$$B_{q_1,q_2}(x_1^i x_2^j;x_1,x_2) - x_1^i x_2^j = \sum_{k=0}^{q_1}\sum_{l=0}^{q_2} \delta_{k,l}^{q_1,q_2}(i,j) b_{k,l}^{(q_1,q_2)}(x_1,x_2). \tag{5.7}$$

Then, from Lemma 5.1 and (5.6) , it follows that

$$\delta_{k,l}^{q_1,q_2}(i,j) = \left(\frac{k}{q_1}\right)^i \left(\frac{l}{q_2}\right)^j - \frac{\binom{k}{i}\binom{l}{j}}{\binom{q_1}{i}\binom{q_2}{j}}, \tag{5.8}$$

$k \in Q_1, l \in Q_2$.

For any fixed $0 \le i \le n_1$ and $0 \le j \le n_2$, we can prove that

$$0 \le \delta_{k,l}^{q_1,q_2}(i,j) \le \left(\frac{q_1-1}{q_1^2}\right)\frac{i(i-1)}{2} + \left(\frac{q_2-1}{q_2}\right)\frac{j(j-1)}{2}, \tag{5.9}$$

for all $k \in Q_1$ and $l \in Q_2$. In order to prove (5.9), it suffices to show that

$$0 \le \varphi_k^{q_1}(i) = \left(\frac{k}{q_1}\right)^i - \frac{\binom{k}{i}}{\binom{q_1}{i}} \le \left(\frac{q_1-1}{q_1^2}\right)\frac{i(i-1)}{2}, \qquad \text{for all } k \in Q_1, \tag{5.10}$$

$$0 \le \varphi_l^{q_2}(j) = \left(\frac{l}{q_2}\right)^j - \frac{\binom{l}{j}}{\binom{q_2}{j}} \le \left(\frac{q_2-1}{q_2^2}\right)\frac{j(j-1)}{2}, \qquad \text{for all } l \in Q_2. \tag{5.11}$$

Since (5.11) is essentially the same as (5.10), we only present the proof of (5.10). Notice that (5.10) clearly holds for $i = 0$, $i = 1$, $k = 0$ and $k = q_1$. Thus, let us consider $1 \le k \le q_1 - 1$ and $i \ge 2$.

If $k < i$, then

$$0 \le \varphi_k^{q_1}(i) = \left(\frac{k}{q_1}\right)^i \le \left(\frac{k}{q_1}\right)^2 \le \left(\frac{q_1-1}{q_1}\right)\left(\frac{i-1}{q_1}\right) \le \left(\frac{q_1-1}{q_1^2}\right)\frac{i(i-1)}{2}.$$

If $k \ge i$, then

$$\varphi_k^{q_1}(i) = \left(\frac{k}{q_1}\right)^i - \frac{\binom{k}{i}}{\binom{q_1}{i}} = \left(\frac{k}{q_1}\right)^i \left[1 - \prod_{r=0}^{i-1}\frac{(1-r/k)}{(1-r/q_1)}\right].$$

Since $0 \le (1-r/k) \le (1-r/q_1) \le 1$ for all $r = 0,\ldots,i-1$, it follows that

$$0 \le \varphi_k^{q_1}(i) \le \left(\frac{k}{q_1}\right)^i \left[1 - \prod_{r=0}^{i-1}\left(1-\frac{r}{k}\right)\right]. \tag{5.12}$$

Using the fact that, for any $z_1,\ldots,z_m \in [0,1]$, we have

$$\prod_{i=1}^{m}(1-z_i) \ge 1 - \sum_{i=1}^{m} z_i,$$

it follows from (5.12) that

$$0 \le \varphi_k^{q_1}(i) \le \left(\frac{k}{q_1}\right)^i \frac{i(i-1)}{2k} = \left(\frac{k}{q_1}\right)^{i-1}\frac{i(i-1)}{2q_1} \le \left(\frac{q_1-1}{q_1^2}\right)\frac{i(i-1)}{2}, \tag{5.13}$$

which finishes the proof of (5.10) and consequently proves (5.9).

Considering the form (1.1) of $p(x_1,x_2)$ and the Bernstein approximation (5.6), we obtain

$$B_{q_1,q_2}(p;x_1,x_2) - p(x_1,x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{i,j}\left[B_{q_1,q_2}(x_1^i x_2^j;x_1,x_2) - x_1^i x_2^j\right],$$

which implies, using (5.7),

$$B_{q_1,q_2}(p;x_1,x_2) - p(x_1,x_2) = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} \left(\sum_{i=0}^{n_1} \sum_{i=0}^{n_2} a_{i,j} \delta_{k,l}^{q_1,q_2}(i,j) \right) b_{k,l}^{q_1,q_2}(x_1,x_2). \tag{5.14}$$

Now, considering the form (5.3), we have

$$B_{q_1,q_2}(p;x_1,x_2) - p(x_1,x_2) = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} \left(p\left(\frac{k}{q_1}, \frac{l}{q_2} \right) - c_{k,l}^{q_1,q_2} \right) b_{k,l}^{q_1,q_2}(x_1,x_2). \tag{5.15}$$

Equating the Bernstein coefficients of expressions (5.14) and (5.15), and using (5.9), we conclude that

$$p\left(\frac{k}{q_1}, \frac{l}{q_2} \right) = c_{k,l}^{q_1,q_2} + \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j} \delta_{k,l}^{q_1,q_2}(i,j)$$

$$\leq c_{k,l}^{q_1,q_2} + \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} |a_{i,j}| \delta_{k,l}^{q_1,q_2}(i,j)$$

$$\leq c_{k,l}^{q_1,q_2} + \gamma_1 \frac{(q_1-1)}{q_1^2} + \gamma_2 \frac{(q_2-1)}{q_2^2},$$

from which follows the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

From Theorems 5.2 and 5.3, it follows that $c^{(q_1,q_2)} \to \lambda$ as $q_1 \to \infty$ and $q_2 \to \infty$ and, therefore, Theorem 1.1 follows as a corollary.

6   CONCLUDING REMARKS

The representation of polynomials that are positive on the unit interval or any compact subset of $\mathbb{R}^n$ is an important subject with direct applications related to moment problems. See [5] for more on this relation. The authors searched for the proof of Theorem 1.1 precisely to prove that the moment problem on the unit square has a solution—i.e. there is a finite representing measure for a sequence of moments—if and only if there is a positive linear functional for all polynomials that are nonnegative on the unit square. Not being aware of the work of Lasserre [5], where the result similar to the one we wanted to prove is demonstrated, we used the univariate results from Bernstein [1] and Hausdorff [4] as a stepping stone to build the proof for the unit square as described in Section 4.

Once our proof was concluded, we have found references [3] and [8], which provided a demonstration for a similar result. Eventually we came across the book by Lasserre [5], where we found a theorem that is similar to Theorem 1.1, proved by Cassier [2]. We briefly present such result, giving the formulation of [5]. Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ be the ring of real multivariate polynomials and $\mathbb{K}$ be a basic semi-algebraic set, subset of $\mathbb{R}^n$

$$\mathbb{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, j = 1, \ldots, m\}, \tag{6.1}$$

where $g_j(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, $j = 1, \ldots, m$. Cassier [2] has proven the following theorem.

Theorem 6.1. *Let $g_j(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ be affine for every $j = 1, \ldots, m$ and assume that $\mathbb{K}$, as defined by (6.1), is compact with nonempty interior. If $f \in \mathbb{R}[\mathbf{x}]$ is strictly positive on $\mathbb{K}$ then*

$$f = \sum_{\alpha \in \mathbb{N}^m} c_\alpha g_1^{\alpha_1} \ldots g_m^{\alpha_m},$$

for finitely many nonnegative scalars (c_α).

If $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $g_1(\mathbf{x}) = x_1$, $g_2(\mathbf{x}) = 1 - x_1$, $g_3(\mathbf{x}) = x_2$ and $g_4(\mathbf{x}) = 1 - x_2$, then $\mathbb{K} = [0,1] \times [0,1] = I$. When f is a positive polynomial on $\mathbb{K}$ the theorem applies and there are nonnegative c_α such that

$$f(x_1, x_2) = \sum_{\alpha \in \mathbb{N}^2} c_\alpha x_1^{\alpha_1} (1 - x_1)^{\alpha_2} x_2^{\alpha_3} (1 - x_2)^{\alpha_4}.$$

The main difference between the above Theorem and Theorem 1.1 is that the latter constructively derives the positive integers q_1 and q_2, the degrees of the Bernstein representation.

Both strategies developed in Sections 4 and 5 can be generalized to prove similar theorems for polynomials that are positive over arbitrary hypercubes.

RESUMO. Apresentamos duas demonstrações, por métodos diferentes, de que polinômios positivos em caixas fechadas no $\mathbb{R}^2$ podem ser escritos como polinômios de Bernstein bivariados com coeficientes estritamente positivos. Ambas as estratégias de demonstração podem ser estendidas para provar o resultado análogo para polinômios que são positivos em caixas fechadas no $\mathbb{R}^n$, $n > 2$.

Palavras-chave: polinômios positivos, hipercubo unitário, polinômios de Bernstein.

REFERENCES

[1] S.N. Bernstein. Sur la représentation des polynômes positifs. *Communications de la Société mathématique de Kharkow*, **14**(5) (1915), 227–228.

[2] G. Cassier. Problème des moments sur un compact de R^n et représentation de polynômes à plusieurs variables. *Journal of Functional Analysis*, **58** (1984), 256–266.

[3] J. Garloff. Convergent bounds for the range of multivariate polynomials. In "International Symposium on Interval Mathematics". Springer (1985), pp. 37–56.

[4] F. Hausdorff. Summationsmethoden und momentfolgen. I. *Mathematische Zeitschrift*, **9**(1-2) (1921), 74–109.

[5] J.B. Lasserre. "Moments, Positive Polynomials and their applications", volume 1. Imperial College Press (2006).

[6] G. Pólya. Über positive darstellung von polynomen. *Vierteljschr. Naturforsch. Ges. Zürich*, **73** (1928), 141–145.

[7] V. Powers & B. Reznick. Polynomials that are positive on an interval. *Transactions of the American Mathematical Society*, **352**(10) (2000), 4677–4692.

[8] T. Rivlin. Bounds on a polynomial. *Journal of Research of the National Bureau of Standards*, **74** (1970), 47–57.