Anais da Academia Brasileira de Ciências (2005) 77(4): 589–594
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

# Complementary Lagrangians in Infinite Dimensional Symplectic Hilbert Spaces

**PAOLO PICCIONE and DANIEL V. TAUSK**

Departamento de Matemática, Universidade de São Paulo
Rua do Matão 1010, 05508-900 São Paulo, SP, Brasil

*Manuscript received on May 19, 2005; accepted for publication on August 15, 2005; presented by* PAULO D. CORDARO

**ABSTRACT**

We prove that any countable family of Lagrangian subspaces of a symplectic Hilbert space admits a common complementary Lagrangian. The proof of this puzzling result, which is not totally elementary also in the finite dimensional case, is obtained as an application of the spectral theorem for unbounded self-adjoint operators.

**Key words:** symplectic Hilbert spaces, Lagrangian subspaces, Lagrangian Grassmannian, unbounded self-adjoint operators, spectral theorem.

## 1 INTRODUCTION

A *real symplectic Hilbert space* is a real Hilbert space $(V, \langle\cdot,\cdot\rangle)$ endowed with a symplectic form; by a *symplectic form* we mean a bounded anti-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$ that is represented by a (anti-self-adjoint) linear isomorphism $H$ of $V$, i.e., $\omega = \langle H\cdot,\cdot\rangle$. If $H = PJ$ is the polar decomposition of $H$ then $P$ is a positive isomorphism of $V$ and $J$ is an orthogonal complex structure on $V$; the inner product $\langle P\cdot,\cdot\rangle$ on $V$ is therefore equivalent to $\langle\cdot,\cdot\rangle$ and $\omega$ is represented by $J$ with respect to $\langle P\cdot,\cdot\rangle$. We may therefore replace $\langle\cdot,\cdot\rangle$ with $\langle P\cdot,\cdot\rangle$ and assume since the beginning that $\omega$ is represented by an orthogonal complex structure $J$ on $V$. A subspace $S$ of $V$ is called *isotropic* if $\omega$ vanishes on $S$ or, equivalently, if $J(S)$ is contained in $S^\perp$. A *Lagrangian subspace* of $V$ is a maximal isotropic subspace of $V$. We have that $L \subset V$ is Lagrangian if and only if $J(L) = L^\perp$. If $L \subset V$ is Lagrangian then a Lagrangian $L' \subset V$ such that $V = L \oplus L'$ is called a *complementary Lagrangian* to $L$. Obviously every Lagrangian $L$ has a complementary

Correspondence to: Paolo Piccione
E-mail: piccione@ime.usp.br
AMS Classification: 53D12.

Lagrangian, namely, its orthogonal complement $L^\perp$. Given a pair $L_1$, $L_2$ of Lagrangians, there are known sufficient conditions for the existence of a common complementary Lagrangian to $L_1$ and $L_2$ (see, for instance, Furutani 2004). In this paper we prove the following:

THEOREM. *If $(V, \langle\cdot,\cdot\rangle, \omega)$ is a real symplectic Hilbert space then any countable family of Lagrangian subspaces of $V$ has a common complementary Lagrangian.*

Associated to each pair of complementary Lagrangians $(L_0, L_1)$ one has a chart $\varphi_{L_0,L_1}$ on the Lagrangian Grassmannian $\Lambda$ whose domain is the set of Lagrangians complementary to $L_1$. Clearly, the charts of the form $\varphi_{L_0,L_1}$ constitute an atlas for $\Lambda$, as $(L_0, L_1)$ runs in the set of all pairs of complementary Lagrangians. Our Theorem implies that, *for fixed* $L_0$, the charts $\varphi_{L_0,L_1}$ also constitute an atlas for $\Lambda$, as $L_1$ runs in the set of Lagrangians complementary to $L_0$. This observation is essential, for instance, to the study of the singularities of the exponential map of infinite dimensional Riemannian manifolds (see Biliotti et al. 2004, Grossman 1965) and, more generally, to the study of spectral properties associated to (not necessarily Fredholm) pairs of curves of Lagrangians in symplectic Hilbert spaces.

The existence of a common complementary Lagrangian is proven first in the case of two Lagrangians $L$ and $L_1$ such that $L \cap L_1 = \{0\}$ (Corollary 4). In this case $L$ is the graph of a densely defined self-adjoint operator on $L_1^\perp$ (Lemma 1), and the result is obtained as an application of the spectral theorem (Lemma 2 and Lemma 3). The existence of a common complementary Lagrangian is then proven in the general case by a reduction argument (Proposition 5), and the final result is an application of Baire's category theorem.

The referee of this article suggested an alternative approach to the problem based on a complexification argument. The complex argumentation is standard in the recent literature (see, for instance, Booss-Bavnbek and Zhu 2005, Zhu 2001, Zhu and Long 1999). We discuss this approach in Section 3.

## 2 PROOF OF THE RESULT

In what follows, $(V, \langle\cdot,\cdot\rangle, \omega)$ will denote a real symplectic Hilbert space such that $\omega$ is represented by an orthogonal complex structure $J$ on $V$. We will denote by $\Lambda(V)$ the set of all Lagrangian subspaces of $V$. It follows from Zorn's Lemma that $V$ indeed has Lagrangian subspaces, i.e., $\Lambda(V) \neq \emptyset$. Given $L_0, L_1 \in \Lambda(V)$ then $(L_0 + L_1)^\perp = J(L_0 \cap L_1)$; in particular, $L_0 \cap L_1 = \{0\}$ if and only if $L_0 + L_1$ is dense in $V$. For $L \in \Lambda(V)$, we denote by $\mathcal{O}(L)$ the subset of $\Lambda(V)$ consisting of Lagrangians complementary to $L$. Given a real Hilbert space $\mathcal{H}$, we denote by $\mathcal{H}^{\mathbb{C}}$ the orthogonal direct sum $\mathcal{H} \oplus \mathcal{H}$ endowed with the orthogonal complex structure $J$ defined by $J(x, y) = (-y, x)$. If $A : D \subset \mathcal{H} \to \mathcal{H}$ is a densely defined linear operator on $\mathcal{H}$ then $J\big(\mathrm{gr}(A)^\perp\big) = \mathrm{gr}(A^*)$. It follows that $\mathrm{gr}(A)$ is Lagrangian in $\mathcal{H}^{\mathbb{C}}$ if and only if $A$ is self-adjoint; in this case, $\mathrm{gr}(A)$ is complementary to $\{0\} \oplus \mathcal{H}$ if and only if $A$ is bounded.

LEMMA 1. *Given $L \in \Lambda(\mathcal{H}^{\mathbb{C}})$ with $L \cap \big(\{0\} \oplus \mathcal{H}\big) = \{0\}$ then $L$ is the graph of a densely defined* *self-adjoint operator* $A : D \subset \mathcal{H} \to \mathcal{H}$.

PROOF. The sum $L + \big(\{0\} \oplus \mathcal{H}\big)$ is dense in $\mathcal{H}^{\mathbb{C}}$; thus, denoting by $\pi_1 : \mathcal{H}^{\mathbb{C}} \to \mathcal{H}$ the projection onto the first summand, we have that $D = \pi_1(L) = \pi_1\big(L + \big(\{0\} \oplus \mathcal{H}\big)\big)$ is dense in $\mathcal{H}$. Hence $L$ is the graph of a densely defined operator $A : D \to \mathcal{H}$, which is self-adjoint by the remarks above. □

Given Lagrangians $L_0, L_1 \in \Lambda(V)$ with $V = L_0 \oplus L_1$ then we have an isomorphism $\rho_{L_1,L_0} : L_1 \to L_0$ defined by $\rho_{L_1,L_0} = P_{L_0} \circ J|_{L_1}$, where $P_{L_0}$ denotes the orthogonal projection onto $L_0$. The map:

$$V = L_0 \oplus L_1 \ni x + y \longmapsto \big(x, -\rho_{L_1,L_0}(y)\big) \in L_0 \oplus L_0 = L_0^{\mathbb{C}} \tag{1}$$

is a symplectomorphism, i.e., it is an isomorphism that preserves the symplectic forms. Thus, we get a one-to-one correspondence $\varphi_{L_0,L_1}$ between Lagrangian subspaces $L$ of $V$ with $L \cap L_1 = \{0\}$ and densely defined self-adjoint operators $A : D \subset L_0 \to L_0$; more explicitly, we set $A = \varphi_{L_0,L_1}(L)$ if the map (1) carries $L$ to the graph of $-A$.

LEMMA 2. *Let* $L_0, L_1, L, L' \in \Lambda(V)$ *be Lagrangians such that* $L_0$ *and* $L'$ *are complementary to* $L_1$ *and* $L \cap L_1 = \{0\}$. *Set* $\varphi_{L_0,L_1}(L) = A : D \subset L_0 \to L_0$ *and* $\varphi_{L_0,L_1}(L') = A' : L_0 \to L_0$. *Then* $L'$ *is complementary to* $L$ *if and only if* $(A - A') : D \to L_0$ *is an isomorphism.*

PROOF. The map (1) carries $L$ and $L'$ respectively to $\mathrm{gr}(-A)$ and $\mathrm{gr}(-A')$. We thus have to show that $L_0^{\mathbb{C}} = \mathrm{gr}(-A) \oplus \mathrm{gr}(-A')$ if and only if $A - A'$ is an isomorphism. This follows by observing that $(x, y) = (u, -Au) + (u', -A'u')$ is equivalent to $\big(u + u', (A' - A)u\big) = (x, y + A'x)$, for all $x, y, u' \in L_0$, $u \in D$. □

LEMMA 3. *If* $A : D \subset \mathcal{H} \to \mathcal{H}$ *is a densely defined self-adjoint operator then for every* $\varepsilon > 0$ *there exists a bounded self-adjoint operator* $A' : \mathcal{H} \to \mathcal{H}$ *with* $\|A'\| \le \varepsilon$ *and such that* $(A - A') : D \to \mathcal{H}$ *is an isomorphism.*

PROOF. By the Spectral Theorem for unbounded self-adjoint operators, we may assume that $\mathcal{H} = L^2(X, \mu)$ and $A = M_f$, where $(X, \mu)$ is a measure space, $f : X \to \mathbb{R}$ is a measurable function and $M_f$ denotes the multiplication operator by $f$ defined on $D = \big\{\phi \in L^2(X, \mu) : f\phi \in L^2(X, \mu)\big\}$. In this situation, the operator $A'$ can be defined as $A' = M_g$, where $g = \varepsilon \cdot \chi_\varepsilon$ and $\chi_\varepsilon$ is the characteristic function of the set $f^{-1}\big([-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]\big)$; clearly $\|A'\| \le \|g\|_\infty = \varepsilon$. The conclusion follows by observing that $A - A' = M_{f-g}$, and $|f - g| \ge \frac{\varepsilon}{2}$ on $X$. □

COROLLARY 4. *Given* $L_1, L \in \Lambda(V)$ *with* $L_1 \cap L = \{0\}$ *then there exists a common complementary Lagrangian* $L' \in \Lambda(V)$ *to* $L_1$ *and* $L$.

PROOF. Set $L_0 = L_1^{\perp}$ and $A = \varphi_{L_0,L_1}(L)$. Lemma 3 gives us a bounded self-adjoint operator $A' : L_0 \to L_0$ with $A - A'$ an isomorphism. Set $L' = \varphi_{L_0,L_1}^{-1}(A')$; $L'$ is a Lagrangian complementary to $L_1$, because $A'$ is bounded. It is also complementary to $L$, by Lemma 2. □

If $V = V_1 \oplus V_2$ is an orthogonal direct sum decomposition into $J$-invariant subspaces $V_1$ and $V_2$, then $V_1$ and $V_2$ are symplectic Hilbert subspaces of $V$. Given subspaces $L_1 \subset V_1$ and $L_2 \subset V_2$ then $L_1 \oplus L_2$ is Lagrangian in $V$ if and only if $L_i$ is Lagrangian in $V_i$, for $i = 1, 2$. A Lagrangian subspace $L \in \Lambda(V)$ is of the form $L = L_1 \oplus L_2$ with $L_i \in \Lambda(V_i)$, $i = 1, 2$, if and only if $L$ is invariant by the orthogonal projection $P_{V_1}$ onto $V_1$. In this case, $L_i = P_{V_i}(L) = L \cap V_i$, $i = 1, 2$. If $S$ is a closed isotropic subspace of $V$ then a decomposition $V = V_1 \oplus V_2$ of the type above can be obtained by setting $V_1 = S \oplus J(S)$ and $V_2 = V_1^{\perp}$. Then, if $L \in \Lambda(V)$ contains $S$, it follows that $P_{V_1}(L) = S$; namely, $S \subset L$ implies $L \subset J(S)^{\perp}$ and $J(S)^{\perp}$ is invariant by $P_{V_1}$. Hence $L = S \oplus P_{V_2}(L)$.

PROPOSITION 5. *Given $L, L' \in \Lambda(V)$ then $\mathcal{O}(L) \cap \mathcal{O}(L') \neq \emptyset$.*

PROOF. Set $S = L \cap L'$, $V_1 = S \oplus J(S)$, and $V_2 = V_1^{\perp}$. Then $L = S \oplus P_{V_2}(L)$, $L' = S \oplus P_{V_2}(L')$, and $P_{V_2}(L) \cap P_{V_2}(L') = (L \cap V_2) \cap (L' \cap V_2) = \{0\}$. By Corollary 4, there exists a Lagrangian $R \in \Lambda(V_2)$ complementary to both $P_{V_2}(L)$ and $P_{V_2}(L')$ in $V_2$. Hence $J(S) \oplus R \in \Lambda(V)$ is in $\mathcal{O}(L) \cap \mathcal{O}(L')$. □

The map $L \mapsto P_L$ is a bijection from $\Lambda(V)$ onto the space of bounded self-adjoint maps $P : V \to V$ with $P^2 = P$ and $PJ + JP = J$. Such bijection induces a topology on $\Lambda(V)$ which makes it homeomorphic to a complete metric space. Moreover, for any $L_0, L_1 \in \Lambda(V)$ with $V = L_0 \oplus L_1$, the set $\mathcal{O}(L_1)$ is open in $\Lambda(V)$ and the map $\mathcal{O}(L_1) \ni L \mapsto \varphi_{L_0,L_1}(L)$ is a homeomorphism onto the space of bounded self-adjoint operators on $L_0$.

LEMMA 6. *For any $L_0 \in \Lambda(V)$, the set $\mathcal{O}(L_0)$ is dense in $\Lambda(V)$.*

PROOF. Given $L \in \Lambda(V)$, Proposition 5 gives us $L_1 \in \mathcal{O}(L_0) \cap \mathcal{O}(L)$. By Lemma 3, the bounded self-adjoint operator $A = \varphi_{L_0,L_1}(L)$ on $L_0$ is the limit of a sequence of bounded self-adjoint isomorphisms $A_n : L_0 \to L_0$. Hence the sequence $\varphi^{-1}_{L_0,L_1}(A_n)$ is in $\mathcal{O}(L_0)$ and it tends to $L$. □

PROOF OF THEOREM. Let $(L_n)_{n \geq 1}$ be a sequence in $\Lambda(V)$. Each $\mathcal{O}(L_n)$ is open and dense in $\Lambda(V)$, hence $\bigcap_{n=1}^{\infty} \mathcal{O}(L_n)$ is dense in $\Lambda(V)$, by Baire's category theorem. □

## 3 AN ALTERNATIVE PROOF OF THE RESULT VIA COMPLEXIFICATION

Let $(V, \langle\cdot,\cdot\rangle, \omega)$ denote a real symplectic Hilbert space such that $\omega$ is represented by an orthogonal complex structure $J$ on $V$. Let $V^{\mathbb{C}}$ denote the complexification of $V$, which is a complex Hilbert space endowed with the unique sesquilinear product $\langle\cdot,\cdot\rangle_{\bar{\mathbb{C}}}$ that extends $\langle\cdot,\cdot\rangle$. We denote by $J^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ the unique complex-linear extension of $J$, so that $\omega^{\bar{\mathbb{C}}} = \langle J^{\mathbb{C}}\cdot,\cdot\rangle_{\bar{\mathbb{C}}}$ is the unique sesquilinear extension of $\omega$ to $V^{\mathbb{C}}$. We have a direct sum decomposition $V^{\mathbb{C}} = Z_{\mathrm{h}} \oplus Z_{\mathrm{a}}$, where $Z_{\mathrm{h}} = \mathrm{Ker}(J^{\mathbb{C}} - i)$ and $Z_{\mathrm{a}} = \mathrm{Ker}(J^{\mathbb{C}} + i)$. The spaces $Z_{\mathrm{h}}$ and $Z_{\mathrm{a}}$ are $\omega^{\bar{\mathbb{C}}}$-orthogonal; moreover, the restriction of $i\omega^{\bar{\mathbb{C}}}$ to $Z_{\mathrm{h}}$ (resp., to $Z_{\mathrm{a}}$) is equal to $-\langle\cdot,\cdot\rangle_{\bar{\mathbb{C}}}$ (resp., equal to $\langle\cdot,\cdot\rangle_{\bar{\mathbb{C}}}$). By a *Lagrangian subspace* $L$ of $V^{\mathbb{C}}$ we mean a complex subspace $L$ of $V^{\mathbb{C}}$ which is equal to its $\omega^{\bar{\mathbb{C}}}$-orthogonal complement; equivalently, $L$ is Lagrangian if $J^{\mathbb{C}}(L)$ is equal to the $\langle\cdot,\cdot\rangle_{\bar{\mathbb{C}}}$-orthogonal complement of $L$ (we observe that every Lagrangian subspace of $V^{\mathbb{C}}$ is maximal $\omega^{\bar{\mathbb{C}}}$-isotropic, but the converse does not hold in the infinite-dimensional case). The Lagrangian subspaces of $V^{\mathbb{C}}$ are precisely the graphs of the complex-linear isometries $U : Z_{\mathrm{h}} \to Z_{\mathrm{a}}$. Given complex-linear isometries $U_1$, $U_2$ from $Z_{\mathrm{h}}$ to $Z_{\mathrm{a}}$ then their graphs are complementary subspaces of $V^{\mathbb{C}}$ if and only if $U_1 - U_2$ is an isomorphism. We have isomorphisms $\mathfrak{i}_{\mathrm{h}} : V \to Z_{\mathrm{h}}$, $\mathfrak{i}_{\mathrm{a}} : V \to Z_{\mathrm{a}}$ defined by $\mathfrak{i}_{\mathrm{h}}(x) = x - iJx$, $\mathfrak{i}_{\mathrm{a}}(x) = x + iJx$. The isomorphism $\mathfrak{i}_{\mathrm{h}}$ carries the complex structure $J$ of $V$ to the complex structure of $Z_{\mathrm{h}}$ (inherited from $V^{\mathbb{C}}$), while the isomorphism $\mathfrak{i}_{\mathrm{a}}$ carries $-J$ to the complex structure of $Z_{\mathrm{a}}$. We observe that $(V, \langle\cdot,\cdot\rangle)$ is the underlying real Hilbert space of a complex Hilbert space whose complex structure is $J : V \to V$ and whose Hermitian product $\langle\cdot,\cdot\rangle_*$ is given by $\langle\cdot,\cdot\rangle - i\omega(\cdot,\cdot)$. The isomorphism $\mathfrak{i}_{\mathrm{h}}$ carries $2\langle\cdot,\cdot\rangle_*$ to $\langle\cdot,\cdot\rangle_{\bar{\mathbb{C}}}$ and the isomorphism $\mathfrak{i}_{\mathrm{a}}$ carries the complex conjugate of $2\langle\cdot,\cdot\rangle_*$ to $\langle\cdot,\cdot\rangle_{\bar{\mathbb{C}}}$. Given a Lagrangian subspace $L_0$ of $V$ then $L_0$ is a real form of $(V, J)$ (i.e., $V = L_0 \oplus J(L_0)$) on which the Hermitian product $\langle\cdot,\cdot\rangle_*$ is real. Thus, the conjugation $\mathfrak{c} : V \to V$ corresponding to the real form $L_0$ (i.e., $\mathfrak{c}(x + Jy) = x - Jy$, $x, y \in L_0$) carries $J$ to $-J$ and $\langle\cdot,\cdot\rangle_*$ to the complex conjugate of $\langle\cdot,\cdot\rangle_*$. Hence each complex-linear isometry $U : Z_{\mathrm{h}} \to Z_{\mathrm{a}}$ can be identified with the unitary operator $T = \mathfrak{c} \circ \mathfrak{i}_{\mathrm{a}}^{-1} \circ U \circ \mathfrak{i}_{\mathrm{h}}$ on $V$ and the set of all Lagrangian subspaces of $V^{\mathbb{C}}$ can be identified with the set of all unitary operators on $V$. The Lagrangian $L_0$ that defines the conjugation $\mathfrak{c}$ corresponds to the identity operator of $V$. By what has been observed above, the Lagrangians corresponding to unitary operators $T_1 : V \to V$, $T_2 : V \to V$ are complementary to each other if and only if $T_1 - T_2$ is an isomorphism of $V$. Notice that the complexification $L^{\mathbb{C}}$ of a Lagrangian subspace $L$ of $V$ is a Lagrangian subspace of $V^{\mathbb{C}}$; moreover, the Lagrangian subspaces of $V^{\mathbb{C}}$ of the form $L^{\mathbb{C}}$ correspond to the unitary operators $T : V \to V$ whose self-adjoint components $\frac{1}{2}(T + T^*)$, $\frac{1}{2i}(T - T^*)$ preserve the real form $L_0$.

We can now give an alternative proof of Lemma 6, which implies our main result.

ALTERNATIVE PROOF OF LEMMA 6. It suffices to show that given $T : V \to V$ a unitary operator whose self-adjoint components preserve the real form $L_0$ and given $\varepsilon > 0$ then there exists another unitary operator $T' : V \to V$ whose self-adjoint components preserve $L_0$, with $\|T - T'\| < \varepsilon$ and such that $T' - \mathrm{Id}$ is an isomorphism. By the "real version" of the Spectral Theorem stated below, we may assume that $V = L^2(X, \mu)$, with $(X, \mu)$ a measure space and that $T$ is a multiplication operator $M_f$, with $f : X \to S^1$ a measurable function taking values in the unit circle $S^1$. Arguing as in the proof of Lemma 3, we may obtain a measurable function $g : X \to S^1$ such that $\|f - g\|_\infty < \varepsilon$ and such that 1 is not in the closure of the range of $g$. We then set $T' = M_g$. □

The following "real version" of the Spectral Theorem can be obtained easily from the standard proof of the complex Spectral Theorem for bounded normal operators.

SPECTRAL THEOREM. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{H}_0$ a real form of $\mathcal{H}$ (i.e., $\mathcal{H} = \mathcal{H}_0 \oplus i\mathcal{H}_0$) on which the Hilbert space Hermitian product of $\mathcal{H}$ is real. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded normal operator whose self-adjoint components*

$$\frac{1}{2}(T + T^*), \quad \frac{1}{2i}(T - T^*)$$

*preserve the real form $\mathcal{H}_0$. Then there exists a measure space $(X, \mu)$, an isometry $\phi$ from $\mathcal{H}$ to $L^2(X, \mu)$ that carries $\mathcal{H}_0$ to the set of real-valued functions on $X$ and such that $\phi \circ T \circ \phi^{-1}$ is a multiplication operator $M_f$, with $f : X \to \mathbb{C}$ a bounded measurable function.*

**ACKNOWLEDGMENTS**

The authors are partially sponsored by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação de Apoio à Pesquisa do Estado de São Paulo (FAPESP); they wish to thank Prof. Kenro Furutani for providing instructive suggestions on the topic. The authors also wish to thank the referee for suggesting the alternative proof of the main result given in Section 3.

**RESUMO**

Nós demonstramos que qualquer coleção enumerável de subespaços Lagrangeanos de um espaço de Hilbert simplético admite um subespaço Lagrangeano complementar. A prova desse intrigante resultado, que também no caso de dimensão finita não é totalmente elementar, é obtida como uma aplicação do teorema espectral para operadores auto-adjuntos ilimitados.

**Palavras-chave:** Espaços de Hilbert simpléticos, subespaços Lagrangeanos, Grassmanniano de Lagrangeanos, operadores auto-adjuntos ilimitados, teorema espectral.

**REFERENCES**

BILIOTTI L, EXEL R, PICCIONE P AND TAUSK D. 2004. On the Singularities of the Exponential Map in Infinite Dimensional Riemannian Manifolds. math.FA/0412108.

BOOSS-BAVNBEK B AND ZHU C. 2005. General Spectral Flow Formula for Fixed Maximal Domain. math.DG/0504125.

FURUTANI K. 2004. Fredholm–Lagrangian–Grassmannian and the Maslov index. J Geom Phys 51: 269–331.

GROSSMAN N. 1965. Hilbert manifolds without epiconjugate points. Proc Amer Math Soc 16: 1365–1371.

ZHU C. 2001. The Morse Index Theorem for Regular Lagrangian Systems. math.DG/0109117.

ZHU C AND LONG Y. 1999. Maslov-type index theory for symplectic paths and spectral flow. I. Chinese Ann Math Ser 20B: 413–424.