A stability result via Carleman estimates for an inverse source problem related to a hyperbolic integro-differential equation

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Abstract. First we prove a Carleman estimate for a hyperbolic integro-differential equation. Next we apply such a result to identify a spatially dependent function in a source term by an (additional) single measurement on the boundary.

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$ and let $\mathbf{v} = \mathbf{v}(x)$ be the outward unit normal vector to $\partial \Omega$ at $x$, $\partial_x u = \nabla u \cdot \mathbf{v}$. We consider a hyperbolic integro-differential equation:

$$
(Pu)(x, t) \equiv \partial_t^2 u(x, t) - p(x) \Delta u(x, t) - \int_0^t K(x, t, \eta) \Delta u(x, \eta)d\eta - L(u)(x, t) = F(x, t), \quad x \in \Omega, \ t > 0,
$$

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where
\[ L(u)(x, t) = \sum_{j=1}^{n} q_j(x) \partial_j u(x, t) + q_{n+1}(x) \partial_t u(x, t) + q_0(x)u(x, t) \]
\[ + \sum_{j=1}^{n} \int_{0}^{t} H_j(x, t, \eta) \partial_j u(x, \eta) d\eta \]
\[ + \int_{0}^{t} H_{n+1}(x, t, \eta) \partial_t u(x, \eta) d\eta \]
\[ + \int_{0}^{t} H_0(x, t, \eta) u(x, \eta) d\eta. \]

Here \( p \in C^2(\overline{\Omega}), p > 0 \) on \( \overline{\Omega}, q_j \in C(\overline{\Omega}), j = 0, \ldots, n+1, \) \( K \in C^2(\overline{\Omega} \times E(T)), \)
\( H_j \in C(\overline{\Omega} \times E(T)), j = 0, \ldots, n+1 \) such that \( \partial_t H_j \in C(\overline{\Omega} \times E(T)) \). Here we set \( E(T) = \{(t, \eta) \in \mathbb{R}^2 : 0 \leq \eta \leq t \leq T\} \).

We set
\[ x = (x_1, ..., x_n), \] \[ \partial_t = \frac{\partial}{\partial t}, \partial_j = \frac{\partial}{\partial x_j}, j = 1, 2, ..., n, \] \[ \nabla_{x,t} = (\nabla, \partial_t) = (\partial_1, ..., \partial_n, \partial_t), \] \[ \Delta = \sum_{j=1}^{n} \partial_j^2. \]

Equation (1.1) appears in various cases such as viscoelasticity.

One of the fundamental questions for (1.1) is the unique continuation: if \( u \) satisfies (1.1) and \( u = \partial_t u = 0 \) on \( \Gamma \times (0, T) \) where \( \Gamma \subset \partial \Omega \), then can we choose a neighbourhood \( U \subset \mathbb{R}^n \) of \( \Gamma \) and an interval \( I \subset (0, T) \) such that \( u = 0 \) in \( U \times I \)?

In order to prove the unique continuation and discuss applications to inverse problems, a Carleman estimate is a main tool. In this paper, we will establish a Carleman estimate for (1.1), and will apply it to determine an unknown source term. We stress that our result is the first step to determine \( x \)-dependent coefficients in (1.1). In a forthcoming paper we will discuss more general inverse problems.

In addition to the assumption that \( p \in C^2(\overline{\Omega}) \) and \( p(x) > 0 \) in \( \overline{\Omega} \), throughout this paper we suppose that there exists \( x_0 \in \mathbb{R}^n \setminus \overline{\Omega} \) such that
\[
\frac{1}{2} p(x)^2 - (\nabla p(x) \cdot (x - x_0)) \geq 0, \quad x \in \overline{\Omega}. \tag{1.2}
\]

We set
\[ \varphi(x, t) = |x - x_0|^2 - \beta t^2, \quad (1.3) \]
where \( \beta > 0 \) is a sufficiently small constant depending on \( \Omega, p, x_0 \). Furthermore, for a fixed \( R > 0 \) and any \( \varepsilon > 0 \), let
\[ Q(\varepsilon) = \left\{ (x, t) \in \Omega \times (0, \infty) : \varphi(x, t) > R^2 + \varepsilon \right\}, \]
\[ \Omega(\varepsilon) = \left\{ x \in \Omega : |x - x_0| > (R^2 + \varepsilon)^{1/2} \right\}. \quad (1.4) \]
Then we can show

**Theorem 1 (Carleman estimate).** Let \( u \in H^2(Q(\varepsilon)) \) satisfy (1.1) and
\[ u(x, 0) = 0 \quad \text{or} \quad \partial_t u(x, 0) = K(x, 0, 0) = 0, \quad x \in \Omega(0). \quad (1.5) \]
Then there exist \( s_0 > 0 \) and a constant \( C = C(s_0) > 0 \) independent of \( u \) such that
\[ \int_{Q(\varepsilon)} (s|\nabla_x u|^2 + s^3 u^2) e^{2s\varphi} dx \, dt \leq C \int_{Q(\varepsilon)} |F|^2 e^{2s\varphi} \, dx \, dt + Ce^{Cs} \|u\|_{1, \Sigma}^2, \quad (1.6) \]
for any \( s \geq s_0 \), where \( \Sigma = \partial Q(\varepsilon) \setminus (\Omega(\varepsilon) \times \{0\}) \) and
\[ \|u\|_{1, \Sigma}^2 = \int_{\partial Q(\varepsilon)(\Omega(\varepsilon) \times \{0\})} (|\nabla_x u|^2 + u^2) \, dS. \]

**Remark 1.** Condition \( K(x, 0, 0) = 0 \) in (1.5) can be erased if we are given the initial conditions \( u(x, 0) = \partial_t u(x, 0) = 0, \quad x \in \Omega(0) \).

**Remark 2.** In the weight function \( \varphi \), we have to choose \( \beta = \beta(\Omega, p, x_0) > 0 \) sufficiently small. In particular, if \( p = 1 \), then we can choose any \( \beta \in (0, 1) \) (e.g., [14], [20]).

Inequality (1.6) is called a Carleman estimate. Carleman estimates are well-known for elliptic, parabolic and hyperbolic operators (e.g., Hörmander [8],

Isakov [12]–[14], Klibanov and Timonov [20], Lavrent’ev, Romanov and Shishat’skiĭ [23]). However our system is involved with the integral term

$$\int_0^T K(x, t, \eta) \Delta u(x, \eta) d\eta,$$

so that a Carleman estimate for (1.1) is not found in the existing papers. In Yong and Zhang [31], an exact controllability problem is considered for a related system.

In order to treat the integral term (1.7), we have to assume the extra information (1.5). In other words, a usual Carleman estimate is proved for the extended domain

$$\{(x, t) \in \Omega \times [-T, T] : \varphi(x, t) > R^2 + \epsilon\},$$

but not for

$$\{(x, t) \in \Omega \times [0, T] : \varphi(x, t) > R^2 + \epsilon\}.$$ 

In order to apply a usual Carleman estimate to the inverse problem in \( t > 0 \), we should extend the solution \( u \) to \( t < 0 \). Such an extension requires an extra argument owing to (1.7). On the contrary, for an inverse problem over a time interval \((0, T)\) under (1.5), we need not extend \( u \) to \((-T, 0)\), and can directly apply our Carleman estimate (1.6). This kind of Carleman estimates in \( t > 0 \) is derived by a pointwise inequality in Klibanov and Timonov [20], Lavrent’ev, Romanov and Shishat’skiĭ [23], and is quite different from the Carleman estimates in Hörmander [8], Isakov [12]–[14], etc.

Next we will consider

The Inverse Source Problem. Let \( \epsilon > 0 \) be arbitrarily fixed and let \( r \in W^{1,\infty}(0, T; L^\infty(\Omega)) \) be a given function. Let us consider

$$ (Pu)(x, t) = r(x, t)f(x), \quad x \in \Omega, \quad 0 < t < T, \quad (1.8) $$

$$ u(x, 0) = \partial_t u(x, 0) = 0, \quad x \in \Omega. \quad (1.9) $$

Our task is to determine function \( f \in \Omega(\delta) \) with \( \delta > 0 \) from the knowledge of

$$ u|_{\Gamma \times (0, T)}, \quad \partial_{\nu} u|_{\Gamma \times (0, T)}. $$

Here \( \Gamma \) is an open subset of \( \partial \Omega \).
The problem to be solved is actually a sort of “double Cauchy” problem, since we are given Cauchy conditions on both \( t = 0 \) and \( \Gamma \). Note that we are given only “incomplete” boundary conditions, since no conditions on \( u \) and its derivatives are prescribed on the whole of \( \partial \Omega \).

Let us assume

\[ \overline{\Omega(0)} \subset \Omega \cup \Gamma. \]  \hfill (1.10)

We are ready to state the stability result for our inverse source problem.

**Theorem 2.** Let \( u \in C^3([0, T]; L^2(\Omega)) \cap C^2([0, T]; H^1(\Omega)) \cap C^1([0, T]; H^2(\Omega)) \) satisfy (1.8) and (1.9), and let us assume in addition to the regularity assumptions for the coefficients in (1.1) that \( \partial_t K \in C^2(\Omega \times E(T)) \). We further assume

\[ |r(x, 0)| > 0, \quad \text{for all} \quad x \in \overline{\Omega} \]  \hfill (1.11)

and

\[ T > \frac{\sup_{x \in \Omega(0)} |x - x_0|}{\sqrt{\beta}}. \]  \hfill (1.12)

Then for any \( \delta > 0 \), there exist two constants \( C = C(\Omega, T, p, x_0, \beta, \delta, r, R) > 0 \) and \( \kappa = \kappa(\Omega, T, p, x_0, \beta, \delta, r, R) \in (0, 1) \), \( \beta \) and \( R \) being as in (1.3) and (1.4), such that

\[
\| f \|_{L^2(\Omega(\delta))} \leq C \left( \| u \|_{H^1(\Omega(0))} + \| \partial_t u \|_{H^1(\Omega(0))} + \| f \|_{L^2(\Omega(0))} \right)^{1-\kappa} \\
\times \left( \| u \|_{H^1(\Gamma \times (0, T))} + \| \partial_t u \|_{H^1(\Gamma \times (0, T))} \right)^\kappa \\
+ C \left( \| u \|_{H^1(\Gamma \times (0, T))} + \| \partial_t u \|_{H^1(\Gamma \times (0, T))} \right). 
\]  \hfill (1.13)

The factor \( \left( \| u \|_{H^1(\Gamma \times (0, T))} + \| \partial_t u \|_{H^1(\Gamma \times (0, T))} \right) \) is the observation datum and (1.13) shows the stability of Hölder type which is conditional under an \textit{a priori} boundedness of \( \left( \| u \|_{H^1(\Omega(0))} + \| \partial_t u \|_{H^1(\Omega(0))} + \| f \|_{L^2(\Omega(0))} \right) \).

Theorem 2 is derived from Theorem 1 by means of the method created by Bukhgeim and Klibanov [3].

As related works on inverse problems by Carleman estimates, see Bellasseued [1], Bukhgeim [2], Imanuvilov and Yamamoto [9]–[11], Isakov [12]–[14], Khaidarov [18], Klibanov [19], Klibanov and Timonov [20], Klibanov and Yamamoto [21], Kubo [22], Yamamoto [30] and the references therein.

The novelty of this paper in comparison with the quoted ones, consists in:

(1) establishing a Carleman estimate for (1.1) with the integral term (Theorem 1).

(2) deriving a Hölder estimate for an unknown factor depending on $x$ in the source term of (1.8).

In particular, we can prove the Lipschitz stability for the unknown function $f$ in terms of the data measured on a suitably large part $\Gamma$ of $\partial \Omega$. The related proof follows some ideas contained in [9] and [10], and makes use of our Carleman estimate (Theorem 1). We stress that Theorem 1 is the starting point for establishing stability also for different inverse problems related to hyperbolic integro-differential equations, such as the determination of $p(x)$ in (1.1), which is physically important. For example, let $v = v(x, t)$ and $w = w(x, t)$ be the solutions to (1.1) corresponding respectively to the coefficients $p$ and $q$. Setting $u = v - w$, we obtain (1.1) where $F(x, t)$ is replaced by $(p(x) - q(x)) \Delta w(x, t)$. Then, on the basis of Theorem 1, we can apply an argument similar to the one used in [11] to prove the stability concerning $p(x)$. In a forthcoming paper, we discuss the details.

Different kinds of inverse problems, which consist in determining time-dependent factors in the kernel $K(x, t, \eta)$, are dealt with, e.g., in the papers by Cavaterra [4], Cavaterra and Grasselli [5], Cavaterra and Lorenzi [6], Janno and Lorenzi [15], Janno and von Wolfersdorf [16], Kabaniikhin and Lorenzi [17], Lorenzi [24], Lorenzi and Messina [25], [26], Lorenzi and Romanov [27], Lorenzi and Yahkno [28], von Wolfersdorf [29] and the references therein.

The rest of this paper is composed of two sections: in Section 2 we will prove Theorem 1, while Section 3 is devoted to the proof of Theorem 2.

2 Proof of Theorem 1

Henceforth $C > 0$ denotes generic constants which are independent of $s > 0$ and may vary from line to line. We first state a pointwise Carleman estimate for a hyperbolic operator (Theorem 2.2.4 in Klibanov and Timonov [20, pp. 45–46]). See also Lemma 2 in [23, p. 128] for the case of $p \equiv 1$ and Cheng, Isakov, Yamamoto and Zhou [7].
Theorem A. Let $p = p(x) \in C^2(\Omega)$ satisfy (1.2) and let $\beta > 0$ be sufficiently small. Then there exist constants $s_0 > 0$ and $C > 0$ such that

\[
( s|\nabla_x, t w(x, t)|^2 + s^3 |w(x, t)|^2 ) e^{2q(x, t)} + \text{div} \ U(x, t) + \partial_t V(x, t) \\
\leq C ( \beta^2 - p(x) \Delta ) w(x, t) |e^{2q(x, t)}), \quad (x, t) \in Q(\varepsilon)
\]

for all $s \geq s_0$ and $w \in C^2(Q(\varepsilon))$. Here $(U, V)$ is a vector-valued function and satisfies

\[
|U(x, t)| + |V(x, t)| \\
\leq C e^{2q(x, t)} ( s|\nabla_x, t w(x, t)|^2 + s^3 |w(x, t)|^2 ) , \quad (x, t) \in Q(\varepsilon).
\]

Moreover $V(x, 0) = 0, x \in \Omega(0)$ if $w(x, 0) = 0$ or $\partial_t w(x, 0) = 0, x \in \Omega(0)$.

Here we modify the statement of Theorem 2.2.4 in [20], the proof being essentially the same. Integrating the first inequality in the above theorem over $Q(\varepsilon)$ and making use of the properties of functions $U$ and $V$ in the proof of the same theorem, we obtain

Theorem B. Let $p = p(x) \in C^2(\Omega)$ satisfy (1.2), $\beta > 0$ be sufficiently small and $w(x, 0) = 0$ or $\partial_t w(x, 0) = 0, x \in \Omega(0)$. Then there exist constants $s_0 > 0$ and $C > 0$ such that

\[
\int_{Q(\varepsilon)} ( s|\nabla_x, t w|^2 + s^3 |w|^2 ) e^{2q} dx \, dt \\
\leq C \int_{Q(\varepsilon)} |( \beta^2 - p(x) \Delta ) w |^2 e^{2q} dx \, dt \\
+ C e^{Cs} \int_{\partial Q(\varepsilon) \setminus (\partial Q(\varepsilon) \cap \{ t = 0 \})} (|\nabla_x, t w|^2 + |w|^2 ) \, dS
\]

for any $s \geq s_0$ and any $w(x, t) \in C^2(Q(\varepsilon))$.

Set

\[
v(x, t) = p(x) u(x, t) + \int_0^t K(x, t, \eta) u(x, \eta) \, d\eta, \quad x \in \Omega, \ t > 0. \quad (2.1)
\]
Then from the formulæ
\[
\partial_t^2 v(x, t) = p(x) \partial_t^2 u(x, t) + \{\partial_t(K(x, t, t)) + \partial_t K(x, t, t)\} u(x, t)
\]
\[
+ K(x, t, t) \partial_t u(x, t) + \int_0^t \partial_t^2 K(x, t, \eta) u(x, \eta) \, d\eta,
\]
\[
\Delta v(x, t) = p(x) \Delta u(x, t) + \int_0^t K(x, t, \eta) \Delta u(x, \eta) \, d\eta
\]
\[
+ 2\nabla p(x) \cdot \nabla u(x, t) + u(x, t) \Delta p(x)
\]
\[
+ 2 \int_0^t \nabla K(x, t, \eta) \cdot \nabla u(x, \eta) \, d\eta + \int_0^t u(x, \eta) \Delta K(x, t, \eta) \, d\eta,
\]
we easily deduce that \( v \) solves the equation
\[
\partial_t^2 v(x, t) - p(x) \Delta v(x, t) = p(x) F(x, t) + p(x) L(u)(x, t)
\]
\[
+ \{\partial_t(K(x, t, t)) + \partial_t K(x, t, t) - p(x) \Delta p(x)\} u(x, t)
\]
\[
+ K(x, t, t) \partial_t u(x, t) - 2 p(x) \nabla p(x) \cdot \nabla u(x, t)
\]
\[
+ \int_0^t [\partial_t^2 K(x, t, \eta) - p(x) \Delta K(x, t, \eta)] u(x, \eta) \, d\eta
\]
\[
- 2 p(x) \int_0^t \nabla K(x, t, \eta) \cdot \nabla u(x, \eta) \, d\eta
\]
\[
= p(x) F(x, t) + L_1(u)(x, t), \quad x \in \Omega, \ t > 0,
\]
and the initial conditions
\[
v(x, 0) = 0 \quad \text{or} \quad \partial_t v(x, 0) = 0, \quad x \in \Omega(0). \quad \text{(2.3)}
\]

Here we note that \((\partial_t K)(x, t, t) = \partial_t K(x, t, \eta))|_{\eta=t}.

In terms of (2.3), we apply Theorem B to (2.2). Consequently there exists some positive constant \( s \geq s_0 \) such that, for \( s \geq s_0 \), we obtain
\[
\int_{Q(e)} (s|\nabla_x t v|^2 + s^3 v^2) e^{2\nu} \, dx \, dt
\]
\[
\leq C \int_{Q(e)} |pF|^2 e^{2\nu} \, dx \, dt \quad \text{(2.4)}
\]
\[
+ C \int_{Q(e)} |L_1(u)|^2 e^{2\nu} \, dx \, dt + Ce^{Cs} \|u\|_{(1, \Sigma)}^2
\]
where $\Sigma = \partial Q(\varepsilon) \setminus (\Omega(\varepsilon) \times \{0\})$.

By our assumptions on the coefficients and the kernels we deduce the estimate
\[
|L_1(u)(x, t)| \leq C \left(|\nabla_x u(x, t)| + |u(x, t)|\right) + C \int_0^t \left(|\nabla_x u(x, \eta)| + |u(x, \eta)|\right) d\eta. \tag{2.5}
\]

Consequently, from (2.4) we obtain, for $s \geq s_0$,
\[
\int_{Q(\varepsilon)} \left(s|\nabla_x v|^2 + s^3v^2\right) e^{2s\varphi} dx dt \leq C \int_{Q(\varepsilon)} |F|^2 e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} (|\nabla_x u|^2 + u^2) e^{2s\varphi} dx dt \\
+ C \int_{Q(\varepsilon)} \left(\int_0^t \left(|\nabla_x u(x, \eta)| + |u(x, \eta)|\right) d\eta\right)^2 e^{2s\varphi} dx dt \tag{2.6}
\]

We need now to show

**Lemma 1.**
\[
\int_{Q(\varepsilon)} \left(\int_0^t |w(x, \xi)| d\xi\right)^2 e^{2s\varphi} dx dt \leq \frac{C}{s} \int_{Q(\varepsilon)} |w(x, t)|^2 e^{2s\varphi} dx dt
\]
for all $w \in L^2(Q(\varepsilon))$.

Lemma 1 is fundamental in order to derive a Carleman estimate for our inverse problem. We note that it was proved in Bukhgeim and Klibanov [3], Klibanov [19], but with a factor not containing $1/s$. On the contrary, for our proof the factor $1/s$ is essential. As for the proof of Lemma 1, see Lemma 3.1.1 (pp.77–78) in [20]. However, for completeness, we will give the proof of it in Appendix.

By (2.1) and $p > 0$ on $\overline{\Omega}$, we obtain
\[
u(x, t) = \frac{1}{p(x)} v(x, t) - \int_0^t \frac{K(x, t, \eta)}{p(x)} u(x, \eta) d\eta. \tag{2.7}
\]

Hence, owing to Lemma 1, we have
\[
\int_{Q(\varepsilon)} u^2 e^{2s\varphi} dx dt \leq C \int_{Q(\varepsilon)} v^2 e^{2s\varphi} dx dt + \frac{C}{s} \int_{Q(\varepsilon)} u^2 e^{2s\varphi} dx dt.
\]

Taking \( s > s_0 \) sufficiently large, we can absorb the second term on the right hand side into the left hand side, and we have
\[
\int_{Q(\varepsilon)} u^2 e^{2s\phi} \, dx \, dt \leq C \int_{Q(\varepsilon)} v^2 e^{2s\phi} \, dx \, dt, \quad s \geq s_0. \tag{2.8}
\]

Similarly, from (2.7) we obtain
\[
\int_{Q(\varepsilon)} |\nabla x, t u|^2 e^{2s\phi} \, dx \, dt \leq C \int_{Q(\varepsilon)} (|\nabla x, t v|^2 + v^2) e^{2s\phi} \, dx \, dt, \quad s \geq s_0. \tag{2.9}
\]

Hence, substituting (2.8) and (2.9) into the left hand side of (2.6) and applying Lemma 1 to the third term on the right hand side of (2.6), we obtain
\[
\int_{Q(\varepsilon)} (s|\nabla x, t u|^2 + s^3 u^2) e^{2s\phi} \, dx \, dt
\]
\[
\leq C \int_{Q(\varepsilon)} (s|\nabla x, t v|^2 + (s + s^3)v^2) e^{2s\phi} \, dx \, dt
\]
\[
\leq C \int_{Q(\varepsilon)} (s|\nabla x, t v|^2 + s^3 v^2) e^{2s\phi} \, dx \, dt
\]
\[
\leq C \int_{Q(\varepsilon)} (|\nabla x, t u|^2 + u^2) e^{2s\phi} \, dx \, dt
\]
\[
+ C \int_{Q(\varepsilon)} F^2 e^{2s\phi} \, dx \, dt + Ce^{C_s} \|v\|_{(1), \Sigma}^2
\]
\[
\leq C \int_{Q(\varepsilon)} (|\nabla x, t u|^2 + u^2) e^{2s\phi} \, dx \, dt
\]
\[
+ C \int_{Q(\varepsilon)} F^2 e^{2s\phi} \, dx \, dt + Ce^{C_s} \|u\|_{(1), \Sigma}^2.
\] \tag{2.10}

In order to derive the last inequality, we used
\[
\|v\|_{(1), \Sigma}^2 \leq C \|u\|_{(1), \Sigma}^2 \tag{2.11}
\]

by (2.1). Taking again \( s > 0 \) sufficiently large, we absorb the first term on the right hand side into the left hand side at (2.10). Thus the proof of Theorem 1 is complete.
3 Proof of Theorem 2

The proof is based on the modification by Imanuvilov and Yamamoto [10] of the original method by Bukhgeim and Klibanov [3]. The main ideas of the proof are as follows:

(1) In order to apply the Carleman estimate, the functions under consideration have to vanish on a part of $\partial(\Omega \times (0, T))$ (see (1.5)). Therefore we introduce a cut-off function given by (3.2).

(2) After taking the $t$-derivative of $u$, an unknown function $f = f(x)$ appears in the initial value and the right hand side $J$ (see (3.11)).

(3) Applying the Carleman estimate with large parameter $s > 0$ to the $t$-differentiated equation, we can estimate the $L^2$-norm of $f(x)$ with the weight $e^{2\rho(x, 0)}$ by $|J|$ and suitable norms of the boundary data on $\Gamma \times (0, T)$ (see (3.16)–(3.17)).

(4) Thanks to the Carleman weight function, the coefficient of $|f(x)|^2$ in $J$ tends to 0 as $s \to \infty$. Thus the term of $f$ in $J$ can be absorbed, so that the proof is complete.

Although our proof originates from [3], the steps (3)–(4) are different and are more convenient for deriving an estimate which is global over the whole domain $\Omega$.

We can prove now Theorem 2. First we modify Theorem 1 as follows.

**Corollary 1.** Let $u \in H^2(Q(\varepsilon))$ satisfy (1.1) and $u(x, 0) = 0, x \in \Omega(\varepsilon)$. Then there exist $s_0 > 0$ and a constant $C = C(s_0) > 0$ independent of $u$ such that

$$
\int_{Q(\varepsilon)} (s|\nabla_{x,t} u|^2 + s^3 u^2) e^{2\rho} dx dt \leq C \int_{Q(\varepsilon)} |F|^2 e^{2\rho} dx dt + C e^{Cs} \int_{\partial Q(\varepsilon) \cap (\Gamma \times (0, \infty))} (|\nabla_{x,t} u|^2 + u^2) dS + Cs^3 e^{2\varepsilon(R^2+3\varepsilon)} \|u\|_{H^1(Q(\varepsilon))}^2
$$

for any $s \geq s_0$. 

Proof of Corollary 1. Let $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfy $0 \leq \chi \leq 1$ in $\mathbb{R}^{n+1}$ and

$$\chi(x, t) = \begin{cases} 1, & (x, t) \in Q(3\varepsilon), \\ 0, & (x, t) \in Q(\varepsilon) \setminus Q(2\varepsilon). \end{cases}$$

(3.2)

We set $v = \chi u$. Then $|v| = |\nabla_{x,t}v| = 0$ on $\partial Q(\varepsilon) \setminus \{(\Gamma \times (0, \infty)) \cup (\Omega(\varepsilon) \times \{0\})\}$ and $v = 0$ on $\Omega(\varepsilon)$. Therefore Theorem 1 yields

$$\int_{Q(\varepsilon)} (s|\nabla_{x,t}(\chi u)|^2 + s^3 |\chi u|^2) e^{2\varphi} dx dt \leq C \int_{Q(\varepsilon)} |F|^2 e^{2\varphi} dx dt$$

$$+Ce^{C_s} \int_{\partial Q(\varepsilon) \cap (\Gamma \times (0, \infty))} (|\nabla_{x,t}(\chi u)|^2 + |\chi u|^2) dS,$$

for any $s \geq s_0$. Since

$$\int_{Q(\varepsilon)} (s|\nabla_{x,t}u|^2 + s^3 u^2) e^{2\varphi} dx dt$$

$$= \left( \int_{Q(3\varepsilon)} + \int_{Q(\varepsilon) \setminus Q(3\varepsilon)} \right) (s|\nabla_{x,t}u|^2 + s^3 u^2) e^{2\varphi} dx dt$$

and $\chi = 1$ in $Q(3\varepsilon)$, $\varphi(x, t) \leq R^2 + 3\varepsilon$ for $(x, t) \in Q(\varepsilon) \setminus Q(3\varepsilon)$, we have

$$\int_{Q(\varepsilon)} (s|\nabla_{x,t}u|^2 + s^3 u^2) e^{2\varphi} dx dt$$

$$\leq \int_{Q(3\varepsilon)} (s|\nabla_{x,t}(\chi u)|^2 + s^3 |\chi u|^2) e^{2\varphi} dx dt + Cs^3 e^{2s(R^2 + 3\varepsilon)} \|u\|^2_{L^1(Q(\varepsilon))}.$$

Thus the proof of Corollary 1 follows from this inequality and (3.3).

Now we proceed to proving Theorem 2. By (1.12), we have $\beta T^2 > |x - x_0|^2$ for $x \in \Omega(0)$. Since $(x, t) \in Q(\varepsilon)$ implies that $x \in \Omega(0)$ and $|x - x_0|^2 - \beta t^2 > 0$, we have $0 < t < T$. Hence $Q(\varepsilon) \subset \Omega \times (0, T)$.

Let $u$ satisfy (1.8) and (1.9). For the sake of simplicity, we will make use of the shorthands:

$$\begin{aligned}
D &= \|u\|^2_{H^1(\Gamma \times (0, T))} + \|\partial_t u\|^2_{H^1(\Gamma \times (0, T))}, \\
M &= \|u\|^2_{H^1(\Omega)} + \|\partial_t u\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(\Omega)},
\end{aligned}$$

(3.4)

where $D$ is a quantity depending only on the data, while $M$ is related to the a priori bound of $u$ and $f$, needed to obtain the stability result (see (3.5) and (3.9)).
Applying Corollary 1 to (1.8), we obtain

\[
\int_{Q(\varepsilon)} (s|\nabla_x u|^2 + s^3 u^2) e^{2s\varphi} dx dt \leq C \int_{Q(\varepsilon)} |f|^2 e^{2s\varphi} dx dt + C e^{Cs^2} D + C s^3 e^{2s(3R^2 + \varepsilon)} M, \quad s \geq s_0.
\]

On the other hand, (1.8) yields

\[
\Delta u(x, t) = -\int_0^t K(x, t, \eta) \Delta u(x, \eta) d\eta + \frac{1}{p(x)} \partial_t^2 u(x, t) \quad - \frac{1}{p(x)} L(u)(x, t) - \frac{1}{p(x)} r(x, t) f(x), \quad (x, t) \in Q(\varepsilon).
\]

Therefore Lemma 1 implies

\[
\int_{Q(\varepsilon)} |\Delta u|^2 e^{2s\varphi} dx dt \leq \frac{C}{s} \int_{Q(\varepsilon)} |\Delta u|^2 e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} |\partial_t^2 u|^2 e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} f^2 e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} (|\nabla_x u|^2 + |u|^2) e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} f^2 e^{2s\varphi} dx dt.
\]

Hence, for \( s > 0 \) sufficiently large, we obtain

\[
\int_{Q(\varepsilon)} |\Delta u|^2 e^{2s\varphi} dx dt \leq C \int_{Q(\varepsilon)} |\partial_t^2 u|^2 e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} f^2 e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} (|\nabla_x u|^2 + |u|^2) e^{2s\varphi} dx dt + C \int_{Q(\varepsilon)} f^2 e^{2s\varphi} dx dt, \quad s \geq s_0.
\]

Next, setting \( w = \partial_t u \), by (1.1) with \( F = rf \) and (1.9) we have

\[
\partial_t^2 w(x, t) - p(x) \Delta w(x, t) = K(x, t, \eta) \Delta u(x, \eta) d\eta + \partial_t L(u)(x, t) + (\partial_t r)(x, t) f(x), \quad (x, t) \in Q(\varepsilon)
\]

and \( w(x, 0) = 0 \) for \( x \in \Omega(\varepsilon) \).
Noting that

\[ \partial_t L(u)(x, t) = \sum_{j=1}^{n} q_j(x) \partial_j \partial_t u(x, t) + q_{n+1}(x) \partial_t^2 u(x, t) + q_0(x) \partial_t u(x, t) + H_{n+1}(x, t) \partial_t u(x, t) + \sum_{j=1}^{n} H_j(x, t) \partial_j u(x, t) + H_0(x, t) u(x, t) + \sum_{j=1}^{n} \int_0^t \partial_j H_j(x, t, \eta) \partial_t u(x, \eta) \, d\eta \\
+ \int_0^t \partial_t H_{n+1}(x, t, \eta) \partial_t u(x, \eta) \, d\eta + \int_0^t \partial_t H_0(x, t, \eta) u(x, \eta) \, d\eta, \quad x \in \Omega, \ 0 < t < T, \]

we have

\[ |\partial_t L(u)(x, t)| \leq C \left( |\nabla_x \partial_t u(x, t)| + |\nabla_x u(x, t)| + |u(x, t)| \right) + C \int_0^t \left( |\nabla_x u(x, \eta)| + |u(x, \eta)| \right) \, d\eta, \quad x \in \Omega, \ 0 < t < T. \] (3.8)

Applying Corollary 1 to the function \( w = \partial_t u \) and to the operator \( \partial_t^2 - p(x) \Delta \), corresponding to (1.1) with \( L = K = 0 \), where \( F \) is the right-hand side of the previous equation, we have

\[ \int_{Q(\varepsilon)} (s|\nabla_x \partial_t u|^2 + s^3 |\partial_t u|^2) e^{2s\varepsilon} \, dx \, dt \leq C \int_{Q(\varepsilon)} |f|^2 e^{2s\varepsilon} \, dx \, dt \\
+ C \int_{Q(\varepsilon)} \left( |\Delta u|^2 + |\nabla_x \partial_t u|^2 + |\nabla_x u|^2 + |u|^2 \right) e^{2s\varepsilon} \, dx \, dt \\
+ Ce^{Cs} D + Cs^3 e^{2s(R^2 + 3\varepsilon)} M, \quad s \geq s_0. \]

Hence, for large \( s > 0 \), we deduce

\[ \int_{Q(\varepsilon)} (s|\nabla_x \partial_t u|^2 + s^3 |\partial_t u|^2) e^{2s\varepsilon} \, dx \, dt \leq C \int_{Q(\varepsilon)} |f|^2 e^{2s\varepsilon} \, dx \, dt \\
+ C \int_{Q(\varepsilon)} \left( |\Delta u|^2 + |\nabla u|^2 + |u|^2 \right) e^{2s\varepsilon} \, dx \, dt \\
+ Ce^{Cs} D + Cs^3 e^{2s(R^2 + 3\varepsilon)} M. \] (3.9)
Combining (3.5), (3.6) and (3.9) and taking \( s > 0 \) sufficiently large, we obtain
\[
\int_{Q(\varepsilon)} (|\Delta u|^2 + s|\nabla_{x,t} u|^2 + s|\nabla_{x,t} \partial_t u|^2 + s^3 |\partial_t u|^2 + s^3 u^2) e^{2s\psi} \, dx \, dt \\
\leq C \int_{Q(\varepsilon)} |f|^2 e^{2s\psi} \, dx \, dt + C e^C s D + C s^3 e^{2s(R^2 + 3\varepsilon)} M, \quad s \geq s_0.
\] (3.10)

We now set \( z = \chi(\partial_t u) e^{s\psi} \). The introduction of the new function \( z \) is convenient for estimating the initial value containing \( f \) with the weight function \( e^{2s\psi} \). Then we compute \( \partial_t^2 z \) and \( \Delta z \):
\[
\partial_t^2 z = e^{s\psi} \chi \partial_t^3 u + e^{s\psi} (\partial_t^2 \chi) \partial_t u + s e^{s\psi} \chi (\partial_t u) \left[ \partial_t^2 \psi + s (\partial_t \psi)^2 \right] + 2 e^{s\psi} (\partial_t \chi) \partial_t^2 u + 2 s e^{s\psi} (\partial_t \chi)(\partial_t u) \partial_t \psi + 2 s e^{s\psi} \chi (\partial_t^2 u) \partial_t \psi,
\]
\[
\Delta z = e^{s\psi} \chi \Delta \partial_t u + e^{s\psi} (\Delta \chi) \partial_t u + s e^{s\psi} \chi \partial_t u \left[ \Delta \psi + s |\nabla \psi|^2 \right] + 2 e^{s\psi} \nabla \chi \cdot \nabla \partial_t u + 2 s e^{s\psi} \chi \nabla (\partial_t u) \cdot \nabla \psi + 2 s e^{s\psi} (\partial_t u) \nabla \chi \cdot \nabla \psi.
\]

By these formulae and (3.7), we deduce that \( z \) solves the equation
\[
\partial_t^2 z - p \Delta z = \left[ \chi(\partial_t L(u) + (\partial_t r)f) + (\partial_t^2 \chi) \partial_t u \right. \\
+s \chi \partial_t u (\partial_t^2 \psi + s (\partial_t \psi)^2) + 2 (\partial_t \chi) \partial_t^2 u + 2 s (\partial_t \chi)(\partial_t u) \partial_t \psi \\
+2 s \chi (\partial_t^2 u) \partial_t \psi - p(x)(\Delta \chi) \partial_t u - s p(x) \chi \partial_t u [\Delta \psi + s |\nabla \psi|^2] \\
\left. - 2 p(x) \nabla \chi \cdot \nabla \partial_t u - 2 p(x)s \chi \nabla (\partial_t u) \cdot \nabla \psi - 2 s p(x)(\partial_t u) \nabla \chi \cdot \nabla \psi \right] e^{s\psi} \\
+ \chi e^{s\psi} \left\{ K(x, t, t) \Delta u(x, t) + \int_0^t (\partial_t K)(x, t, \eta) \Delta u(x, \eta) \, d\eta \right\} \equiv J(u).
\] (3.11)

Then we have
\[
|J(u)(x, t)| \leq C e^{s\psi} \left( s |\nabla_{x,t} u(x, t)| + |u(x, t)| \right) \\
+ s |\nabla_{x,t}(\partial_t u)(x, t)| + s^2 |\partial_t u(x, t)| + |\Delta u(x, t)| \\
+ C e^{s\psi} |f(x)| + C e^{s\psi} \int_0^t \left( |\nabla_{x,t} u(x, \eta)| + |u(x, \eta)| + |\Delta u(x, \eta)| \right) \, d\eta,
\] (3.12)

\((x, t) \in Q(\varepsilon)\).

Multiply
\[
- \partial_t^2 z + p \Delta z = - J(u) \quad \text{by} \quad 2 \partial_t z
\]
and integrate over $Q(\varepsilon)$ to obtain
\[
- \int_{Q(\varepsilon)} 2(\partial_t^2 z) \partial_t z \, dx \, dt + \int_{Q(\varepsilon)} 2(\partial_t z) p \Delta z \, dx \, dt
= -2 \int_{Q(\varepsilon)} J(u)(\partial_t z) \, dx \, dt.
\]

(3.13)

We see that
\[
|\partial_t z(x, t)| \leq Cs |\partial_t u(x, t)| e^{\varepsilon p} + C |\partial_t^2 u(x, t)| e^{\varepsilon p}, \quad (x, t) \in Q(\varepsilon)
\]

and
\[
|\nabla z(x, t)| \leq Cs |\partial_t u(x, t)| e^{\varepsilon p} + C |\nabla_x \partial_t u(x, t)| e^{\varepsilon p}, \quad (x, t) \in Q(\varepsilon).
\]

Henceforth let $(\nu, \nu_{n+1}) = (\nu_1, \ldots, \nu_n, \nu_{n+1})$ denote the unit outward normal vector to $\partial Q(\varepsilon)$. Hence, in terms of (1.9) and (3.2), we obtain that $z = |\nabla_{x,t} z| = 0$ on $\partial Q(\varepsilon) \setminus (\Gamma \times (0, T)) \setminus (\Omega(\varepsilon) \times \{0\}), \nabla z = 0$ on $\Omega(\varepsilon) \times \{0\}$ and $\nu_{n+1} = 0$ on $\partial Q(\varepsilon) \cap (\Gamma \times (0, T))$. An integration by parts gives
\[
- \int_{Q(\varepsilon)} 2(\partial_t^2 z) \partial_t z \, dx \, dt + \int_{Q(\varepsilon)} 2(\partial_t z) p \Delta z \, dx \, dt
= - \int_{Q(\varepsilon)} \partial_t (|\partial_t z|^2) \, dx \, dt - \int_{Q(\varepsilon)} p \partial_t (|\nabla z|^2) \, dx \, dt
+ \int_{\partial Q(\varepsilon)} 2(\partial_t z) p \nabla z \cdot \nu \, dS - 2 \int_{Q(\varepsilon)} \nabla p \cdot (\nabla z)(\partial_t z) \, dx \, dt
\]
\[
= \int_{\Omega(\varepsilon)} |\partial_t z(x, 0)|^2 \, dx + 2 \int_{\partial Q(\varepsilon) \cap (\Gamma \times (0, T))} p(\partial_t z) \nabla z \cdot \nu \, dS
- 2 \int_{Q(\varepsilon)} \nabla p \cdot (\nabla z)(\partial_t z) \, dx \, dt
\]
\[
\geq \int_{\Omega(\varepsilon)} |\partial_t z(x, 0)|^2 \, dx + 2 \int_{\partial Q(\varepsilon) \cap (\Gamma \times (0, T))} p(\partial_t z) \nabla z \cdot \nu \, dS
- C \int_{Q(\varepsilon)} (|z|^2 + |\partial_t z|^2) \, dx \, dt.
\]
Hence
\[
\int_{\Omega(t)} |\partial_t z(\cdot, 0)|^2 \, dx \\
\leq -2 \int_{Q(\varepsilon)} J(u)(\partial_t z) \, dx dt + 2 \int_{\partial Q(\varepsilon) \cap (\Gamma \times (0, T))} |p||\partial_t z||\nabla z \cdot \nu| \, dS 
\]
\[+ \int_{Q(\varepsilon)} (s^2|\partial_t u|^2 + |\nabla_x, \partial_t u|^2)e^{2s\phi} \, dx dt.
\]
By (3.12) we have
\[
| -2 \int_{Q(\varepsilon)} J(u)\partial_t z \, dx dt |
\leq C \int_{Q(\varepsilon)} (|u|^2 + |\Delta u|^2 + s|\nabla_x, \partial_t u|^2) (|\partial_t^2 u| + s|\partial_t u|) e^{2s\phi} \, dx dt
\]
\[+ C \int_{Q(\varepsilon)} |f| (|\partial_t^2 u| + s|\partial_t u|) e^{2s\phi} \, dx dt
\]
\[+ C \int_{Q(\varepsilon)} e^{2s\phi} (|\partial_t^2 u| + s|\partial_t u|) \left( \int_0^t (|\nabla_x, \partial_t u(x, \eta)| + |u(x, \eta)| + |\Delta u(x, \eta)|) \, d\eta \right) \, dx dt.
\]
On the other hand, the Cauchy-Schwarz inequality yields
\[s^2|\nabla_x, \partial_t u| \leq s|\nabla_x, \partial_t u|^2 + s^3|\partial_t u|^2\]
and
\[|f| (|\partial_t^2 u| + s|\partial_t u|) \leq |f|^2 + 2|\partial_t^2 u|^2 + 2s^2|\partial_t u|^2,
\]
etc. Taking advantage of Lemma 1, we derive the estimate
\[
\left| -2 \int_{Q(\varepsilon)} J(u)\partial_t z \, dx dt \right|
\leq C \int_{Q(\varepsilon)} (|u|^2 + |\Delta u|^2 + s|\nabla_x, \partial_t u|^2 + s^3|\partial_t u|^2) e^{2s\phi} \, dx dt
\]
\[+ C \int_{Q(\varepsilon)} |f|^2 e^{2s\phi} \, dx dt.
\]
Hence inequality (3.10) yields
\[
\left| -2 \int_{Q(\varepsilon)} J(\partial_t z) \, dx dt \right|
\leq C \int_{Q(\varepsilon)} f^2 e^{2s\phi} \, dx dt + Ce^{C_s} D + Cs^3 e^{2s(K^2+3\varepsilon)} M, \quad s \geq s_0.
\]
Consequently, recalling definition (3.4) of $D$, from (3.13)–(3.15), we derive
\[
\int_{\Omega(\varepsilon)} |\partial_t z(x, 0)|^2 \, dx \leq C \int_{\Gamma \times (0, T)} (|\partial_t z|^2 + |\nabla z|^2) \, dS
+C \int_{\Omega(\varepsilon)} f^2 e^{2\psi} \, dxdt + C e^{Cs} D + Cs^3 e^{2s(R^2+3\varepsilon)} M
\]
(3.16)
\[
\leq C \int_{\Omega(\varepsilon)} f^2 e^{2\psi} \, dxdt + C e^{Cs} D + Cs^3 e^{2s(R^2+3\varepsilon)} M, \quad s \geq s_0.
\]
By (1.8) and (1.9), we have\[(\partial_t z)(x, 0) = \chi(x, 0)(\partial_t^2 u)(x, 0)e^{i\psi(x, 0)} = \chi(x, 0)r(x, 0)f(x)e^{i\psi(x, 0)}\]for $x \in \Omega(\varepsilon)$. Hence, (1.11), (3.2) and (3.16) imply
\[
\int_{\Omega(\varepsilon)} f^2 e^{2\psi} \, dx \leq C \int_{\Omega(\varepsilon)} \chi(x, 0)^2 \, dx
\]
\[
\leq C \int_{\Omega(\varepsilon)} f^2 e^{2\psi} \, dxdt + C e^{Cs} D + Cs^3 e^{2s(R^2+3\varepsilon)} M, \quad s \geq s_0.
\]
Consider now the inequalities
\[
\int_{\Omega(\varepsilon)} f^2 e^{2\psi} \, dxdt
= \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2\psi(x, 0)} \left( \int_0^{(\varepsilon|x_0|^2-(R^2+3\varepsilon))^{1/2}} e^{\frac{1}{2s}(\psi(x, t) - \psi(x, 0))} \, dt \right) \, dx
\]
\[
\leq \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2\psi(x, 0)} \left( \int_0^{+\infty} e^{-2\beta t^2} \, dt \right) \, dx
= \frac{\sqrt{\pi}}{2\sqrt{2}\beta} \frac{1}{\sqrt{s}} \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2\psi(x, 0)} \, dx
\]
and
\[
\int_{\Omega(\varepsilon) \setminus Q(3\varepsilon)} f^2 e^{2\psi} \, dxdt \leq C Me^{2s(R^2+3\varepsilon)}.
\]
Hence
\[
\int_{\Omega(\varepsilon)} |f(x)|^2 e^{2\psi} \, dxdt = \left( \int_{Q(3\varepsilon)} + \int_{Q(\varepsilon) \setminus Q(3\varepsilon)} \right) |f(x)|^2 e^{2\psi} \, dxdt
\]
\[
\leq \frac{C}{\sqrt{s}} \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2\psi(x, 0)} \, dx + C Me^{2s(R^2+3\varepsilon)}.
\]
Therefore from (3.17), we deduce
\[ \int_{\Omega(3\epsilon)} f^2 e^{2\psi(x,0)} \, dx \leq C \sqrt{s} \int_{\Omega(3\epsilon)} f^2 e^{2\psi(x,0)} \, dx + C e^{Cs} D + C s^3 e^{2s(R^2+3\epsilon)} M, \quad s \geq s_0. \]
Hence, for sufficiently large \( s \), we obtain
\[ \int_{\Omega(3\epsilon)} f^2 e^{2\psi(x,0)} \, dx \leq Ce^{Cs} D + Cs^3 e^{2s(R^2+3\epsilon)} M, \quad s \geq s_0. \]
Consequently
\[ e^{2s(R^2+4\epsilon)} \| f \|^2_{L^2(\Omega(4\epsilon))} \leq \int_{\Omega(4\epsilon)} |f(x)|^2 e^{2\psi(x,0)} \, dx \leq \int_{\Omega(3\epsilon)} |f(x)|^2 e^{2\psi(x,0)} \, dx \leq Ce^{Cs} D + Cs^3 e^{2s(R^2+3\epsilon)} M, \quad s \geq s_0, \]
that is,
\[ \| f \|^2_{L^2(\Omega(4\epsilon))} \leq Ce^{Cs} D + Cs^3 e^{-2\epsilon s} M \leq Ce^{Cs} D + Ce^{-\epsilon s} M, \quad s \geq s_0 \]
for a suitable \( C > 0 \). Then we replace \( C > 0 \) with \( Ce^{C_s} \) so that (3.18) holds for all \( s > 0 \). Assume \( M > D \) and choose \( s = \frac{1}{4\beta} \log \frac{M}{D} > 0 \). Then we obtain
\[ \| f \|^2_{L^2(\Omega(4\epsilon))} \leq 2CD^{-\epsilon} D^{\epsilon s}. \]
If \( M \leq D \), then the proof is already complete. Choosing \( \delta = 4\epsilon \), we conclude the proof of Theorem 2.

**Appendix. Proof of Lemma 1.**

First we have
\[ te^{2\psi(x,t)} = -\frac{1}{4\beta s} \partial_t (e^{2\psi}). \]
Therefore, by the Cauchy–Schwarz inequality, we obtain
\[
\left( \int_{Q(\epsilon)} \left( \int_0^t |w(x,\xi)|^2 \, d\xi \right)^2 e^{2\psi} \, dx \, dt \right)^{1/2} \leq \int_{Q(\epsilon)} \left( \int_0^t |w(x,\xi)|^2 \, d\xi \right)^{1/2} e^{2\psi} \, dx \, dt \leq \| f \|_{L^2(\Omega(4\epsilon))} \leq 2CD^{-\epsilon} D^{\epsilon s}.
\]

Here we have set \( \ell(x) = \left( \frac{|x - x_0|^2 - R^2 - \epsilon}{\beta} \right)^{1/2} \).

An integration by parts yields

\[
\int_{Q(\epsilon)} \left( \int_0^t |w(x, \xi)| d\xi \right)^2 e^{2s\phi} \, dx \, dt \\
\leq \frac{1}{4\beta s} \int_{Q(\epsilon)} \left( \int_0^t |w(x, \xi)|^2 d\xi \right) dx + \int_{Q(\epsilon)} |w(x, \xi)|^2 e^{2s\phi} \, dx \, dt \\
\leq \frac{1}{4\beta s} \int_{Q(\epsilon)} |w(x, \xi)|^2 e^{2s\phi} \, dx \, dt.
\]

The proof of Lemma 1 is complete.

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