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Free Vibration Analysis of Multiple Cracked Functionally Graded Timoshenko Beams

Abstract

In this paper, authors present the study of free vibration of bending multiple cracked functionally graded material (FGM) beam. Vibration equations of multiple cracked FGM beam were established by using the rotational spring model of cracks, dynamic stiffness method (DSM) and actual position of neutral plane. The frequency equation obtained was in a simple form, that provides an effective approach to study not only free vibration of the beams but also inverse problems like identification of material and crack parameters in structure. The obtained numerical results show good agreement with other previous published results. Thence, numerical computation has been carried out to investigate the effect of each crack, the number of cracks, material and geometric parameters on the natural frequencies of multiple cracked Timoshenko FGM beams.

Keywords

Timoshenko beam; FGM; Crack; Rotational spring model; DSM; Natural frequency.

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1 INTRODUCTION

Functionally graded materials (FGMs) are inhomogeneous composites characterized by smooth and continuous variations in both compositional profile and material properties. FGMs are widely used in many scientific and engineering fields, such as aerospace, automobile, electronics, optics, chemistry, biomedical engineering, nuclear engineering and mechanical engineering. FGMs have been proved to be advanced materials by their advantaged properties compared to the laminate composites and by the wide application in the high-tech industries.

Crack in a structure usually leads to reduction of stiffness and consequently to change the dynamic characteristics of structure. The cracked problem in FGM is greatly important to evaluate FGM structure serviceability and integrity. Based on fracture mechanics (F. Erdogan, B.H. Wu

1997), the stiffness reduction of FGM beam caused by the presence of cracks can be modeled by continuous stiffness model (A.S.J. Swamidas, X. Yang, R. Seshadri 2004) and rotational spring model (L.L. Ke, J. Yang, S. Kitipornchai, Y. Xiang 2009).

Various methods have been developed for free vibration analysis of cracked FGM beam. The analytical methods have been shown to be most accurate and efficient for dynamic analysis of FGM beam-like structures. A number of authors have developed approximate methods such as Finite Element Method (FEM), Galerkin and Ritz method, dynamic stiffness method.

Yang et al. (2008) used an analytical method to calculate natural frequencies of cracked FGM beam based on Euler-Bernoulli theories and spring model of cracks. Authors also studied free and forced vibration of inhomogeneous Euler-Bernoulli beams under an axial force and a transverse moving load. Ke et al. (2009) studied effects of open edge cracks to vibration of FGM Timoshenko beam with different boundary conditions. This method also solved the discontinuity caused by the presence of cracks in (J. Yang, Y. Chen, Y. Xiang, X.L. Jia 2008; J. Yang, Y. Chen 2008). For FGM Timoshenko beam, Wei et al. (2012) established equations of motion with rotary inertia and shear deformation included. Because of ignoring axial inertia, the bending vibration was still independence from axial vibration. Authors used the transfer matrix method to obtain frequency equations of beam with arbitrary number of cracks only in the form of third-order determinant. This is a remarkable improvement in order to study free vibration of multiple cracked FGM beam. Aydin (2013) presented analytical expression for bending vibration of FGM Euler-Bernoulli beam. Author established frequency equations in the form of third-order determinant without using the transfer matrix method in position of cracks. Sherafatnia et al. (2014) analyzed natural frequencies and mode shapes of cracked beam according to Euler-Bernoulli, Rayleigh, shear deformation and Timoshenko theories. Rakideh et al. (2013) used an analytical method to obtain natural frequencies of the cracked Timoshenko beam. The obtained data have used to design a neural network which can identify characteristics of cracks on beam.

Using Galerkin's procedure with theoretical formulations based on mode shapes of FGM Timoshenko beam and combined with Newmark direct integral method, Yan et al. (2011) obtained dynamic deflections of cracked FGM beam on an elastic foundation under a transverse moving load. Authors showed that the elastic foundation makes dynamic deflections of FGM beam more sensitive to the presence of cracks. Matbuly et al. (2009) investigated the free vibration of an elastically support cracked FGM beam rests on Winkler-Pasternak foundation. Differential quadrature method was employed to determine the natural frequencies and the mode shapes of the beam. Besides, Ritz method was used to analyze nonlinear vibration of cracked FGM Timoshenko beam in (S. Kitipornchai, L.L. Ke, J. Yang, Y. Xiang 2009).

Ziou (2016) used FEM to analyze the response of isotropic and FGM beam. Akbas (2014) studied free vibration and wave propagation analysis of an cracked FGM cantilever beam. However, these results were only applied to Euler-Bernoulli beam. FEM (Z.G. Yu, F.L. Chu 2009; A. Banerjee, B. Panigrahi, G. Pohit 2015) was also used to calculate frequency paragraphs based on the location and size of cracks which is called frequency contours. These paragraphs were not only used to analysis changes of frequencies because of cracks but also employed to identify the crack in FGM beam by measuring natural frequencies.

As the FEM was formulated by using frequency independent polynomial shape function, the FEM could not capture all necessary high frequencies and shape modes of interest. An alternative approach improved the solution accuracy was to use the shape functions that depended on vibration frequency. This elegant concept has led to the so - called dynamic stiffness method (DSM) (N.T. Khiem, T.V.Lien 2002; H.Su, J.R. Banerjee 2015; N.T.Khiem, N.D.Kien, N.N. Huyen 2014; T.V. Lien, N.T. Duc and N.T. Khiem 2016). DSM has used the frequency dependent shape functions obtained from the exact solution of the governing differential equations of free vibration, so the obtained frequencies and mode shapes are accurate. Although finding solutions of the differential equations of motion is very difficult but it is the basic difference between DSM and FEM.

Because of grading material properties, the neutral plane and mid plane of the FGM beam are different. The effect of neutral plane position on static and dynamic behavior of the beam was investigated in some studies. Eltaher et al. (2013) studied free vibration of FGM beam base on Euler-Bernoulli theory include variation of neutral plane position. The numerical results showed that the natural frequencies of the beam are higher when ignoring effect of neutral plane. Moreover, it is emphasized that for the neutral plane theory the governing differential equations of FGM beam were simplified so that the axial and flexural vibration could be uncoupled likely to those of homogeneous beam.

In this paper, authors present free vibration analysis of multiple cracked FGM Timoshenko beam. Vibration equations of multiple cracked FGM beam were established base on rotational spring model of cracks, DSM method and actual position of neutral plane. The frequency equation obtained provides a simple and effective approach to study not only free vibration of the beam but also inverse problem like identification of material and crack parameters in structures.

2 GOVERNING EQUATIONS

Consider a FGM beam of length L, cross section area $A = b \times h$ (Fig. 1). It is assumed that the material properties of FGM beam vary along the thickness direction by power law distribution as follows

$$\begin{Bmatrix} E(z) \\ G(z) \\ \rho(z) \end{Bmatrix} = \begin{Bmatrix} E_b \\ G_b \\ \rho_b \end{Bmatrix} + \begin{Bmatrix} E_t - E_b \\ G_t - G_b \\ \rho_t - \rho_b \end{Bmatrix} \left(\frac{z}{h} + \frac{1}{2} \right)^n, -h/2 \le z \le h/2 \tag{1}$$

Where E, G and ϱ stand for Young's, shear modulus and material density, n is power law exponent, z is co-ordinate of point from the mid plane at high $h/2$. Suppose that the beam is subjected to distributed loads: axial $n(x,t)$, flexural $p(x,t)$ and bending moment $m(x,t)$. Based on the Hamilton's principle, the equations of motion can be established in the time domain (N.T.Khiem, N.D.Kien, N.N. Huyen 2014) as follow

$$\begin{gathered} (I_{11}\ddot{u} - A_{11}u'') - (I_{12}\ddot{\theta} - A_{12}\theta'') = n(x,t) \\ (I_{12}\ddot{u} - A_{12}u'') - (I_{22}\ddot{\theta} - A_{22}\theta'') + A_{33}(w' - \theta) = m(x,t) \\ I_{11}\ddot{w} - A_{33}(w'' - \theta') = p(x,t) \end{gathered} \tag{2}$$

where $\{U,\Theta,W\}^T$ is the amplitudes of axial displacement, rotation, deflection and loads

$$\{U,\Theta,W\} = \int_{-\infty}^{\infty} \{u_0(x,t),\theta(x,t),w_0(x,t)\}e^{-i\omega t}dt$$
$$\{\overline{N}(x,\omega),\overline{M}(x,\omega),P(x,\omega)\} = \int_{-\infty}^{\infty} \{n(x,t),m(x,t),p(x,t)\}e^{-i\omega t}dt \tag{3}$$

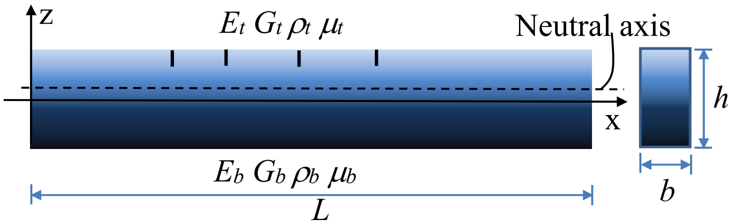


Figure 1: A multiple cracked FGM beam.

Then, vibration equations (2) with amplitudes of displacement, rotation and deflection are in the form

$$\begin{gathered}(\omega^2 I_{11}U + A_{11}U'') - \omega^2 I_{12}\Theta - A_{12}\Theta'' = -\overline{N} \\ (\omega^2 I_{22}\Theta + A_{22}\Theta'') - \omega^2 I_{12}U - A_{12}U'' + A_{33}(W' - \Theta) = -\overline{M} \\ \omega^2 I_{11}W + A_{33}(W'' - \Theta') = -P\end{gathered} \tag{4}$$

Using the following matrices and vector notations

$$\mathbf{A} = \begin{bmatrix} A_{11} & -A_{12} & 0 \\ -A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}; \mathbf{\Pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{33} \\ 0 & -A_{33} & 0 \end{bmatrix}; \mathbf{C}(\omega) = \begin{bmatrix} \omega^2 I_{11} & -\omega^2 I_{12} & 0 \\ -\omega^2 I_{12} & \omega^2 I_{22} - A_{33} & 0 \\ 0 & 0 & \omega^2 I_{11} \end{bmatrix} \tag{5}$$

$$\mathbf{z} = \{U,\Theta,W\}^T, \mathbf{q} = \{\overline{N},\overline{M},P\}^T \tag{6}$$

Equations (4) are rewritten in form

$$\mathbf{Az}'' + \mathbf{\Pi z}' + \mathbf{Cz} = -\mathbf{q} \tag{7}$$

In the case of free vibration, equation (7) is

$$\mathbf{Az}'' + \mathbf{\Pi z}' + \mathbf{Cz} = 0 \tag{8}$$

A continuous solution of equation (8) can be sought in the form $\mathbf{z}_0 = \mathbf{d}e^{\lambda x}$ that yields

$$\det[\lambda^2\mathbf{A} + \lambda\mathbf{\Pi} + \mathbf{C}] = 0 \tag{9}$$

In fact, a cubic algebraic equation with respect to $\eta=\lambda^2$ that can be elementarily solved and gives three roots η_1,η_3,η_3. Therefore, solutions of equation (9) are

$$\lambda_{1,4} = \pm k_1; \lambda_{2,5} = \pm k_2; \lambda_{3,6} = \pm k_3; k_j = \sqrt{\eta_j}, j = 1,2,3 \tag{10}$$

Now, general continuous solution of Eq. (8) can be represented as

$$\mathbf{z}_0(x,\omega) = \sum_{j=1}^{6} \mathbf{d}_j e^{\lambda_j x} \tag{11}$$

or

$$\mathbf{z}_0(x,\omega) = \mathbf{G}(x,\omega)\mathbf{C} \tag{12}$$

With $\mathbf{C} = (C_1,...,C_6)^T$ are constants and $\mathbf{G}(x,\omega) = [\mathbf{G}_1(x,\omega)\ \ \mathbf{G}_2(x,\omega)]$ are function matrices:

$$\mathbf{G}_1(x,\omega) = \begin{bmatrix} e^{k_1x} & e^{k_2x} & e^{k_3x} \\ \alpha_1 e^{k_1x} & \alpha_2 e^{k_2x} & \alpha_3 e^{k_3x} \\ \beta_1 e^{k_1x} & \beta_2 e^{k_2x} & \beta_3 e^{k_3x} \end{bmatrix}; \mathbf{G}_2(x,\omega) = \begin{bmatrix} e^{-k_1x} & e^{-k_2x} & e^{-k_3x} \\ \alpha_1 e^{-k_1x} & \alpha_2 e^{-k_2x} & \alpha_3 e^{-k_3x} \\ -\beta_1 e^{-k_1x} & -\beta_2 e^{-k_2x} & -\beta_3 e^{-k_3x} \end{bmatrix} \tag{13}$$

It is assumed that the beam has been cracked at different position $e_1,...,e_n$ and the cracks are modeled by equivalent springs of stiffness K_j. Therefore, conditions that must satisfied at the crack are (L.L. Ke, J. Yang, S. Kitipornchai, Y. Xiang 2009; N.T. Khiem, T.V.Lien 2002)

$$\begin{gathered} U(e_j+0) = U(e_j-0); \Theta(e_j+0) = \Theta(e_j-0) + M(e_j)/K_j; W(e_j+0) = W(e_j-0) \\ N(e_j) = N(e_j+0) = N(e_j-0); Q(e_j+0) = Q(e_j-0); M(e_j+0) = M(e_j-0) = M(e_j) \end{gathered} \tag{14}$$

where N, Q, M are internal axial, shear forces and bending moment respectively

$$N = A_{11}U'_x - A_{12}\Theta'; M = A_{12}U'_x - A_{22}\Theta'_x; Q = A_{33}(W'_x - \Theta) \tag{15}$$

Substituting (15) into (14), one can rewrite the conditions (14) as follows

$$\begin{gathered} U(e_j+0) = U(e_j-0); \Theta(e_j+0) = \Theta(e_j-0) + \gamma_j^{\theta}\Theta'_x(e_j); W(e_j+0) = W(e_j-0) \\ U'_x(e_j+0) = U'_x(e_j-0); \Theta'_x(e_j+0) = \Theta'_x(e_j-0); W'_x(e_j+0) = W'_x(e_j-0) + \gamma_j\Theta'_x(e_j) \\ \gamma_j = A_{22}/K_j; j = 1,2,3,...,n \end{gathered} \tag{16}$$

Magnitudes γ_j introduced in (16) are function of the material properties such as Young's modulus, power law exponent n and cross sectional dimensions. With the FGM beam, we can present crack magnitudes in the form (L.L. Ke, J. Yang, S. Kitipornchai, Y. Xiang 2009)

$$\gamma_j = \gamma_b \theta_2(R_E, n)$$
$$\gamma_b = E_b I / R; I = bh^3/12\,;\, \theta_2(R_E,n) = 12\left(\frac{3R_E+n}{3(3+n)} - \frac{2R_E+n}{(2+n)}\alpha + \frac{R_E+n}{(1+n)}\alpha^2\right); R_E = \frac{E_t}{E_b} \tag{17}$$

In case of homogenous beam $E_t=E_b=E_0$ ($R_E=1$), the crack magnitudes can be calculated from crack depth a_j as

$$\gamma_{j0} = E_0 I / R_0 = 6\pi(1-\nu^2).h.f(z)\,;\, z = a_j/h$$
$$f(z) = z^2(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 + 47.1063z^6 - 40.7556z^7 + 19.6z^8) \tag{18}$$

Therefore, for modal analysis of cracked FGM beam, we can choose crack magnitudes of equivalent spring in the form (12), it means

$$\gamma_j = F(z) = 6\pi.(1-\nu^2).h.\theta_2(R_E,n)f(z) \tag{19}$$

These functions would be used below for determining spring stiffness from given crack depth.

First, we seek solution $\mathbf{S}$(x) of equation (8) in the form (12) with the left boundary condition

$$\mathbf{S}(0) = (0,1,0)^T; \mathbf{S}'(0) = (0,0,1)^T \tag{20}$$

Then we have to solve equations $[\mathbf{\Pi}]\{\mathbf{C}\} = \{\mathbf{v}\}$, where

$$[\mathbf{\Pi}] = \begin{bmatrix} \mathbf{G}_1(0,\omega) & \mathbf{G}_2(0,\omega) \\ \mathbf{G}_1'(0,\omega) & \mathbf{G}_2'(0,\omega) \end{bmatrix}; \{\mathbf{v}\} = \{0,1,0,0,0,1\}^T$$

These equations have solutions

$$C_1 = C_4 = \delta_1/2; C_2 = C_5 = \delta_2/2; C_3 = C_6 = \delta_3/2 \tag{21}$$

where

$$\delta_1 = \frac{k_2\beta_2 - k_3\beta_3 + (\alpha_3-\alpha_2)}{\delta}; \delta_2 = \frac{k_3\beta_3 - k_1\beta_1 + (\alpha_1-\alpha_3)}{\delta}; \delta_3 = \frac{k_1\beta_1 - k_2\beta_2 + (\alpha_2-\alpha_1)}{\delta}$$
$$\delta = k_1\beta_1(\alpha_3-\alpha_2) + k_2\beta_2(\alpha_1-\alpha_3) + k_3\beta_3(\alpha_2-\alpha_1) \tag{22}$$

So, we obtained

$$S_1(x) = \delta_1 \cosh k_1x + \delta_2 \cosh k_2x + \delta_3 \cosh k_3x$$
$$S_2(x) = \delta_1\alpha_1 \cosh k_1x + \delta_2\alpha_2 \cosh k_2x + \delta_3\alpha_3 \cosh k_3x \tag{23}$$
$$S_3(x) = \delta_1\beta_1 \sinh k_1x + \delta_2\beta_2 \sinh k_2x + \delta_3\beta_3 \sinh k_3x$$

Denoting solutions of equation (8) in the interval (e_j, e_{j+1}) by $\mathbf{z}_j(x)$, it is easy to verify that

$$\mathbf{z}_j(x) = \mathbf{z}_{j-1}(x) + \gamma_j \Theta'_{j-1}(e_j)\mathbf{S}(x - e_j) \tag{24}$$

where $\mathbf{z}_{j-1}(x)$ is solution in (e_{j-1}, e_j) being continuously expended to the subsequent interval (e_j, e_{j+1}) and $\mathbf{S}(x)$ is solution of equation (8) conducted in the form (23). Namely, since both functions $\mathbf{z}_{j-1}(x)$, $\mathbf{S}(x-e_j)$ are solutions of equation (8) in (e_j, e_{j+1}), so their combination in (24) would be solution of that equation in the interval. Moreover, solution (24) satisfies also the conditions

$$\begin{aligned} U_j(e_j) = U_{j-1}(e_j);\ \Theta_j(e_j) = \Theta_{j-1}(e_j) + \gamma_j \Theta'_{j-1}(e_j); W_j(e_j) = W_{j-1}(e_j) \\ U'_j(e_j) = U'_{j-1}(e_j); \Theta'_j(e_j) = \Theta'_{j-1}(e_j); W'_j(e_j) = W'_{j-1}(e_j) + \gamma_j \Theta'_j(e_j) \end{aligned} \tag{25}$$

These conditions ensure that solution of equation (8) in the form of (24) satisfy condition at crack position (16). Based on the recurrent connection, one can express solution of equation (8) for beam with n cracks in the form

$$\mathbf{z}_c(x) = \mathbf{z}_0(x) + \sum_{j=1}^{n} \mu_j \mathbf{K}(x - e_j) \tag{26}$$

$$\mu_j = \gamma_j [\Theta'_0(e_j) + \sum_{k=1}^{j-1} \mu_k S'_2(e_j - e_k), j = 1,2,3,...,n \tag{27}$$

In above equations, $\mathbf{z}_0(x)$ is continuous solution in the form (12) and function K(x) is

$$\mathbf{K}(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ \mathbf{S}(x) & \text{for} \quad x > 0 \end{cases}; \mathbf{K}'(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ \mathbf{S}'(x) & \text{for} \quad x > 0 \end{cases} \tag{28}$$

To complete solving Eq. (8), solution obtained in (26) must satisfy boundary conditions in the beam ends

$$\mathbf{B}_0 \{\mathbf{z}\}|_{x=0} = 0 ; \mathbf{B}_\mathrm{L} \{\mathbf{z}\}|_{x=L} = 0 \tag{29}$$

where $\mathbf{B}_0$, $\mathbf{B}_\mathrm{L}$ are differential matrix operators of dimension 3×3. Since the second term of solution (26) satisfy any trivial condition at $x = 0$, the first condition in (29) is only applied for $\mathbf{z}_0(x)$.

Separating constant vector $\mathbf{C} = \{\mathbf{C}_1, \mathbf{C}_2\}^T$ into $\mathbf{C}_1 = \{C_1, C_2, C_3\}^T; \mathbf{C}_2 = \{C_4, C_5, C_6\}^T$, the boundary condition at left end of the beam can be presented as

$$\mathbf{B}_{01}\mathbf{C}_1 + \mathbf{B}_{02}\mathbf{C}_2 = 0 \tag{30}$$

where

$$\mathbf{B}_{01}(\omega) = \mathbf{B}_0 \{\mathbf{G}_1(x,\omega)\}|_{x=0}; \mathbf{B}_{02}(\omega) = \mathbf{B}_0 \{\mathbf{G}_2(x,\omega)\}|_{x=0}$$

Eq. (30) allows eliminating one of the vectors $\mathbf{C}_1, \mathbf{C}_2$. And as the result the solution $\mathbf{z}_0(x)$ can be reassembled as $\mathbf{z}_0(x,\omega) = \mathbf{G}_0(x,\omega)\mathbf{D}$ with $\mathbf{G}_0(x,\omega)$ is 3×3 dimension matrix function and arbitrary constant vector $\mathbf{D} = \{D_1, D_2, D_3\}^T$. In particular, one has

$$\Theta_0(x) = [g_{21}(x,\omega)D_1 + g_{22}(x,\omega)D_2 + g_{23}(x,\omega)D_3] \tag{31}$$

where $g_{2k}(x,\omega), k = 1,2,3$ are element on second row of matrix $\mathbf{G}_0(x,\omega)$. So, solution (26) can be rewritten as

$$\mathbf{z}_c(x) = \mathbf{G}_0(x,\omega)\mathbf{D} + \sum_{j=1}^{n} \mu_j \mathbf{K}(x - e_j) \tag{32}$$

In order to solution of (32) satisfy boundary condition at right end of the beam, we have

$$\begin{gathered} [\mathbf{B}_{L0}(\omega)]\{\mathbf{D}\} + \sum_{j=1}^{n} \mu_j \{\mathbf{b}(e_j)\} = 0 \\ \mathbf{B}_{\mathbf{L}0}(\omega) = \mathbf{B}_{\mathbf{L}}\{\mathbf{G}_0(x,\omega)\}\big|_{x=L}; \mathbf{b}_c(e_j) = \mathbf{B}_L\{\mathbf{S}(x - e_j)\}\big|_{x=L} \end{gathered} \tag{33}$$

In case of intact beam, when $\mu_j = 0, j = 1,..., n$, equation (30) is reduced to

$$[\mathbf{B}_{L0}(\omega)]\{\mathbf{D}\} = 0 \tag{34}$$

That enables to determine undamaged natural frequencies by solving the equation

$$L_0(\omega) = \det[\mathbf{B}_{L0}(\omega)] = 0 \tag{35}$$

Each roots ω_j^0 of this equation is related to mode shape

$$\Phi_j^0(x) = C_j^0 \mathbf{G}_0(x, \omega_j^0)\overline{\mathbf{D}}_j \tag{36}$$

where C_j^0 is an arbitrary constant and $\overline{\mathbf{D}}_j$ is the normalized solution of (34) corresponding to ω_j^0.

For cracked beam, the constant vector $\mathbf{D}$ is sought in the form $\mathbf{D} = \sum_{j=1}^{n} \mu_j \mathbf{D}_j$ that leads the equation (33) to $[\mathbf{B}_{L0}(\omega)]\{\mathbf{D}_j\} = -\{\mathbf{b}_c(e_j)\}$. Hence, one is able to calculate

$$\mathbf{D}_j = -[\mathbf{B}_{L0}(\omega)]^{-1}\{\mathbf{b}_c(e_j)\} = -(1/L_0)\{\overline{\mathbf{b}}_c(e_j)\} \tag{37}$$

Therefore, solution (32) gets the form

$$\mathbf{z}_c(x) = (1/L_0)\sum_{j=1}^{n}\mu_j[L_0(\omega)\mathbf{K}(x-e_j) - \mathbf{G}_0(x,\omega)\overline{\mathbf{b}}_c(e_j)] \tag{38}$$

The above solution still contains unknown parameters $\mu_j,\ j = 1,..., n$. Substituting (31) together with constant vector **D** found above into (27), one gets

$$\mu_j = (\gamma_j / L_0)\sum_{k=1}^{n}\mu_k[L_0(\omega)K_2'(e_j - e_k) - g'_{jk}],\ j = 1,2,3,...,n$$
$$g'_{jk} = g'_{21}(e_j,\omega)\overline{b}_{c1}(e_k) + g'_{22}(e_j,\omega)\overline{b}_{c2}(e_k) + g'_{23}(e_j,\omega)\overline{b}_{c3}(e_k)$$

The above equation can be rewritten in the matrix form

$$\begin{gathered}[L_0(\omega)\mathbf{I} - \Gamma(\boldsymbol{\gamma})\mathbf{A}(\mathbf{e},\omega)]\{\boldsymbol{\mu}\} = 0\\ \Gamma(\boldsymbol{\gamma}) = diag\{\gamma_1,...,\gamma_n\};\mathbf{A}(\mathbf{e},\omega) = [a_{jk} = L_0(\omega).K_2'(e_j - e_k) - g'_{jk};\ j,k = 1,2,...,n]\\ \boldsymbol{\mu} = \{\mu_1,...,\mu_n\}^T;\boldsymbol{\gamma} = \{\gamma_1,...,\gamma_n\}^T;\mathbf{e} = \{e_1,...,e_n\}^T\end{gathered} \tag{39}$$

Condition for existence of non-trivial solution of equation (39) is

$$f(\omega,\boldsymbol{\gamma},\mathbf{e}) \equiv \det[L_0(\omega)\mathbf{I} - \Gamma(\boldsymbol{\gamma})\mathbf{A}(\mathbf{e},\omega)] = 0 \tag{40}$$

This is so-called frequency equation for FGM beam with arbitrary number of cracks, solution of which gives natural frequencies *(ω_j, j=1,2,3,...)*. With each obtained natural frequency, we can determine one parameter vector $\boldsymbol{\mu}_j$ from the equation (39).

3 ANALYSIS OF THE FREE VIBRATION OF MULTIPLE CRACKED FGM TIMOSHENKO BEAM

3.1 Comparison with published numerical results

3.1.1 Homogeneous beam with an open edge crack

Consider a homogeneous beam with material parameters: $E_t=E_b=210$GPa, $\rho=7800$kg/m^3, $\mu=0.3$ and geometric parameters: L=0.8m, b=0.02m, h=0.02m. Beam has one open edge crack with variable location and crack depth/height ratio is a/h=0.2.

Table 1 shows the fundamental frequency ratios between cracked beam and intact one. These published results were taken from studies of Khiem & Lien (2002) and Aydin (2013). Khiem & Lien (2002) used transfer matrix method to calculate natural frequencies of multiple cracked beam while Aydin (2013) used analytical method to obtain natural frequencies of FGM beam. The obtained numerical results show good agreement with previous announced results.

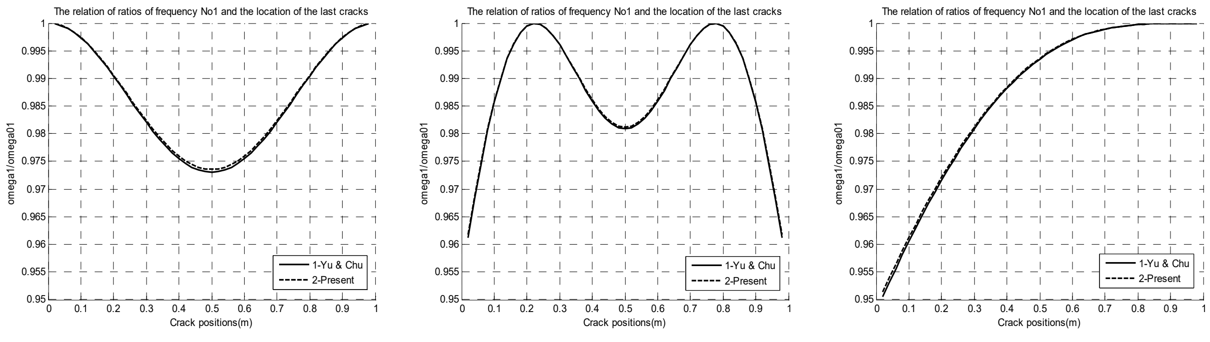
Simple support beam	$X_1/L=0.2$	$X_1/L=0.4$	$X_1/L=0.7$
Khiem & Lien	0.995	0.991	0.998
Aydin	0.9959	0.9916	0.9985
Present	0.9953	0.9908	0.9986
Clamped end beam	$X_1/L=0.1$	$X_1/L=0.3$	$X_1/L=0.4$
Khiem & Lien	0.997	0.996	0.993
Aydin	0.9971	0.9963	0.9943
Present	0.9968	0.9959	0.9933
Cantilever beam	$X_1/L=0.2$	$X_1/L=0.4$	$X_1/L=0.6$
Khiem & Lien	0.990	0.996	0.998
Aydin	0.9906	0.9958	0.9982
Present	0.9920	0.9978	0.9998

Table 1: Fundamental frequency ratios between beam that has an open edge crack and intact beam.

3.1.2 Comparison with cracked FGM beam

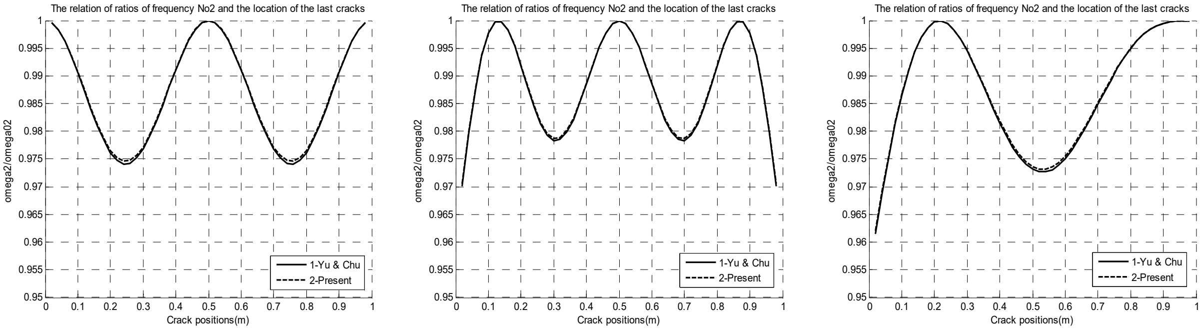
Consider a FGM beam with geometric parameters: L=1.0m, b=0.1m, h=0.05m and material parameters: E_t=70GPa, ρ_t=2780kg/m^3, μ_t=0.33, E_b/E_t=0.2, ρ_b=7800kg/m^3, μ_t=0.33, n=0.1 (Z.G. Yu, F.L. Chu 2009).

Fig 2 and 3 present variation of the fundamental and the secondary frequency ratios of FGM beam which has one open edge crack (a/h=0.2) and an intact one when crack location move from the left end to the right end of the beam (hidden line) with announced results of Yu & Chu (2009) (continuous line). The boundary conditions are simple support, clamped end and cantilever. The present results are the same as announced ones using FEM in (Z.G. Yu, F.L. Chu 2009).



a. Simple support beam b. Clamped end beam c. Cantilever beam

Figure 2: Variation of the fundamental frequency ratio of FGM beam which has one open edge crack (a/h=0.2) and an intact one when crack location move along beam length.



a. Simple support beam b. Clamped end beam c. Cantilever beam

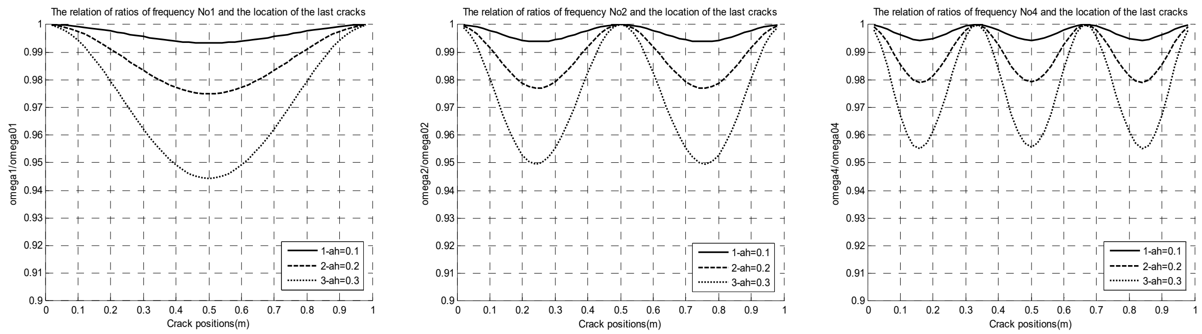
Figure 3: Variation of the secondary frequency ratio of FGM beam which has one open edge crack (a/h=0.2) and an intact one when crack location move along beam length.

3.2 Effects of material parameters to natural frequencies of cracked FGM beam

Consider a FGM beam with geometric parameters: L=1.0m, b=0.1m, h=0.1m and material parameters: E_t=70GPa, ρ_t=2780kg/m^3, μ_t=0.33, E_b/E_t=5, ρ_b=7800kg/m^3, μ_t=0.33, n=0.5. Beam has one crack that location move along beam length and crack depth/height ratios are a/h=0.1,0.2,0.3.

3.2.1 Effect of the crack depth

Fig 4 shows variation of the first three natural frequency ratios of simple support FGM beam that has one open edge crack and an intact one when the crack depth/height ratios are a/h=0.1,0.2,0.3. When crack depth increases, the natural frequencies of the beam decrease remarkable. A fact that might be derived from graphics given in the figures is that there exist certain positions in the beam, at which cracks do not affect certain natural frequencies. Such positions are called here *critical points* for a given frequency.



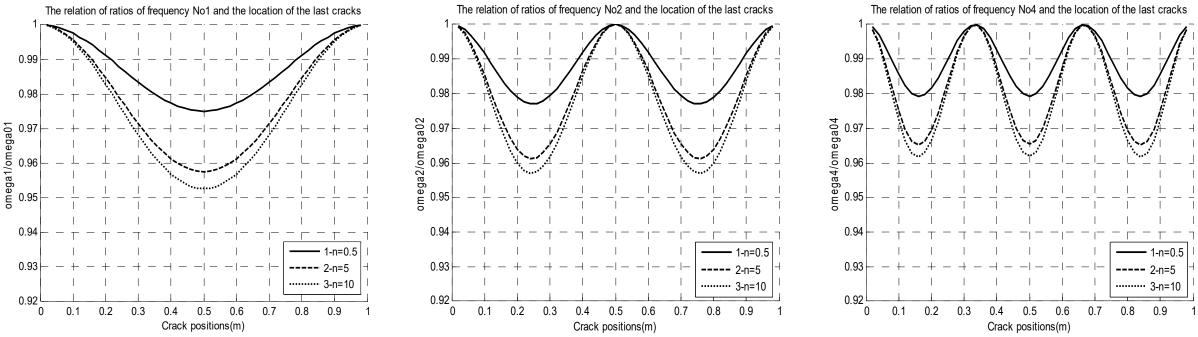
a. Fundamental frequency b. Secondary frequency c. Third frequency

Figure 4: Variation of first three natural frequency ratios of simple support FGM beam that has one crack and an intact one when the crack depths are 10%, 20%, 30% beam height.

3.2.2 Effect of power law index n, E_b/E_t and L/h ratios

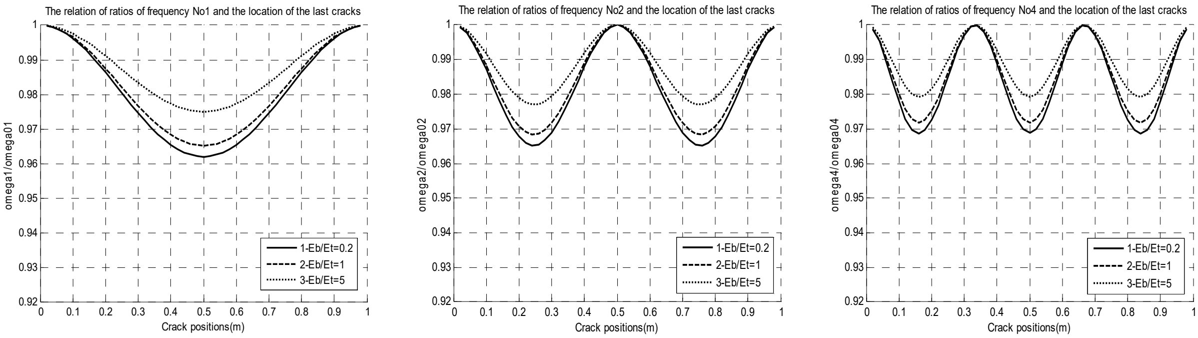
Figs 5-7 show variation of the first three natural frequency ratios of simple support FGM beam that has one open edge crack (a/h=0.2) and an intact one with different power law index n, E_b/E_t and L/h ratios.

When power law index n or L/h ratio increase or E_b/E_t decrease, the beam is more sensitive to presence of crack. With n<1 (or E_b/E_t>1), changing these parameters will make frequencies change much more than with n>1 (or E_b/E_t<1). Especially, we can see that in Figs 7.b-c, there are horizontal line (frequency ratio is 1) which mean the presence of crack at any position of the beam doesn't effect to natural frequency because changing L/h ratio will change the order of axial frequency. In case of simple support beam, with L/h=5, the axial frequency is the secondary frequency but with L/h=20 it is the third one. The same problems are met in clamped end and cantilever beam.



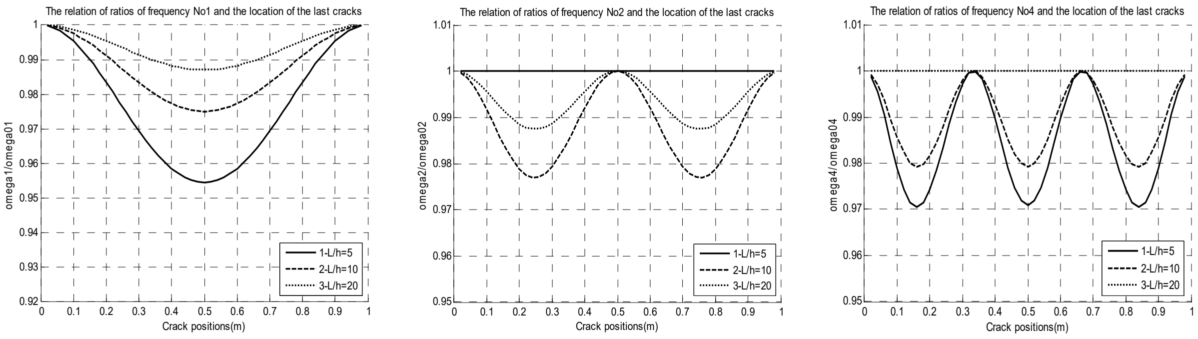
a. Fundamental frequency b. Secondary frequency c. Third frequency

Figure 5: Variation of the first three natural frequency ratios of simple support FGM beam that has one open edge crack (a/h=0.2) and an intact one with different power law index n=0.5, 5, 10.



a. Fundamental frequency b. Secondary frequency c. Third frequency

Figure 6: Variation of the first three natural frequency ratios of simple support FGM beam that has one open edge crack (a/h=0.2) and an intact one when n=0.5 and E_b/E_t=0.2,1,5.

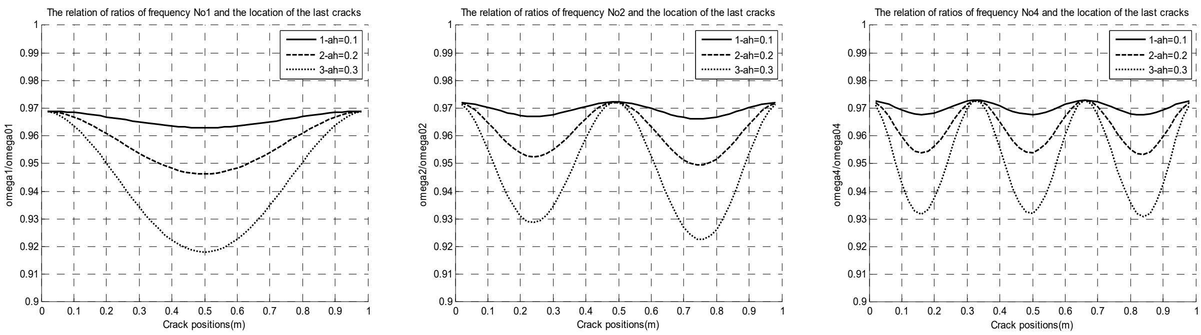


a. Fundamental frequency b. Secondary frequency c. Third frequency

Figure 7: Variation of the first three natural frequency ratios of simple support FGM beam that has one open edge crack (a/h=0.2) and an intact one when n=0.5 and L/h=5,10,20.

3.2.3 Effect of number of cracks

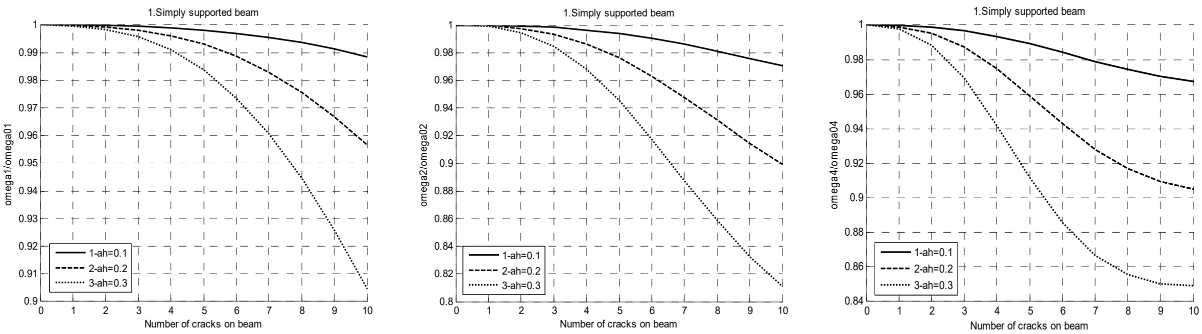
Fig 8 shows variation of the first three natural frequency ratios of simple support FGM beam which has 3 cracks and an intact one. The first two cracks are located at 0.2L, 0.4L and crack depth/height ratio is 0.2. The third crack move along beam length and has different depth as: 10%, 20%, 30% beam height. Compare to Fig 4, we can see that when number of crack increases frequency ratio decreases and paragraphs are not symmetric from the midpoint of the beam because of unsymmetric presence of cracks.



a. Fundamental frequency b. Secondary frequency c. Third frequency

Figure 8: Variation of the first three natural frequency ratio of simple support FGM beam that has three open edge crack and an intact one when n=0.5. The first two crack locates at $X_1/L=0.2$, $X_2/L=0.4$ (a/h=0.2), the third crack has the depth of 10%, 20%, 30% beam height.

Fig 9 shows variation of the first three natural frequency ratios of simple support FGM beam that has 10 cracks and intact one. Equidistant cracks are located in left quarter part of the beam with the depth of 10%, 20%, 30% beam height. It is easy to see that when number of cracks on beam increases, the frequency will decrease significantly.



a. Fundamental frequency b. Secondary frequency c. Third frequency

Figure 9: Variation of the first three natural frequency ratios of simple support FGM beam that has 10 cracks and an intact one when n=0.5. Equidistant cracks are located in left quarter part of the beam with the depth of 10%, 20%, 30% beam height.

4 CONCLUSIONS

In this paper, authors established free vibration equation of multiple cracked FGM beam based on Timoshenko beam model, power law distribution of FGM material, rotation spring model, DSM and taking into account actual position of neutral axis. The frequency equation obtained is in a simple form, which provides an effective approach to study not only free vibration of the beam but also inverse problem like identification of material and crack parameter in structure. The obtained numerical results show good agreement with other previous announced results.

Authors studied changes of natural frequencies of FGM beam with different material, geometric and crack parameters. Bending frequencies are more sensitive to cracks than axial one and much depend on material, geometric parameters of the beam. Increasing the depth and number of cracks on FGM Timoshenko beam significantly decreases natural frequencies.

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